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The character sum of polynomials with k variables and two-term exponential sums

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Abstract: The main purpose of this paper is using the properties of the classical Gauss sums and the analytic methods to study the computational problem of one kind of hybrid power mean involving the character sums of polynomials with k variables and the two-term exponential sums, and give an identity and asymptotic formula for it.
Keywords: Character sums of polynomials with k variables, Two-term exponential sums, Hybrid power mean, Analytic method, Identity, Asymptotic formula.
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1 Introduction

Let $q > 1$ be an integer, k and h be integers with $k > h \geq 1$. For any integers m and n, the two-term exponential sums $K(m, n, k, h; q)$ are defined as follows:

$$K(m, n, k, h; q) = \sum_{a=1}^{q-1} e\left(\frac{ma^k + na^h}{q}\right),$$

where as usual, $e(y) = e^{2\pi i y}$.

This sum occupies a very important position in the research of analytic number theory, and many classical problems in analytic number theory are closely related to it. For example, Waring's problem, exponential sum problem, and so on. Therefore, any substantial progress in this field will certainly promote the development of analytic number theory. It is for these reasons that many scholars have studied the properties of $K(m,n,k,h;q)$, and obtained a series of important results. For example, Zhang H. and Zhang W. P. [13] proved a precise formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3-p^2, & \text{if } 3 \nmid p-1, \\ 2p^3-7p^2, & \text{if } 3|p-1, \end{cases}$$

where p indicates an odd prime, and n is any integer with $(n,p)=1$.

On the other hand, Lv X. X. and Zhang W. P. [7] first introduced a sum analogous to Kloosterman sum as follows:

$$K(m,n,r,\chi;q) = \sideset{}{'}\sum_{a=1}^{q} \chi\left(ma+n\overline{a}\right) e\left(\frac{ra}{q}\right),$$

where $m,\ n$ and r are integers, χ denotes any Dirichlet character $\bmod\ q$, $\overline{a}$ denotes the solution of the congruence equation $xa \equiv 1 \bmod q$, and $\sideset{}{'}\sum_{a=1}^{q}$ denotes the summation over all integers $1 \le a \le q$ such that $(a,q)=1$.

Then they used the analytic methods and the properties of the classical Gauss sums to prove that for any odd prime p with $p \equiv 3 \bmod 4$ and integers m and n with $(mn,p)=1$, one has the identity

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2 \cdot \left|\sum_{b=1}^{p-1}\chi\left(b+\overline{b}\right)\right|^2 = (p-1)\left(3p^2-6p-1\right)$$

and

$$\begin{aligned} &\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2 \cdot \left|\sum_{b=1}^{p-1}\chi\left(b+\overline{b}\right)e\left(\frac{nb}{p}\right)\right|^2 \\ = \ &(p-1)\left(p^2-2p-1\right)+p(p-1)\left(\sum_{b=2}^{p-2} e\left(\frac{n\left(b+\overline{b}\right)}{p}\right)+\sum_{b=2}^{p-2} e\left(\frac{n\left(b-\overline{b}\right)}{p}\right)\right). \end{aligned}$$

From these identities and the estimate for Kloosterman sum, Lv X. X. and Zhang W. P. [7] deduced the asymptotic formula

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2 \cdot \left|\sum_{b=1}^{p-1}\chi\left(b+\overline{b}\right)e\left(\frac{nb}{p}\right)\right|^2 = p^3+O\left(p^{\frac{5}{2}}\right).$$

Smith R. A. [10] studied the properties of the n-dimensional Kloosterman sums, and obtained a sharp upper bound estimate for

$$S(m,n;q) = \sideset{}{'}\sum_{a_1=1}^{q}\cdots\sideset{}{'}\sum_{a_n=1}^{q} e\left(\frac{a_1+a_2+\cdots+a_n+m\overline{a_1}\cdot\overline{a_2}\cdots\overline{a_n}}{q}\right).$$

Zhang W. P. and Han D. [16] studied the fourth power mean of the 2-dimensional Kloosterman sums, and proved the identity

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{ma+b+\overline{ab}}{p}\right)\right|^4\\ =&\left\{\begin{array}{ll} 7p^5-18p^4-(b_p+6)\,p^3-6p^2-3p, & \text{if } p\equiv 1 \bmod 6;\\ 7p^5-22p^4-(b_p-14)\,p^3-6p^2-3p, & \text{if } p\equiv 5 \bmod 6,\end{array}\right.\end{aligned}$$

where b_p is an integer satisfying $|b_p| < 2p^{\frac{3}{2}}$.

