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Finiteness of lattice points on varieties $F(y) = F(g(\mathbb{X})) + r(\mathbb{X})$ over imaginary quadratic fields

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Abstract: We construct affine varieties over $\mathbb{Q}$ and imaginary quadratic number fields $\mathbb{K}$ with a finite number of α-lattice points for a fixed $\alpha \in \mathcal{O}_{\mathbb{K}}$, where $\mathcal{O}_{\mathbb{K}}$ denotes the ring of algebraic integers of $\mathbb{K}$. These varieties arise from equations of the form $F(y) = F(g(x_1, x_2, \ldots, x_k)) + r(x_1, x_2 \ldots, x_k)$, where F is a rational function, g and r are polynomials over $\mathbb{K}$, and the degree of r is relatively small. We also give an example of an affine variety of dimension two, with a finite number of algebraic integral points. This variety is defined over the cyclotomic field $\mathbb{Q}(\xi_3) = \mathbb{Q}(\sqrt{-3})$.

Keywords: Diophantine equations, Algebraic integral points, Higher dimension affine varieties.
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1 Introduction

For decades mathematicians have been interested in Diophantine equations, i.e., in looking for integral points on algebraic varieties. In this paper we identify algebraic varieties with solutions of a system of rational equations over a fixed field. Generally an algebraic variety can have a finite or an infinite number of integral points. The pivotal question arises: does there exist a general method for finding nontrivial integral points, that is points that do not come from factors of the polynomials defining the variety? There are various methods which are used to count the number of solutions of Diophantine equations. For example, the theory of heights on algebraic varieties implies that there exist elliptic curves, which have a finite number of integral points which are non-trivial in the above sense, cf. [7].

On the other hand, affine surfaces which come from projective curves $C_n : X^n + Y^n = Z^n$ for $n \geq 3$ from the Fermat's Last Theorem have an infinite number of integral points over $\mathbb{Q}$ and all these points are trivial. This theorem proved Andrew Wiles in 1995, cf. [8], and in the proof he used the theory of modular forms. Another example of varieties that have only trivial integral points are the varieties that arise from the fact that a non-zero product of at least two consecutive integers is never a perfect power, cf. [3, 4]. The proof is based on a prime factorization of the product and the distribution of prime numbers. On the other hand, the set of Pythagorean triples is an infinite set of solutions of the equation $x^2+y^2 = z^2$ which can be proved by a parametrization of rational points of the circle. In a similar way, one can prove that if an elliptic curve defined by the equation $F(x, y) = 0$ has a rank greater than 0, then the equation $F(x/z, y/z) = 0$, defining an affine surface, has an infinite number of integral solutions.

In this paper, we describe a family of algebraic varieties (over quadratic imaginary number fields) with a finite number of lattice points. We consider these fields hence as stated in the [1, Theorem 4.4] the rational number field and imaginary quadratic fields are the only one for which a ring of integers has a least positive element. These varieties are defined by equations of the form $F(y) = F(g(\mathbb{X}))+r(\mathbb{X})$, where F is an almost rational function, g is a polynomial which asymptotically is greater than a power of maximal norm on the subset of complex numbers, and the degree of polynomial r is small. The construction of this family is inspired by two results. The first result is a very well known theorem of Siegel, which says that an algebraic curve of genus greater than one has a finite number of integral points, cf. [6, Thm. D.9.1]. The second result is a theorem of Vojtech Jarnik about the maximal asymptotic number of n-lattice points on convex curves, cf. [2]. To obtain our results, we use tools of topological and analytic nature. In particular, our methods are based on the fact that the set $F^{-1}(F(g(X)) + r(X))$ is contained in a sum of finite family of balls, which contain few lattice points. In order to obtain such a property, we investigate the function $X \mapsto F^{-1}(F(X))$ and control the growth rate of functions g and r. Functions F, g, r are described in technical Lemma 2.1 and Lemma 5.1.

In order to state our results, we need several definitions and a bit of notation.

All varieties in this paper are over the field $\mathbb{K} := \mathbb{Q}(\sqrt{-D})$, where $D \geq 1$ is a square-free integer or $\mathbb{K} := \mathbb{Q}$.

Definition 1.1. *Let α be an algebraic integer of a number field $\mathbb{K}$. Then a complex number z is called an α-lattice number over $\mathbb{K}$ if z belongs to a subset $\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}} \subset \mathbb{C}$.*

Similarly, we say that the point $X := (x_1, x_2, \ldots, x_k)$ is an α-lattice point if X belongs to the set

$$\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}^k := \underbrace{\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}} \times \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}} \times \ldots \times \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}}_{k \text{ times}},$$

which we consider as a subset of $\mathbb{C}^k$, for $k \geq 1$.

Definition 1.2. *We say that a subset $\mathcal{S}$ of complex numbers is uniformly discrete if there exists $\varepsilon > 0$ such that the distance in maximal metric of every two different points of $\mathcal{S}$ is at least equal ε.*

In the proof of Theorem 1.1, it is crucial that a set of the form

$$\bigcup_{j=0}^{n-1} \xi_n^j \mathcal{O}_{\mathbb{K}} \tag{1}$$

is a uniformly discrete set, where $\xi_n = e^{2\pi i/n}$. What is equivalent to $n \big| N(\mathbb{K})$, where

$$N(\mathbb{K}) := \begin{cases} 6, & \text{if } \mathbb{K} = \mathbb{Q}(\sqrt{-3}) \\ 4, & \text{if } \mathbb{K} = \mathbb{Q}(\sqrt{-1}) \\ 0, & \text{if } \mathbb{K} = \mathbb{Q} \\ 2, & \text{otherwise.} \end{cases} .$$

Next definitions capture the notion of a function that has large values for large arguments.

Definition 1.3. *Let k be a natural number and $\mathcal{S} \subset \mathbb{C}^k$ be an infinite set. Let $f, g : \mathcal{S} \to \mathbb{C}$. We say that the function g **dominates** the function f on the set $\mathcal{S}$ if there exist constants $M, C > 0$ satisfying*

$$|g(X)| \geq C|f(X)|$$

for $X \in \mathcal{S}$ such that $\|X\| > M$.