Zhang W. P. and Li X. X. [17] obtained the identity

$$\begin{aligned}&\sum_{\chi \bmod p}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(ab)e\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^4\\ =&\ (p-1)(2p^5-7p^4+2p^3+8p^2+4p+1).\end{aligned}$$

Of course, many papers related to generalized Kloosterman sums, high-dimensional Kloosterman sums, character sums of polynomials and two-term exponential sums can also be found in [2–7, 11–21] (here we will not list them one by one).

Recently, Lv X. X. and Zhang W. P. [8] studied the hybrid power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\bar{a}\bar{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb+\bar{b}}{p}\right)\right|^2,$$

and proved the following results:

Let p be an odd prime with $3\nmid(p-1)$. Then for any non-principal character $\chi \bmod p$, one has the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\bar{a}\bar{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb+\bar{b}}{p}\right)\right|^2=p^2\cdot\left(p^2-p-1\right)$$

and

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\bar{a}\bar{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2=p^2\cdot\left(p^2-p-1\right).$$

If p is an odd prime with $3\mid(p-1)$. Then for any third character $\chi \bmod p$ (i.e., there exists a character $\chi_1 \bmod p$ such that $\chi=\chi_1^3$), then one has the asymptotic formulae

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\bar{a}\bar{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb+\bar{b}}{p}\right)\right|^2=3p^4+E(p)$$

and

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\bar{a}\bar{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2=3p^4+E_1(p),$$

where $E(p)$ and $E_1(p)$ satisfy $|E(p)|\le 9\cdot p^{\frac{7}{2}}$ and $|E_1(p)|\le 15\cdot p^3$, respectively.

In this paper, as a note of [8], we consider the following generalized hybrid power mean:

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{h+1}+a}{p}\right)\right|^2, \tag{1}$$

where p is an odd prime, χ is a character $\bmod p$, and h is any fixed positive integer.

It is clear that the results in [8] are some special cases of (1). That is, $h=2$. Han D. [4] also studied a similar problem, and she proved that for any integer k, one has the asymptotic formula:

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^k+na}{p}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}\chi\left(ma+\overline{a}\right)\right|^2=\begin{cases}2p^3+O\left(|k|p^2\right), & \text{if } 2\mid k;\\ 2p^3+O\left(|k|p^{\frac{5}{2}}\right), & \text{if } 2\nmid k.\end{cases}$$

The main purpose of this paper is using the analytic methods and the properties of the classical Gauss sums to study the computational problem of (1), and give two generalized conclusions. That is, we will prove the following:

Theorem 1.1. *Let p be an odd prime, and $h\geq 1$ be an integer with $(h+1,p-1)=1$. Then for any non-principal character $\chi \bmod p$, we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{h+1}+a}{p}\right)\right|^2$$

$$=p^h\cdot\left(p^2-p-1\right).$$

Theorem 1.2. *Let p be an odd prime, h be an integer with $(h+1)\mid(p-1)$, and χ be any non-principal character $\bmod p$. If χ is the $(h+1)$-st character $\bmod p$ (that is, there exists a character $\chi_1 \bmod p$ such that $\chi=\chi_1^{h+1}$), then we have the asymptotic formula*

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{h+1}+a}{p}\right)\right|^2$$

$$=(h+1)\cdot p^{h+2}+O\left(h^2\cdot p^{h+1}\right);$$

If χ is not the $(h+1)$-st character $\bmod p$, then we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{h+1}+a}{p}\right)\right|^2=0.$$

Some notes: Let p be an odd prime, $h\geq 1$ be an integer with $(h+1)\mid(p-1)$, and χ be any $(h+1)$-st character $\bmod p$. It seems that using our methods we cannot obtain any asymptotic formula for the fourth power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right|^4.$$

Whether there exits a sharp asymptotic formula for this fourth power mean is an open problem. Interested readers are suggested to study it with us.