Definition 1.4. *Any rational function $F(z)$ we can write as*

$$f(z) + \frac{f_1(z)}{f_2(z)} = \sum_{i=0}^{n} a_i z^i + \frac{f_1(z)}{f_2(z)},$$

*where f, f_1 and f_2 are polynomials and $\deg f_1 < \deg f_2$, then we define **degree of rational function** F by equality $\deg F := \deg f$.*

Definition 1.5. *For any $0 \leq j < n = \deg F$ we put $A_j := \frac{(\xi_n^j - 1)a_{n-1}}{n a_n}$ and define a rational function of k variables*

$$L_j(\mathbb{X}) := F(g(\mathbb{X})) - F\left(\xi_n^j g(\mathbb{X}) + A_j\right) + r(\mathbb{X}).$$

Our first main result provides a sufficient condition for a variety given by the equation

$$F(z) = F(g(\mathbb{X})) + r(\mathbb{X})$$

to have a finite number of α-lattice points over $\mathbb{K}$, where $\mathbb{X} = (x_1, x_2, \ldots, x_k)$ and $k \geq 1$.

Theorem 1.1. *Let k and m be a natural number and $F(z) \in \mathbb{K}(z)$ be a rational function of degree $n \geq 2$ such that $n \mid N(\mathbb{K})$. Let $\mathcal{S}$ be a subset of the $\mathcal{O}_{\mathbb{K}}^k$ and $g(\mathbb{X}), r(\mathbb{X}) \in \mathbb{K}[\mathbb{X}]$ be polynomials of k variables such that $|g(\mathbb{X})|$ dominates $\|\mathbb{X}\|^m$ on the set $\mathcal{S}$ and the degree of $r(\mathbb{X})$ is smaller than $(n-1)m$. Then the number of solutions of the equation*

$$F(z) = F(g(\mathbb{X})) + r(\mathbb{X})$$

in the set $\mathcal{S} \times \mathcal{O}_{\mathbb{K}}$ is finite, assuming that the number of zeros of the rational function

$$\prod_{i=0}^{n-1} L_j(\mathbb{X}) \tag{2}$$

is finite in the set $\mathcal{S}$.

Remark 1.1.1. *At this point, we would like to note some of the special cases of the Theorem 1.1. If $\mathbb{K} = \mathbb{Q}$, then instead of zeros of the function (2) we can consider zeros of $L_0(\mathbb{X})L_{\lfloor n/2 \rfloor}(\mathbb{X})$. If $k = 1$, then every non-constant polynomial $g \in \mathbb{C}[x]$ dominates the m-th power of $|x|$. Moreover, we can take $m = \deg g$. In addition, if $k = 1$, then the function (2) has a finite number of zeros, if and only if, rational functions L_i are all non-zero, since every non-zero rational function of one variable has a finite number of zeros.*

In the last section of the paper, we will use Theorem 1.1 to prove that hyperelliptic curves over $\mathbb{Q}(\sqrt{-D})$ have a finite number of $\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}$ points over $\mathbb{Q}(\sqrt{-D})$. In addition, we do not use Siegel's theorem to prove the following

Corollary 1.1.2. *Let $\alpha \in \mathcal{O}_{\mathbb{Q}(\sqrt{-D})}$ be an algebraic integral number and $h \in \mathbb{Q}(\sqrt{-D})[x]$ be a monic polynomial of degree $2m$ which has $2m$ distinct roots in $\mathbb{C}$. Then a hyperelliptic curve over $\mathbb{Q}(\sqrt{-D})$ defined by the equation*

$$y^2 = h(x) = x^{2m} + \sum_{i=0}^{2m-1} a_i x^i$$

has a finite number of α-lattice points over $\mathbb{Q}(\sqrt{-D})$.

After this proof, we apply the main result to obtain the following construction of higher dimensional varieties over imaginary quadratic field with a finite number of α-lattice points. But to construct such varieties we need polynomials of many variables which dominate a positive power of maximal norm on the set of α-lattice points. Unfortunately, such polynomials have large degree which increases as the square of the number of variables.

Theorem 1.2. *Let $\mathbb{K}$, k, n, α and $F := f + \frac{f_1}{f_2}$ be as in Theorem 1.1. Let $r \in \mathbb{K}[\mathbb{X}]$ be a polynomial of k variables. If $r(\mathbb{X}) = 0$ has a finite number of α-lattice solutions, and for any $0 \leq j \leq n-1$ a rational function $F(z) - F(\xi_n^j z + A_j)$ has a positive degree or $\equiv 0$, then there exists a polynomial $g(\mathbb{X}) \in \mathbb{K}[\mathbb{X}]$ such that the variety defined by the equation*

$$F(z) = F(g(\mathbb{X})) + r(\mathbb{X})$$

has a finite number of α-lattice points over $\mathbb{K}$.
Moreover, the polynomial g can be chosen in such a way that its degree is smaller than $k(m_0 + k - 1)$, where m_0 is any integral number greater than

$$\frac{\deg r}{\min_{\deg L_j > 0, j=1,\dots,n}\{\deg L_j\}}.$$

2 Preliminaries

We start by reminding a big and a small o notation, cf. [5].

Definition 2.1. *Let d be a natural number and $f, g : \mathbb{C}^d \supset S \to \mathbb{C}$, then we write*

i) $f = o(g)$, *if* $\lim_{\|X\| \to \infty} \left| \frac{f(X)}{g(X)} \right| = 0$, *ii)* $f \asymp g$, *if there exist* $C_1, C_2, M > 0$ *such that* $C_1|f(X)| < |g(X)| < C_2|f(X)|$ *for* $\|X\| > M$ *which are contained in* S.

Definition 2.2. *Let* $\varepsilon > 0$, $k \in \mathbb{N}$ *and* $X_0 \in \mathbb{C}^k$, *then the following set*

$$\mathcal{B}(X_0, \varepsilon) := \left\{ X \in \mathbb{C}^k \mid \max_{1 \le i \le k} |X_i - X_{0i}| < \varepsilon \right\}$$

is a ball in the maximum metric.

Before we introduce the crucial lemma, we need the definition of functions which asymptotically behave as monomials $x \mapsto a_n x^n$.

Definition 2.3. *We say that an open and continuous function* $F : \mathbb{C} \backslash P \to \mathbb{C}$ *is an* ***almost-rational*** *function of degree* n *if its domain is the set of all complex numbers except a finite set, and there exist* $a_{n-1} \in \mathbb{C}$ *and a non-zero* $a_n \in \mathbb{C}$ *such that an asymptotic equation*

$$F(y) - a_n y^n - a_{n-1} y^{n-1} = o\left(y^{n-1}\right)$$

holds, i.e.,

$$\lim_{|y| \to \infty} \frac{|F(y) - a_n y^n - a_{n-1} y^{n-1}|}{|y|^{n-1}} = 0.$$

Now we can describe a distribution of an inverse image of an image of almost-rational functions for sufficiently large arguments.

Lemma 2.1. *We fix natural numbers* k, m. *Let* F *be an almost-rational function of degree* n, $\mathcal{S}$ *be a subset of* $\mathcal{O}_{\mathbb{K}}^k$ *and* $g(\mathbb{X}) \in \mathcal{O}_{\mathbb{K}}[\mathbb{X}]$ *be a function which dominates* $\|\mathbb{X}\|^m : \mathbb{C}^k \to \mathbb{C}$ *on the set* $\mathcal{S}$. *If we denote a set of points excluded from a domain of* F *by* P, *then the following statements hold.*

(i) If $r : \mathcal{S} \to \mathbb{C}$ *is a function such that* $r(\mathbb{X}) = o\left(\|\mathbb{X}\|^{m(n-1)}\right)$ *and* $\varepsilon_1 > 0$, *then there exists a constant* $M_1 := M_1(\varepsilon_1, F, r, g, \mathcal{S}) > 0$ *such that inclusion*

$$r(X) \in \{F(z) - F(g(X)) | z \in \mathcal{B}(g(X), \varepsilon_1)\}. \tag{3}$$

holds for $X \in \mathcal{S}$ *satisfying an inequality* $|g(X)| > M_1$

(ii) For any $\varepsilon_2, \varepsilon_3 > 0$, *there exists a constant* $M_2 := M_2(\varepsilon_2, \varepsilon_3, F)$ *such that the following inclusion holds*

$$F^{-1}(F(y_0)) \subset \bigcup_{j=0}^{n-1} \mathcal{B}(\xi_n^j y_0 + A_j, \varepsilon_2) \cup \bigcup_{p \in P} \mathcal{B}(p, \varepsilon_3),$$

whenever $|y_0| > M_2$.

Remark 2.1.1. *Lemma 2.1.(i) is true when* $\deg r = m(n-1)$, *assuming an extra condition on the polynomial* $r(\mathbb{X})$. *More precisely, we require the existence of constants* C_3 *and* M_3 *such that*

$$\left| \frac{r(X)}{g(X)^{n-1}} \right| \le C_3 < |n a_n \varepsilon_1|,$$

whenever $g(X) > M_3$.

Before the proof of Lemma 2.1 we state and prove two more lemmas. The first one says that a translation of almost-rational function asymptotically behaves as a monomial. The second lemma describes a family of loops which are not contractible on $\mathbb{C}\setminus\{z_0\}$.

Lemma 2.2. *Let P, n, a_n, a_{n-1}, F be as in Lemma 2.1. Then the following statements are true:*
(i) There exists a linear transformation $L : y \mapsto y - \frac{a_{n-1}}{na_n}$ such that $F(L(y)) = a_n y^n + o\left(y^{n-1}\right)$.
(ii) For any pair $a, b \in \mathbb{C}$ the equality

$$F(ay+b) = F(y) + o\left(y^{n-1}\right)$$

implies:

$$(a,b) \in \left\{ \left(\xi_n^j, \frac{\left(\xi_n^j - 1\right) a_{n-1}}{na_n}\right) \,\Big|\, 1 \le j \le n \right\}.$$

Proof. (i) We calculate $F(L(y))$. We use the formula for the n-th power of sum to obtain

$$\begin{aligned} F\left(y - \tfrac{a_{n-1}}{na_n}\right) &= a_n\left(y - \tfrac{a_{n-1}}{na_n}\right)^n + a_{n-1}\left(y - \tfrac{a_{n-1}}{na_n}\right)^{n-1} + \mathrm{o}\left(\left(y - \tfrac{a_{n-1}}{na_n}\right)^{n-1}\right) \\ &= a_n y^n - na_n \tfrac{a_{n-1}}{na_n} y^{n-1} + \mathrm{o}\left(y^{n-1}\right) + a_{n-1}y^{n-1} + \mathrm{o}\left(y^{n-1}\right) + \mathrm{o}\left(y^{n-1}\right) \\ &= a_n y^n + \mathrm{o}\left(y^{n-1}\right). \end{aligned}$$

(ii) First we reduce the coefficient a_{n-1} using a linear transformation. Next we solve the problem for functions of the form $a_n y^n + \mathrm{o}\left(y^{n-1}\right)$ and finally we use linear transformations to return to the original function. $F\left(y - \frac{a_{n-1}}{na_n}\right) = a_n y^n + \mathrm{o}\left(y^{n-1}\right)$, therefore, without loss of generality we assume $F(y) = a_n y^n + \mathrm{o}\left(y^{n-1}\right)$. To solve this case it is enough to calculate the following limits

$$\lim_{|y|\to\infty}\left|\frac{F(ay+b)-F(y)}{y^{n-1}}\right| = \lim_{|y|\to\infty}\left|\frac{a_n((ay+b)^n - y^n) + \mathrm{o}\left(y^{n-1}\right)}{y^{n-1}}\right| = \begin{cases} \infty & a \neq \xi_n^j \\ na_n a^{n-1} b & a = \xi_n^j \end{cases}.$$

Which implies that the following limit

$$\lim_{|y|\to\infty}\left|\frac{F(ay+b)-F(y)}{y^{n-1}}\right| = 0$$

holds, if and only if $(a,b) \in \left\{\left(\xi_n^j, 0\right) \,\big|\, 1 \le j \le n\right\}$. Then we come back to original form of function F

$$\xi_n^j y \to \left(y - \frac{a_{n-1}}{na_n}\right) \circ \xi_n^j y \circ \left(y + \frac{a_{n-1}}{na_n}\right) = \xi_n^j y + \frac{\left(\xi_n^j - 1\right) a_{n-1}}{na_n}. \qquad \square$$