2 Some lemmas

At many places of this paper, we need to use the definition and properties of the Gauss sums $\tau(\chi)$ and character sums, these contents can be found in some analytic number theory books, such as [1] or [9], here we will not repeat the related contents. First we have the following lemma.

Lemma 2.1. *Let p be an odd prime, $k \geq 1$ be an integer with $(k+1) \mid (p-1)$, χ be any non-principal character* $\bmod p$*, and m be an integer with $(m,p)=1$. If χ is not the $(k+1)$-st character* $\bmod p$*, then we have the identity*

$$\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2=0.$$

If χ is the $(k+1)$-st character $\bmod p$*, then we have the identity*

$$\begin{aligned}&\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2\\ =\ &(k+1)\cdot p^k+\frac{1}{p}\sum_{j=1}^{k}\tau^{k+1}\left(\lambda^j\right)\overline{\lambda}^j(m)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)\\ &\times\overline{\lambda}^j(b-1)\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right),\end{aligned}$$

where λ denotes any $(k+1)$-order character $\bmod p$.

Proof. If $(k+1) \mid (p-1)$, and χ is not the $(k+1)$-st character $\bmod p$, then there exists an integer $1<r<p-1$ such that $r^{k+1}\equiv\overline{r}^{k+1}\equiv 1 \bmod p$ and $\chi(r)\neq 1$. So from the properties of the reduced residue system $\bmod p$ we have

$$\begin{aligned}&\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\\ =\ &\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(ra_1+ra_2+\cdots+ra_k+m\overline{r}^k\overline{a_1\cdots a_k}\right)\\ =\ &\chi(r)\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{r}^{k+1}\overline{a_1\cdots a_k}\right)\\ =\ &\chi(r)\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right).\end{aligned}\tag{2}$$

Since $\chi(r)\neq 1$, so from (2) we have the identity

$$\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2=0.\tag{3}$$

If $(k+1) \mid (p-1)$, and χ is the $(k+1)$-st character $\bmod p$, then for any integer m with $(m,p)=1$, note that the identity

$$
\begin{aligned}
&\sum_{a=0}^{p-1} e\left(\frac{ma^{k+1}}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1+\lambda(a)+\lambda^2(a)+\cdots+\lambda^k(a)\right) e\left(\frac{ma}{p}\right) \\
= \;& \overline{\lambda}(m)\tau\left(\lambda\right) + \overline{\lambda}^2(m)\tau\left(\lambda^2\right) + \cdots + \overline{\lambda}^k(m)\tau\left(\lambda^k\right), \qquad (4)
\end{aligned}
$$

where λ denotes the $(k+1)$-order character $\bmod\ p$.

From the properties of the reduced residue system $\bmod\ p$ and the trigonometric identity

$$
\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{if } q \nmid n \end{cases} \qquad (5)
$$