Lemma 2.3. [About non-trivial loops on $\mathbb{C}\setminus\{z_0\}$] *Let $R > R' > 0$, $z_0 \in \mathbb{C}$ and $a \in \mathbb{C}$ be such that $|z_0| < R - R'$ and $|a| = 1$. If in addition a loop $f : S^1 \to \mathbb{C}$ satisfies $f(z) \in B(azR, R')$, then the loop f is not contractible on $\mathbb{C}\setminus\{z_0\}$.*

Proof. Without loss of generality we can assume that $a = 1$. Then, it is enough to take a homotopy $f_t(x) = (1-t)f(z) + tzR$, which shows that the loop f is in the same class of homotopy as a circle centered at 0 with radius equals $R - R'$ on $\mathbb{C}\setminus\{z_0\}$, but such circle is not contractible. $\square$

The idea of the proof of Lemma 2.1.(i) is following. When $r(X) = 0$, then the claim is obvious. To prove inclusion (3) for $r(X) \neq 0$, we will find an upper estimate of $|r(X)|$ and a lower estimate of $|F(y) - F(g(X))|$ for $y \in \mathbb{C}$ such that $|y - g(X)| = \varepsilon_1$. Next, we use these estimates to parametrize a loop $F(y) - F(g(X))$. On the power of Lemma 2.3 the loop is not contractible on $\mathbb{C} \setminus \{r(\mathbb{X})\}$. On the other hand, if $r(X) \notin \{F(y) - F(g(X)) \mid y \in \mathcal{B}(g(X), \varepsilon_1)\}$, then we can show that the loop $F(y) - F(g(X))$ is contractible on $\mathbb{C} \setminus \{r(\mathbb{X})\}$. It implies the claim of Lemma 2.1.(i).

Proof of Lemma 2.1.(i). Assume that $r(X) \neq 0$. We fix positive numbers C', C'' such that $C' + C'' < |na_n\varepsilon_1|$. Since the asymptotic equation $|r(\mathbb{X})| = \mathrm{o}\left(\|\mathbb{X}\|^{(n-1)m}\right)$ and the assumption on function g, for any constant $C' > 0$ the inequality

$$|r(X)| < C'C_0^{n-1}\|X\|^{(n-1)m} < C'|g(X)|^{n-1} \tag{4}$$

holds if $X \in \mathcal{S}$, $\|X\| > M_0$ and $|g(X)|$ is sufficiently large. We denote $e(t) := e^{2\pi i t}$, where $t \in (0, 1]$. Then after some calculations for $t \in (0, 1]$, we obtain

$$F(g(\mathbb{X}) + \varepsilon_1 e(t)) - F(g(\mathbb{X})) = na_n\varepsilon_1 e(t) g(\mathbb{X})^{n-1} + \mathrm{o}\left(g(\mathbb{X})^{n-1}\right).$$

Next we define a function $\mathcal{F} : \mathbb{C}^k \times (0, 1] \to \mathbb{C}$ by

$$\mathcal{F}(g(\mathbb{X}), t) := F(g(\mathbb{X}) + \varepsilon_1 e(t)) - F(g(\mathbb{X})) - na_n\varepsilon_1 e(t) g(\mathbb{X})^{n-1}.$$

We use the above asymptotic inequality to obtain that

$$|\mathcal{F}(g(X), t)| < C''|g(X)|^{n-1} \tag{5}$$

for $X \in \mathcal{S}$ with sufficiently large $|g(X)|$, where $C'' > 0$ is defined earlier. Therefore, for sufficiently large $\|X\|$ from the set $\mathcal{S}$ the inequality

$$|F(g(X) + \varepsilon_1 e(t)) - F(g(X))| \geq |na_n\varepsilon_1 e(t) g(X)^{n-1}| - C''|g(X)|^{n-1} \tag{6}$$

holds. Finally, we define a constant M_1 as the smallest value for which equations (4) and (6) hold, whenever $X \in \mathcal{S}$ and $|g(X)| > M_1$. We set $X \in \mathcal{S}$ such that $|g(X)| > M_1$. Then we obtain a loop

$$f : S^1 \to \mathbb{C} \setminus \{r(X)\}$$

defined by

$$e(t) \mapsto F(g(X) + \varepsilon_1 e(t)) - F(g(X)).$$

Note that (5) implies the inclusion

$$F(g(X) + \varepsilon_1 e(t)) - F(g(X)) \in \mathcal{B}(na_n\varepsilon_1 e(t) g(X)^{n-1}, C''|g(X)|^{n-1}).$$

Hence on the power of Lemma 2.3 we obtain that f is not homotopic to a constant map on the set $\mathbb{C} \setminus \{r(X)\}$, since $|r(X)| < |na_n\varepsilon_1 g(X)^{n-1}| - C''|g(X)|^{n-1}$. Assume to the contrary that $r(X) \notin \{F(y) - F(g(X)) \mid y \in \mathcal{B}(g(X), \varepsilon_1)\}$. Then we can define a homotopy

$$f_s(t) = F(g(X) + (1 - s)\varepsilon_1 e(t)) - F(g(X)), \text{ where } s \in [0, 1]$$

so that $f_0 = f$ and f_1 is a constant map to $0 \in \mathbb{C} \setminus \{r(X)\}$. This contradicts the fact that the loop f is not contractible on the set $\mathbb{C} \setminus \{r(X)\}$. Therefore, $r(X) \in \{F(y) - F(g(X)) \mid y \in \mathcal{B}(g(X), \varepsilon_1)\}$ holds, whenever $|g(X)| > M_1$. $\square$

Proof of Lemma 2.1.(ii). By the Lemma 2.2.(i) we can assume without loss of generality that $F(y) - y^n = \mathrm{o}\,(|y|^{n-1})$, since we can use a linear transformation to reduce coefficients a_n and a_{n-1} to 1 and 0. From these linear transformations arise constants A_j's, see Lemma 2.2.(ii). In order to investigate the set $F^{-1}(F(y_0))$, we have to solve the equation

$$F(y) = F(y_0)$$

for $y \in \mathbb{C} \setminus P$. Let $\varepsilon_2, \varepsilon_3$ be positive real numbers. By B_P we denote the sum of the discs with centers at points of the finite set P,

$$B_P := \bigcup_{p \in P} \mathcal{B}(p, \varepsilon_3).$$

We fix the compact set $\overline{B_P} \setminus B_P$. Since F is a continuous function on $\overline{B_P} \setminus B_P$, the image $F(\overline{B_P} \setminus B_P)$ is a bounded subset of $\mathbb{C}$. We fix a constant $C > 0$ which will be specified later. We choose a sufficiently large ball $\mathcal{B} := \mathcal{B}(0, M) \subset \mathbb{C}$ such that $\overline{B_P} \setminus B_P \subset \mathcal{B}$ and the function F restricted to $\mathbb{C} \setminus \mathcal{B}$ is close to a polynomial function y^n. More precisely there exists a constant $M > 0$ such that

$$|F(y) - y^n| < C|y|^{n-1}, \text{ whenever } |y| > M.$$

By $\mathcal{M}$ we denote the maximum of the absolute value of the function F on the compact set $\overline{\mathcal{B}(0, M)} \setminus B_P$,

$$\mathcal{M} := \max_{y \notin B_P \text{ and } |y| \le M} |F(y)|.$$

We define the constant M_2 as the minimal positive real number for which

$$|F(y)| > \mathcal{M}, \text{ whenever } |y| > M_2.$$

Assume to the contrary that for y_0 such that $|y_0| > M_2$, there exists a complex number $y' \notin B_P$ such that $F(y') = F(y_0)$ and $\min_{1 \le j \le n} |y' - y_0| \ge \varepsilon_2$. Then by the choice of M_2 we would have $|y'| > M$, hence using the triangle inequality we obtain

$$|F(y') - F(y_0)| \ge |y'^n - y_0^n| - C|y'|^{n-1} - C|y_0|^{n-1} \ge \prod_{j=1}^{n} |y' - \xi_n^j y_0| - 2C(\max |y'|, |y_0|)^{n-1}, \quad (7)$$

where we have used the formula $y'^n - y_0^n = \prod_{j=1}^{n} (y' - \xi_n^j y_0)$. Let us fix $1 \le j_0 \le n$ such that

$$|y' - \xi_n^{j_0} y_0| = \min_{1 \le j_0 \le n} |y' - \xi_n^j y_0|.$$

Then for any $1 \le j \le n$ such that $j \ne j_0$ we have obvious inequalities

$$|y' - \xi_n^j y_0| + |y' - \xi_n^{j_0} y_0| \ge |(\xi_n^{j_0} - \xi_n^j) y_0| = |\sin(\pi(j - j_0)/n) y_0|$$

and

$$|\xi_n^{n-j_0} y' - y_0| + |\xi_n^{n-j} y' - y_0| \ge |(\xi_n^{n-j_0} - \xi_n^{n-j}) y'| = |\sin(\pi(j - j_0)/n) y'|.$$

Therefore, for $j \ne j_0$

$$|y' - \xi_n^j y_0| \ge \frac{\sin(\pi/n)}{2} \max\{|y'|, |y_0|\}. \quad (8)$$

We specify a positive number C such that $C < \frac{\varepsilon_2 \sin^{n-1} \pi/n}{2}$. Finally, we use inequalities (7) and (8) to obtain following inequalities

$$\begin{aligned} |F(y') - F(y_0)| &\geq \prod_{j=1}^{n} |y' - \xi_n^j y_0| - 2C(\max |y'|, |y_0|)^{n-1} \\ &\geq (\min_{1\leq j\leq n} |y' - \xi_n^j y_0|)(\sin \pi/n \max |y'|, |y_0|)^{n-1} - 2C(\max |y'|, |y_0|)^{n-1} \\ &\geq (\varepsilon_2 \sin^{n-1} \pi/n - 2C)(\max |y'|, |y_0|)^{n-1} \\ &> 0 \end{aligned}$$

This is a contradiction, since $F(y_0) = F(y')$. □

Proposition 2.3.1. *Any rational function of degree n is an almost-rational function of degree n.*

In the proof of Theorem 1.1 we use Lemma 2.1 for rational functions of the form $F = f + \frac{f_1}{f_2}$ where $f, f_1, f_2 \in \mathbb{C}[y]$, $\deg f > 0$ and $\deg f_1 < \deg f_2$. We also use a property of an image of α-lattice points which is described in the following lemma.

Lemma 2.4. *Let $g \in \mathbb{K}[\mathbb{X}]$ and let $\alpha \in \mathcal{O}_{\mathbb{K}}$ be a non-zero algebraic integer. Then there exists $\alpha' \in \mathcal{O}_{\mathbb{K}}$ such that $g(X)$ is an α'-lattice for any point $X \in \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}$.*

Proof. Let

$$g(X) = \sum_{I=(i_1,i_2,\ldots,i_k)} a_I x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k},$$

where the sum runs over all I such that $a_I \neq 0$. We put

$$a_I = \frac{p_I}{q_I}, \qquad x_j = \frac{r_j}{\alpha}$$

for some $p_I, q_I, r_j \in \mathcal{O}_{\mathbb{K}}$. Then it is enough to set $\alpha' = \alpha^{\deg g} \cdot \prod_I q_I$. □

3 Proof of Theorem 1.1

Proof. Recall that $\xi_n := e^{2\pi i/n}$ is the n-th primitive root of unity, $A_j := \frac{(\xi_n^j - 1)a_{n-1}}{na_n}$ and $\mathcal{S}$ is an infinite subset of the set of α-lattice points over the field $\mathbb{K}$. For the sake of simplicity, we introduce the following notation. If $\mathbb{K} = \mathbb{Q}(\sqrt{-D})$, then we denote by $\alpha' := \alpha'(\mathbb{K}, f, g, \mathcal{S})$ an algebraic integer of $\mathbb{K}$ such that

$$\{\xi_n^j g(X) + A_j \mid 0 \leq j \leq n-1, X \in \mathcal{S}\} \subset \frac{1}{\alpha'}\mathcal{O}_{\mathbb{K}}.$$

If $\mathbb{K} = \mathbb{Q}$, then we denote by $\alpha' := \alpha'(\mathbb{K}, f, g, \mathcal{S})$ an algebraic integer of $\mathbb{K}$ such that

$$\{\xi_n^j g(X) + A_j \mid j \in \{0, n/2\} \cap \mathbb{Z}, X \in \mathcal{S}\} \subset \frac{1}{\alpha'}\mathcal{O}_{\mathbb{K}}.$$