we have

$$
\begin{aligned}
&\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} \chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2 \\
= \;& \frac{1}{p}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\left(b\right) e\left(\frac{b\left(a_1+a_2+\cdots+a_k+m\overline{a_1a_2\cdots a_k}\right)}{p}\right)\right|^2 \\
= \;& \frac{1}{p}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\left(ba_1\cdots a_k\right) e\left(\frac{ba_1\cdots a_k\left(a_1+\cdots+a_k\right)+mb}{p}\right)\right|^2 \\
= \;& \frac{1}{p}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\left(ba_1a_2\cdots a_k\right)\sum_{c_1=1}^{p-1}\cdots\sum_{c_k=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
& \times e\left(\frac{1}{p}\sum_{i=1}^{k} dc_1\cdots c_{i-1}c_i^2\cdot c_{i+1}\cdots c_k\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right) \\
= \;& \frac{1}{p}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\left(ba_1a_2\cdots a_k\right)\sum_{c_1=1}^{p-1}\cdots\sum_{c_k=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
& \times e\left(\frac{1}{p}\left(c_1^2\overline{c_2}\cdots\overline{c_k}\overline{d}\left(ba_1^2a_2a_3\cdots a_k-1\right)\right)\right) \\
& \times e\left(\frac{1}{p}\sum_{i=2}^{k} c_1c_i\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right) \\
= \;& \frac{1}{p}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\left(ba_1a_2\cdots a_k\right)\sum_{c_1=0}^{p-1}\cdots\sum_{c_k=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
& \times e\left(\frac{1}{p}\left(c_1^{k+1}\overline{c_2}\cdots\overline{c_k}\overline{d}\left(ba_1^2a_2\cdots a_k-1\right)\right)\right) \\
& \times e\left(\frac{1}{p}\sum_{i=2}^{k} c_i\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right) \\
& -\frac{1}{p}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\left(ba_1a_2\cdots a_k\right)\sum_{c_2=1}^{p-1}\cdots\sum_{c_k=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
& \times e\left(\frac{1}{p}\sum_{i=2}^{k} c_i\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right) = E_1 - E_2. \qquad (6)
\end{aligned}
$$

Note that the identity

$$\sum_{a_1=1}^{p-1} \chi(a_1) = 0,$$

from the properties of the reduced residue system $\bmod p$ we have

$$\begin{aligned}
E_2 &= \frac{1}{p} \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(b a_1 a_2 \cdots a_k\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
&\times e\left(\frac{1}{p} \sum_{i=2}^{k} c_i \left(b a_1 \cdots a_{i-1} a_i^2 a_{i+1} \cdots a_k - 1\right)\right) \\
&= \frac{1}{p} \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(a_1\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) e\left(\sum_{i=2}^{k} \frac{c_i \left(a_1 a_i - 1\right)}{p}\right) \\
&= \frac{1}{p} \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(a_1\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) e\left(\sum_{i=2}^{k} \frac{c_i \left(a_i - 1\right)}{p}\right) \\
&= 0. \qquad (7)
\end{aligned}$$

From (4) and (5) we have

$$\begin{aligned}
E_1 &= \frac{1}{p} \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(b a_1 a_2 \cdots a_k\right) \sum_{c_1=0}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
&\quad \times e\left(\frac{1}{p} \left(c_1^{k+1} \overline{c_2} \cdots \overline{c_k} \overline{d} \left(b a_1^2 a_2 \cdots a_k - 1\right)\right)\right) \\
&\quad \times e\left(\frac{1}{p} \sum_{i=2}^{k} c_i \left(b a_1 \cdots a_{i-1} a_i^2 a_{i+1} \cdots a_k - 1\right)\right) \\
&= \sum_{\substack{a_1=1 \\ b a_1^2 a_2 \cdots a_k \equiv 1 \bmod p}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(b a_1 a_2 \cdots a_k\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
&\times e\left(\frac{1}{p} \sum_{i=2}^{k} c_i \left(b a_1 \cdots a_{i-1} a_i^2 a_{i+1} \cdots a_k - 1\right)\right) \\
&\quad + \frac{1}{p} \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(b a_1 a_2 \cdots a_k\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) \\
&\times e\left(\frac{1}{p} \sum_{i=2}^{k} c_i \left(b a_1 \cdots a_{i-1} a_i^2 a_{i+1} \cdots a_k - 1\right)\right) \\
&\times \sum_{j=1}^{k} \overline{\lambda}^{j} \left(\overline{c_2} \cdots \overline{c_k} \overline{d} \left(b a_1 \cdots a_{i-1} a_i^2 a_{i+1} \cdots a_k - 1\right)\right) \tau\left(\lambda^j\right) \\
&= \sum_{\substack{a_1=1 \\ b a_1^{k+1} a_2 \cdots a_k \equiv 1 \bmod p}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \chi\left(a_1\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) e\left(\frac{1}{p} \sum_{i=2}^{k} c_i \left(a_i - 1\right)\right)
\end{aligned}$$

$$+\frac{1}{p}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)\sum_{j=1}^{k}\tau^{k+1}\left(\lambda^j\right)$$

$$\times\overline{\lambda}^j(m)\overline{\lambda}^j(b-1)\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)$$

$$= p^k\cdot\sum_{\substack{a_1=1\\ a^{k+1}\equiv 1 \bmod p}}^{p-1}\chi\left(a_1\right)+\frac{1}{p}\sum_{j=1}^{k}\tau^{k+1}\left(\lambda^j\right)\overline{\lambda}^j(m)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)$$

$$\times\overline{\lambda}^j(b-1)\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)$$

$$= (k+1)\cdot p^k+\frac{1}{p}\sum_{j=1}^{k}\tau^{k+1}\left(\lambda^j\right)\overline{\lambda}^j(m)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)$$

$$\times\overline{\lambda}^j(b-1)\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right). \qquad (8)$$

Now Lemma 2.1 follows from (3), (6), (7) and (8). □

Lemma 2.2. *Let* p *be an odd prime and* $k>1$ *be an integer with* $(k+1)\nmid(p-1)$. *Then for any non-principal character* $\chi \bmod p$ *and integer* m *with* $(m,p)=1$, *we have the identity*

$$\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2=p^k.$$

Proof. Note that $(k+1)\nmid(p-1)$, so if a passes through a reduced residue system $\bmod p$, then a^{k+1} also passes through a reduced residue system $\bmod p$. Then from the method of proving (6) and the trigonometric identity (5) we have

$$\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2$$

$$=\frac{1}{p}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)\sum_{c_1=0}^{p-1}\cdots\sum_{c_k=1}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{md(b-1)}{p}\right)$$

$$\times e\left(\frac{1}{p}\left(c_1^{k+1}\overline{c_2}\cdots\overline{c_k}\overline{d}\left(ba_1^2a_2\cdots a_k-1\right)\right)\right)$$

$$\times e\left(\frac{1}{p}\sum_{i=2}^{k}c_i\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right)$$

$$-\frac{1}{p}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)\sum_{c_2=1}^{p-1}\cdots\sum_{c_k=1}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{md(b-1)}{p}\right)$$

$$\times e\left(\frac{1}{p}\sum_{i=2}^{k}c_i\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right)$$

$$= \sum_{\substack{a_1=1 \\ ba_1^2a_2\cdots a_k\equiv 1 \bmod p}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(ba_1a_2\cdots a_k\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right)$$

$$\times\, e\left(\frac{1}{p}\sum_{i=2}^{k} c_i\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right)$$

$$-\frac{1}{p}\sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(a_1\right) \sum_{c_2=1}^{p-1} \cdots \sum_{c_k=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{md(b-1)}{p}\right) e\left(\sum_{i=2}^{k}\frac{c_i\left(a_i-1\right)}{p}\right)$$

$$= \sum_{\substack{a_1=1 \\ ba_1^2a_2\cdots a_k\equiv 1 \bmod p}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \chi\left(a_1\right) \sum_{c_2=0}^{p-1} \cdots \sum_{c_k=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{md(b-1)}{p}\right) e\left(\sum_{i=2}^{k}\frac{c_i\left(\overline{a}_1a_i-1\right)}{p}\right)$$

$$= p^k \cdot \sum_{\substack{a_1=1 \\ a_1^{k+1}\equiv 1 \bmod p}}^{p-1} \chi\left(a_1\right) = p^k.$$

This proves Lemma 2.2. □

Lemma 2.3. *Let p be an odd prime and $h > 1$ be an integer. Then we have the identities*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^h+a}{p}\right)\right|^2 = \begin{cases} p^2-hp-1, & \text{if } h \mid (p-1), \\ p^2-p-1, & \text{if } (h,p-1)=1. \end{cases}$$