Observe that the number α' defined above exists by Lemma 2.4, since $n \big| N(\mathbb{K})$ implies that $\xi_n \in \mathbb{K}$ or $\mathbb{K} = \mathbb{Q}$. Recall also that we are interested in varieties in $\mathbb{A}_{\mathbb{K}}^{k+1}$ given by the equation

$$F(y) = F(g(\mathbb{X})) + r(\mathbb{X}). \tag{9}$$

Let P be the set of poles of function F. We choose positive real numbers $\varepsilon_1, \varepsilon_2$ and ε_3, such that $\varepsilon_1 + \varepsilon_2 = \varepsilon < |1/\alpha'(\mathbb{K}, f, g, \mathcal{S})|$ and $\varepsilon_3 < d(P \setminus \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}, \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}})$, where $d(A, B)$ denotes a distance between sets A and B in the maximum metric. By assumptions $g(\mathbb{X})$ dominates $\|\mathbb{X}\|^m$ on the set $\mathcal{S}$ and the degree of $r(\mathbb{X})$ is smaller than $(n-1)m$, hence we obtain asymptotic formula

$$r(\mathbb{X}) = \mathrm{o}\left(g(\mathbb{X})^{n-1}\right) \tag{10}$$

on the set $\mathcal{S}$, i.e.,

$$\lim_{X \in \mathcal{S} \text{ and } \|X\| \to \infty} \frac{r(X)}{g(X)^{n-1}} = 0.$$

Proposition 2.3.1. implies that a rational function $F(z)$ satisfies the conditions of Lemma 2.1 with $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ as in the beginning of the proof. Then by Lemma 2.1(i) we obtain the inclusion

$$F(g(X)) + r(X) \in F(\mathcal{B}(g(X), \varepsilon_1)) \tag{11}$$

for $X \in \mathcal{S}$, such that $|g(X)| > M_1 = M_1(\varepsilon_1, F, r, g, \mathcal{S})$. Recall that P denotes the set of poles of rational function $F(z)$. In order to simplify notation, we put $\mathcal{B}_j := \mathcal{B}(\xi_n^j g(X) + A_j, \varepsilon)$ and $\mathcal{B}_p = \mathcal{B}(p, \varepsilon_3)$. Then by Lemma 2.1.(ii) there exists $M_2 = M_2(\varepsilon_2, \varepsilon_3, F)$ such that we obtain

$$F^{-1}(F(\mathcal{B}(g(X), \varepsilon_1))) \subset \bigcup_{j=0}^{n-1} \left(\bigcup_{y \in \mathcal{B}(g(X), \varepsilon_1)} \mathcal{B}(\xi_n^j y + A_j, \varepsilon_2) \right) \cup \bigcup_{p \in P} \mathcal{B}_p = \bigcup_{j=0}^{n-1} \mathcal{B}_j \cup \bigcup_{p \in P} \mathcal{B}_p \tag{12}$$

for $X \in \mathcal{S}$ such that $|g(X)| > M_\varepsilon := \max\{M_1, M_2\}$. By (11) and (12) we obtain

$$F^{-1}(F(g(X)) + r(X)) \subset \bigcup_{j=0}^{n-1} \mathcal{B}_j \cup \bigcup_{p \in P} \mathcal{B}_p \tag{13}$$

for $X \in \mathcal{S}$ such that $|g(X)| > M_\varepsilon$. We choose $\varepsilon_3 < d(P \setminus \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}, \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}})$. This implies that

$$\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}} \cap \left(\bigcup_{p \in P} \mathcal{B}_p \right) \subset P, \tag{14}$$

hence

$$\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}} \cap F^{-1}(F(g(X)) + r(X)) \subset \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}} \cap \left(\bigcup_{j=0}^{n-1} \mathcal{B}_j \right), \tag{15}$$

for $|g(X)| > M_\varepsilon$, since P consists of poles of F and the inclusions (13) and (14) hold.

In order to conclude the thesis, we will show that the α-lattice point (X, y_0) on the variety defined by

$$F(y_0) = F(g(X)) + r(X)$$

such that $|g(X)| > M_\varepsilon$ satisfies equation

$$L_j(X) = 0$$

for some $1 \leq j \leq n$. Assume that $(X, y_0) \in \mathcal{S} \times \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}$ is a point of the variety

$$F(y) = F(g(\mathbb{X})) + r(\mathbb{X}) = 0$$

such that $|g(X)| > M_\varepsilon$. Then the inclusion (15) implies

$$y_0 \in \mathcal{B}_{j_0} \text{ for some } 0 \le j_0 < n.$$

We have chosen $\varepsilon < \frac{1}{\alpha'(\mathbb{K},f,g,\mathcal{S})}$ with $\alpha'(\mathbb{K}, f, g, \mathcal{S})$ as in the beginning of the proof. This implies that $\xi_n^{j_0} g(X) + A_{j_0}$ is the only α'-lattice point in $\mathcal{B}_{j_0} = \mathcal{B}(\xi_n^{j_0} g(X) + A_{j_0}, \varepsilon)$. Therefore,

$$y_0 = \xi_n^{j_0} g(X) + A_{j_0},$$

since inclusion $\frac{1}{\alpha}\mathcal{O}_\mathbb{K} \subset \frac{1}{\alpha'}\mathcal{O}_\mathbb{K}$ holds. Then X has to be an α-lattice point which is a zero of the rational function $L_{j_0}(\mathbb{X}) = f(g(\mathbb{X})) - f\left(\xi_n^{j_0} g(\mathbb{X}) + A_{j_0}\right) + r(\mathbb{X})$. But the number of zeros in the set $\mathcal{S}$ of the rational function

$$\prod_{i=1}^{k} L_j(\mathbb{X})$$

is finite, hence the variety (9) has finitely many α-lattices points (X, y_0) satisfying $|g(X)| > M_\varepsilon$. On the other hand, the number of points such that $|g(X)| \le M_\varepsilon$ is finite, since $|g(\mathbb{X})|$ dominates $\|\mathbb{X}\|^m$ on the set $\mathcal{S}$. This completes the proof. □

4 Results for varieties over $\mathbb{Q}$

In this section we use Theorem 1.1 to prove Corollary 4.0.1, which is a useful fact about polynomials with integral coefficients. We will consider the following variety

$$F(y) = F(g(\mathbb{X})) + r(\mathbb{X}) \tag{16}$$

over rationals, where $g[\mathbb{X}] \in \mathbb{Z}[\mathbb{X}]$ is the polynomial with k variables.