Proof. If $h \mid (p-1)$, then the congruence equation $x^h \equiv 1 \bmod p$ has h different solutions $\bmod\ p$. Notice that this point, Lemma 2.3 can easily be obtained by using the trigonometric identity (5), so its proved process is omitted. □

Lemma 2.4. *Let p be an odd prime and $h > 1$ be an integer with $h \mid (p-1)$. Then for any h-order character $\lambda \bmod p$, we have the estimate*

$$\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^h+a}{p}\right)\right|^2 \ll p.$$

Proof. If $h \mid (p-1)$, note that $\lambda^h = \chi_0$, the principal character $\bmod\ p$, from the properties of the Gauss sums and the reduced residue system $\bmod\ p$ we have

$$\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^h+a}{p}\right)\right|^2 = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\lambda(m)e\left(\frac{ma^h-mb^h+a-b}{p}\right)$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\lambda(m)e\left(\frac{mb^h\left(a^h-1\right)+b(a-1)}{p}\right)$$

$$= \tau\left(\lambda\right)\sum_{a=1}^{p-1}\overline{\lambda}\left(a^h-1\right)\sum_{b=1}^{p-1}\overline{\lambda}^h\left(b\right)e\left(\frac{b(a-1)}{p}\right)$$

$$= \tau\left(\lambda\right)\sum_{a=1}^{p-1}\overline{\lambda}\left(a^h-1\right)\sum_{b=1}^{p-1}e\left(\frac{b(a-1)}{p}\right) = -\tau\left(\lambda\right)\sum_{a=1}^{p-1}\overline{\lambda}\left(a^h-1\right)$$

$$= -\tau\left(\lambda\right)\sum_{a=1}^{p-1}\left(1+\lambda(a)+\lambda^2(a)+\cdots+\lambda^{h-1}(a)\right)\overline{\lambda}\left(a-1\right). \tag{9}$$

It is clear that for any $0 \leq i \leq h-1$, we also have

$$\sum_{a=1}^{p-1} \lambda^i(a)\overline{\lambda}(a-1) = \frac{1}{\tau(\lambda)} \sum_{b=1}^{p-1} \lambda(b) \sum_{a=1}^{p-1} \lambda^i(a) e\left(\frac{b(a-1)}{p}\right)$$
$$= \frac{\lambda^{i-1}(-1)\tau\left(\lambda^i\right)\tau\left(\overline{\lambda}^{i-1}\right)}{\tau(\lambda)}. \tag{10}$$

Note that $|\tau\left(\lambda^i\right)| \leq \sqrt{p}$, from (9) and (10) we may deduce the estimate

$$\sum_{m=1}^{p-1} \lambda(m) \left|\sum_{a=1}^{p-1} e\left(\frac{ma^h + a}{p}\right)\right|^2 \ll p.$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let p be an odd prime, and k be a positive integer with $(k+1) \mid (p-1)$. Then for any non-principal character $\chi \bmod p$, we have the estimate*

$$\sum_{j=1}^{k} \left|\sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(ba_1a_2\cdots a_k\right) \overline{\lambda}^j(b-1) \prod_{i=2}^{k} \overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2 a_{i+1}\cdots a_k - 1\right)\right|$$
$$\leq k\cdot(k+1)\cdot p^{\frac{k+1}{2}}.$$

Proof. It is clear that if χ is not the $(k+1)$-st character $\bmod p$, then from the method of proving (4) we have the identity

$$\sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}\left(ba_1a_2\cdots a_k\right) \overline{\lambda}^j(b-1) \prod_{i=2}^{k} \overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2 a_{i+1}\cdots a_k - 1\right)$$
$$= 0. \tag{11}$$

If χ is the $(k+1)$-st character $\bmod p$, then for any non-$(k+1)$-st residue $h \bmod p$, note that $1+\lambda(h)+\lambda^2(h)+\cdots+\lambda^k(h) = 0$ and for any $1 \leq j \leq k$,

$$\sum_{i=0}^{k} \lambda^j\left(h^i\right) = \sum_{i=0}^{k} \lambda^{ij}(h) = 0, \tag{12}$$

from Lemma 2.1 we have

$$\sum_{s=0}^{k} \left|\sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi\left(a_1+a_2+\cdots+a_k+mh^s\overline{a_1\cdots a_k}\right)\right|^2 = (k+1)^2\cdot p^k. \tag{13}$$

So for any integer m with $(m,p)=1$, from (13) we have the estimate

$$\sum_{s=0}^{k} \left|\sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi\left(a_1+a_2+\cdots+a_k+mh^s\overline{a_1\cdots a_k}\right)\right|^4$$
$$\leq (k+1)^4\cdot p^{2k}. \tag{14}$$

On the other hand, from (12) and Lemma 2.1 we also have

$$\sum_{s=0}^{k}\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+mh^s\overline{a_1\cdots a_k}\right)\right|^4=(k+1)^3\cdot p^{2k}$$
$$+(k+1)p^{k-1}\sum_{j=1}^{k}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)\overline{\lambda}^j(b-1)\right|^2$$
$$\times\left|\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right|^2 \tag{15}$$

Now combining (14) and (15) we have the estimate

$$\sum_{j=1}^{k}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)\overline{\lambda}^j(b-1)\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right|^2$$
$$<k\cdot(k+1)^2\cdot p^{k+1}. \tag{16}$$

Applying (16) and Cauchy's inequality we have

$$\sum_{j=1}^{k}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1a_2\cdots a_k\right)\overline{\lambda}^j(b-1)\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\right|$$
$$<k\cdot(k+1)\cdot p^{\frac{k+1}{2}}. \tag{17}$$

It is clear that Lemma 2.5 follows from (11) and (17). □

3 Proofs of the theorems

Now we use the several basic lemmas of the previous section to complete the proofs of our theorems. First we prove Theorem 1.1. If $(k+1,p-1)=1$, then for any non-principal $\chi \bmod p$, from Lemma 2.2 and Lemma 2.3 we have the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{k+1}+a}{p}\right)\right|^2$$
$$=p^k\cdot\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{k+1}+a}{p}\right)\right|^2=p^k\cdot\left(p^2-p-1\right).$$

This proves Theorem 1.1.

Now we prove Theorem 1.2. If $(k+1)\mid(p-1)$, χ is not the $(k+1)$-st character $\chi \bmod p$, from Lemma 1 we get the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{k+1}+a}{p}\right)\right|^2=0. \tag{18}$$

If $(k+1)\mid(p-1)$, and χ is the $(k+1)$-st character $\chi \bmod p$, then from Lemma 2.1, Lemma 2.3, Lemma 2.4 and Lemma 2.5 we have the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{k+1}+a}{p}\right)\right|^2$$

$$\begin{aligned}
&= \frac{1}{p}\sum_{j=1}^{k}\tau^{k+1}\left(\lambda^j\right)\sum_{m=1}^{p-1}\overline{\lambda}^j(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{k+1}+a}{p}\right)\right|^2\\
&\quad\times\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(ba_1\cdots a_k\right)\overline{\lambda}^j(b-1)\prod_{i=2}^{k}\overline{\lambda}^j\left(ba_1\cdots a_{i-1}a_i^2a_{i+1}\cdots a_k-1\right)\\
&\quad+(k+1)\cdot p^k\cdot\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^{k+1}+a}{p}\right)\right|^2\\
&= (k+1)p^k\left(p^2-(k+1)p-1\right)+O\left(p^{\frac{k+1}{2}}\cdot k\cdot(k+1)\cdot p^{\frac{k+1}{2}}\right)\\
&= (k+1)\cdot p^{k+2}+O\left(k^2\cdot p^{k+1}\right). \qquad (19)
\end{aligned}$$

Combining (18) and (19), we complete the proof of Theorem 1.2.

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