Corollary 4.0.1. *Let $P \in \mathbb{Z}[x]$ be a monic polynomial of even degree $2m$. If $P(a)$ is a square of an integer for infinitely many integers a, then P is a square of a polynomial with integer coefficients.*

Proof. Let

$$P(x) = x^{2m} + c_{2m-1}x^{2m-1} + \ldots + c_0.$$

Then we construct a polynomial

$$g(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_0,$$

such that its coefficients are given by the following equalities

$$\begin{aligned} b_{m-1} \cdot 1 + 1 \cdot b_{m-1} &= c_{2m-1} \\ b_{m-2} \cdot 1 + b_{m-1}b_{m-1} + 1 \cdot b_{m-2} &= c_{2m-2} \\ &\vdots \\ b_0 \cdot 1 + b_1 b_{m-1} + \ldots + b_{m-1}b_1 + 1 \cdot b_0 &= c_{2m-m}. \end{aligned}$$

Then polynomial $r(x) := P(x) - g(x)^2$ has a degree smaller than m and $g, r \in \mathbb{Q}[x]$. If $r(x) \not\equiv 0$, then Theorem 1.1 with $\mathbb{K} = \mathbb{Q}$, $\mathcal{S} = \mathbb{Z}$ and $F(y) = y^2$ implies that $y^2 = g(x)^2 + r(x) = P(x)$ has a finite number of solutions in integers, since $L_1 = L_2 = r$. Moreover, by the Gauss Lemma we have $g(x) \in \mathbb{Z}[x]$. □

5 Varieties over $\mathbb{Q}(\sqrt{-D})$

In the first part of this section we will show a proof of Corollary 1.1.2. from the Corollary 5.0.1 which is a consequence of Theorem 1.1. In the second part we derive a family of varieties over $\mathbb{Q}(\xi_3) = \mathbb{Q}(\sqrt{-3})$ with a finite number of α-lattice points. In the last part we prove Theorem.1.2 which allows us construct varieties with a finite number of $\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}$ points.

Corollary 5.0.1. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{-D})$. We fix numbers $m, n \in \mathbb{N}$. Let $\alpha \in \mathcal{O}_{\mathbb{K}}$ be a non-zero integral number and $f(x)$, $g(x)$, and $r(x) \in \mathbb{K}[x]$ be polynomials such that:*

1. $\deg f = n \geq 2$, $\deg g = m$ *and* $\deg r < nm - m$*;*
2. n *divides* $N(\mathbb{K})$*;*
3. *for every* $0 \leq j \leq n-1$

$$f(\xi_n^j g(x) + A_j) - f(g(x)) \not\equiv r(x),$$

where $A_j = \frac{\left(1-\xi_n^{j(n-1)}\right)a_{n-1}(f)}{n\xi_n^{j(n-1)}a_n(f)}$*. Then the variety defined by the equation*

$$f(y) = f(g(x)) + r(x) \tag{17}$$

has a finite number of α-lattice solutions (x, y).

Proof of Corollary 1.1.2. It is sufficient to take $f(y) = y^2$ and construct $g(x)$ as in Corollary 4.0.1 such that $\deg g > \deg r$, where $r(x) := x^{2k} + \sum_{i=0}^{2k-1} a_i x^i - (g(x))^2$. In addition, $r(x)$ has to be a non-zero polynomial, since

$$x^{2k} + \sum_{i=0}^{2k-1} a_i x^i$$

is not a square in $\mathbb{K}[x] \subset \mathbb{C}[x]$. The claim follows if one applies Corollary 5.0.1. □

In this part, we derive a family of varieties over $\mathbb{Q}(\xi_3)$ with a finite number of α-lattice points, where $\alpha \in \mathcal{O}_{\mathbb{Q}(\sqrt{-D})}$. For this purpose, we need an appropriate polynomial $g(\mathbb{X})$ which dominates a positive power of $\|\mathbb{X}\|$ on the set $\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}^k$, where $\mathbb{K} = \mathbb{Q}(\sqrt{-D})$ and $\alpha \in \mathcal{O}_{\mathbb{K}}$. Such polynomials come from the following lemma.

Lemma 5.1. *We fix the natural numbers k, $m_1, m_2, \ldots, m_k$. Let $\mathbb{K} = \mathbb{Q}(\sqrt{-D})$ and let $g(\mathbb{X}) \in \mathcal{O}_{\mathbb{K}}[x_1, \ldots, x_k]$ be a polynomial of the form*

$$g(\mathbb{X}) := \prod_{i=1}^{k} x_i^{m_i} + \sum_{l_i < m_i} a_{l_1,\ldots,l_k} x_1^{l_1} \cdots x_k^{l_k}.$$

We define $m := \min\{m_1, \ldots, m_k\}$. Then there exist constants $C, M > 0$ such that the equality

$$|g(X)| \geq C\|X\|^m$$

holds, for α-lattice points $X = (x_1, x_2 \ldots, x_k)$, where $\prod_{i=1}^{k} x_i \neq 0$ and $\|X\| > M$.

Proof. Recall that the norm of non-zero algebraic integers of $\mathbb{Q}(\sqrt{-D})$ is greater or equal to 1, which gives that $|x_i| \geq 1/|\alpha|$. This implies the inequality

$$|a_{l_1,\ldots,l_k}||\alpha|^{\sum(m_i-l_i-1)}\prod_{i=1}^{k}|x_i^{m_i-1}| \geq |a_{l_1,\ldots,l_k}||x_i^{l_i}|.$$

We use the triangle inequality and the above inequality to obtain

$$|g(X)| \geq \prod_{i=1}^{k}|x_i^{m_i}| - C'\prod_{i=1}^{k}|x_i^{m_i-1}| = \left(1 - \frac{C'}{\prod_{i=1}^{k}|x_i|}\right)\prod_{i=1}^{k}|x_i^{m_i}|,$$

where $C' = \sum_{l_i<m_i}|a_{l_1,\ldots,l_k}||\alpha|^{\sum(m_i-l_i-1)}$. Moreover,

$$\frac{C'}{\prod_{i=1}^{k}|x_i|} \leq \frac{C'|\alpha|^{k-1}}{\|X\|} \xrightarrow{\|X\|\to\infty} 0.$$

This implies that for any $0 < \varepsilon < 1$ and the point $\|X\| > \frac{C'|\alpha|^{k-1}}{\varepsilon}$ the following inequalities

$$|g(X)| > (1-\varepsilon)\prod_{i=1}^{k}|x_i^{m_i}| \geq (1-\varepsilon)\frac{\|X\|^m}{|\alpha|^{m_1+m_2+\ldots+m_k-m}},$$

hold. In order to finish the proof, we put $C := \frac{1-\varepsilon}{|\alpha|^{m_1+m_2\cdots+m_k-m}}$ and $M := \frac{C'|\alpha|^{k-1}}{\varepsilon}$. $\square$

Example 5.1. *Let $\mathbb{K} := \mathbb{Q}(\xi_3) = \mathbb{Q}(\sqrt{-3})$ and let $a, b, c, d \in \mathbb{Z}[\xi_3]$ be algebraic integers such that $c \neq 0, ab$. We fix polynomials $f(z) = z^3 + dz$, $g(x, y) = x^3y^3 + x^2 + y^2$ and $r(x, y) = xy + ax + by + c$. Then the variety*

$$f(z) = f(g(x, y)) + r(x, y)$$

has a finite number of points in $\mathbb{Z}[\xi_3] \times \mathbb{Z}[\xi_3] \times \mathbb{Z}[\xi_3]$. In order to show this, we consider four cases. We show that in each case the number of points on the variety is finite.

Case 1: $x = y = 0$. *In order to find points on the variety it is sufficient to solve the equation $z^3 + dz = c$. But this equation has a finite number of solutions.*

Case 2: $y = 0$ *and* $x \neq 0$. *Then the equation which defines a variety reduces to $f(z) = f(g_1(x)) + r_1(x)$, where $g_1(x) = x^2$ and $r_1(x) = ax + c$. Polynomials f, g_1 and r_1 obviously satisfy the assumptions of Theorem 1.1, when $\mathcal{S} = \mathbb{Z}[\xi_3]$. To check the statement (A) of Theorem 1.1 it is enough to check that polynomials L_j are non-zero. Indeed, in this case we have:*

$$L_j(x) = f(g_1(x)) - f(\xi_3^j g_1(x) + 0) + r_1(x) = d(1 - \xi_3^j)x + ax + c \not\equiv 0,$$

because $c \neq 0$. Hence by Theorem 1.1 the number of points of the form $(x, 0, z)$ is finite.

Case 3: $x = 0$ *and* $y \neq 0$. *This case is completely analogous to the previous one.*

Case 4: $xy \neq 0$. *In view of Lemma 5.1, the assumptions of Theorem 1.1 are satisfied for* $\mathcal{S} = \{(x, y) \in \mathbb{Z}[\xi_3] \times \mathbb{Z}[\xi_3] | xy \neq 0\}$. *In this case we have*

$$L_j(x, y) = D(1 - \xi_3^j)(x^3y^3 + x^2 + y^2) + xy + Ax + By + C,$$

where $j \in \{0, 1, 2\}$. *Then the equation* $L_0(x, y) = 0$ *is equivalent to* $(x + b)(y + a) = ab - c$. *This equation has only a finite number of solutions in* $\mathbb{Z}[\xi_3] \times \mathbb{Z}[\xi_3]$*, since* $ab \neq c$ *and the norm of non-zero* α*-lattice points is bounded from below by* $1/|\alpha|$.

For $j = 1, 2$ *we use Lemma 5.1 to obtain that there exists a constant* C' *such that*

$$|L_j(x, y)| \geq C' \|(x, y)\|^3 \text{ for large } \|(x, y)\|, \text{ where } xy \neq 0.$$

Hence $L_1(x, y)$ *and* $L_2(x, y)$ *also have a finite number of zeros in the set*

$$\{(x, y) \in \mathbb{Z}[\xi_3] \times \mathbb{Z}[\xi_3] | xy \neq 0\}.$$

We expand the last example to Theorem 1.2, which says that instead of the variety $0 = r(x, y) = xy + ax + by + c$, we can use any variety $r(x_1, x_2, \ldots, x_k) = 0$, with a finite number of α-lattice points to construct a variety of higher dimension with a finite number of α-lattice points.

Proof of Theorem 1.2. We first write α-lattice points as a sum of family of sets.

$$\frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}^k = \bigcup_{I \subset \{1,2,\ldots k\}} \mathcal{S}_I, \text{ where } \mathcal{S}_I := \left\{ X \in \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}^k \mid x_i = 0 \text{ if and only if } i \in I \right\}.$$

In each set $\mathcal{S}_I$ we put 0 on fixed coordinates. Let

$$m_0 > \frac{\deg r}{\min_{\deg L_j > 0, j=1,\ldots,n}\{\deg L_j\}}.$$

Consider the function

$$g(\mathbb{X}) := \sum_{i=1}^{k} \sum_{1 \leq h_1 < h_2 < \ldots < h_i \leq k} \prod_{h \in \{h_1, h_2, \ldots, h_i\}} x_h^{m_0+i-1}.$$

By Lemma 5.1 we deduce that for any $I \subset \{1, \ldots, k\}$ there exist constants $C_I > 0$ and $M_I > 0$ such that

$$|g(X)| > C_I \|X\|^{m_0+|I|-1} \geq C_I \|X\|^{m_0},$$

for $X \in S_I$, such that $\|X\| > M_I$. This implies that there exist constants $C > 0$ and $M > 0$ such that $|g(X)| > C\|X\|^{m_0}$, for $X \in \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}^k$, such that $\|X\| > M$. Then $\mathcal{S} := \frac{1}{\alpha}\mathcal{O}_{\mathbb{K}}^k$, f, g and r satisfies assumptions of Theorem 1.1. In addition for $1 \leq j \leq n$ the polynomial $L_j(\mathbb{X})$ is equal to $r(\mathbb{X})$ or dominates $\|\mathbb{X}\|$ to the power $m_0 \cdot \min_{\deg L_j > 0, j=1,\ldots,n}\{\deg L_j\}$. Hence, the variety $f(g(\mathbb{X})) - f(\xi_n^j g(\mathbb{X}) + A_j) + r(\mathbb{X}) = 0$ has a finite number of α-lattice points. This implies statement (A) of Theorem 1.1. Therefore, by Theorem 1.1 we obtain the claim. $\square$

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