The Gauss product and Raabe’s integral for $k$-gamma functions

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Abstract: We obtain an extension of the famous Gauss product formula to the case of $k$-gamma functions. The Sándor–Tóth short product formula [16] is also attended to these functions. An asymptotic formula and Raabe’s integral analogue are also considered.

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1 Introduction

As a generalization of the classical Euler gamma function $\Gamma(x)$, in 2007 R. Diaz and E. Pariguan [6] have introduced and studied the notion of $k$-gamma function.

For $k > 0$, the $\Gamma_k$-function is defined by

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n! h^n(nk)^{\frac{x}{k} - 1}}{(x)_{n,k}}, \quad (1)$$

for $x \in \mathbb{C} \setminus k\mathbb{Z}^-$, where $\mathbb{C}$ is the set of complex numbers, $\mathbb{Z}^-$ is the set of negative integers, $(x)_{n,k}$ denotes the classical Pochhammer symbol $(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k)$.

For $x \in \mathbb{C}$, with $\text{Re}(x) > 0$, it can be proved the integral representation [6]

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t}{k}} dt. \quad (2)$$
Also, it satisfies the following properties [6]:

\begin{align}
(i) \quad & \Gamma_k(x + k) = x \Gamma_k(x), \\
(ii) \quad & \frac{\Gamma_k(x + nk)}{\Gamma_k(x)} = (x)_{n,k}, \\
(iii) \quad & \Gamma_k(k) = 1, \\
(iv) \quad & \frac{1}{\Gamma_k(x)} = x.k^{-\frac{x}{k}}.e^{\frac{\pi}{2} \gamma} \prod_{n=1}^{\infty} \left(1 + \frac{x}{nk}\right) e^{-\frac{\pi}{nk}}.
\end{align}

It is obvious also that \( \Gamma_1(x) \equiv \Gamma(x) \).

One of the motivations of introduction of the \( \Gamma_k(x) \)-function is in its connection with the symbol \( (x)_{n,k} \) which appears in a variety of contexts (see [5] and the references). In the recent years, there is an increasing interest about the \( k \)-gamma function (see, e.g., [5, 6, 8–11]).

The famous short product formula of Gauss for the Euler gamma function states that one has the identity

\[
\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \cdots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{n-1}}{\sqrt{n}}.
\]

In 1989, J. Sándor and L. Tóth [16] studied the short product

\[
\prod_{l=1,(l,n)=1}^{n} \Gamma\left(\frac{l}{n}\right) = \left(\frac{2\pi}{k}\right)^{\frac{\varphi(n)}{2}} e^{\frac{\Lambda(n)}{2k}},
\]

where \( \varphi(n) \) is the Euler totient function, and \( \Lambda(n) \) is the von Mangoldt function. This paper has evoked large interest, see, e.g. [1–4, 12–15]. Particularly, the recent paper by M. E. Bachraoui and J. Sándor [2] offers an extension of (5) to the \( \Gamma_q \)-function, which is a classical extension of gamma function, due to F. H. Jackson (see the references from [2]).

The aim of this paper is to extend (4) and (5) to the case of \( k \)-gamma functions.

## 2 Main results

The main results are contained in the following.

**Theorem 2.1.** One has the identity

\[
\prod_{l=1}^{n-1} \Gamma_k\left(\frac{kl}{n}\right) = \left(\frac{2\pi}{k}\right)^{\frac{\varphi(n)}{2}} \frac{1}{\sqrt{n}}.
\]

**Theorem 2.2.** One has the identity

\[
P_k(n) = \prod_{l=1,(l,n)=1}^{n} \Gamma_k\left(\frac{kl}{n}\right) = \frac{(2\pi)^{\varphi(n)}}{k^{\varphi(n)}}, \quad \text{for } n = p^m,
\]

\[
= \left(\frac{2\pi}{k}\right)^{\varphi(n)} \exp\left(\frac{\Lambda(n)}{2k}\right), \quad \text{for } n \neq p^m
\]

where \( p \) is an arbitrary prime, and \( m \) is an arbitrary positive integer.
Theorem 2.3. One has the following Raabe type integral formula
\[ \int_{0}^{1} \log \Gamma_k(kx) \, dx = \log \sqrt{\frac{2\pi}{k}}. \] (8)

Theorem 2.4. One has the following asymptotic formula
\[ \sum_{n \leq x} \log P_k(n) = \frac{3 \log \left( \frac{2\pi}{k} \right)}{2\pi^2} x^2 + O(x \log x), \] (9)
where \( P_k(n) \) is defined in Theorem 2.2.

First, one needs the following auxiliary result.

Lemma 2.1. The following extension of the Euler reflexion formula holds true:
\[ \Gamma_k(x)\Gamma_k(k-x) = \pi \frac{k}{\sin \left( \frac{\pi x}{k} \right)}. \] (10)
Proof. By using the fundamental identity (i) of (3) one can write that \( \Gamma_k(k-x) = -x \Gamma_k(-x) \).
By the Weierstrass type relation (iv) of (3) one gets
\[ \frac{1}{\Gamma_k(x)\Gamma_k(k-x)} = x \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2k^2} \right) \]
(where we have omitted some obvious computations). Now, by the classical Euler formula
\[ \frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2} \right) \] (11)
with the application for \( x := \frac{x}{k} \), identity (10) follows. \( \square \)

The following auxiliary result was stated first by A. Hurwitz ([7, 16]).

Lemma 2.2 Let \( s : [0, 1] \to \mathbb{C} \) be an arbitrary function, and put
\[ f(n) = \sum_{k \in A(n)} s \left( \frac{k}{n} \right), \quad g(n) = \sum_{k=1}^{n} s \left( \frac{k}{n} \right), \]
where \( A(n) = \{ l : 1 \leq l \leq n, (l, n) = 1 \} \). Then one has
\[ f(n) = \sum_{d|n} \mu(d) g \left( \frac{n}{d} \right), \] (12)
where \( \mu \) is the classical Möbius function.

Corollary 2.1 If \( F(n) = \prod_{k \in A(n)} s \left( \frac{k}{n} \right) \) and \( G(n) = \prod_{k=1}^{n} s \left( \frac{k}{n} \right) \), then
\[ F(n) = \prod_{d|n} \left( G \left( \frac{n}{d} \right) \right)^{\mu(d)}. \] (13)
Proof. This follows by letting \( f = \ln F \) and \( g = \ln G \) in Lemma 2.2. \( \square \)
3 Proofs of the theorems

Proof of Theorem 2.1. Letting \( x = \frac{kl}{n} \) in identity (10), we get

\[
\Gamma_k \left( \frac{kl}{n} \right) \Gamma_k \left( k \left( 1 - \frac{l}{n} \right) \right) = \frac{\pi}{k} \cdot \frac{1}{\sin \frac{\pi l}{n}}.
\]  

(14)

By remarking that, when \( l = 1, 2, \ldots, n - 1 \) one has

\[
\prod_{l=1}^{n-1} \Gamma_k \left( k \left( 1 - \frac{l}{n} \right) \right) = \prod_{l=1}^{n-1} \Gamma_k \left( k \frac{l}{n} \right),
\]

as \( 1 - \frac{l}{n} = \frac{n-l}{n} \), and applying identity (14) to \( l = 1, 2, \ldots, n - 1 \), by term-by-term multiplication of the of the obtained relation, we get

\[
\left( \prod_{l=1}^{n-1} \Gamma_k \left( \frac{kl}{n} \right) \right)^2 = \left( \frac{\pi}{k} \right)^{n-1} \frac{1}{\prod_{l=1}^{n-1} \sin \frac{\pi l}{n}} = \left( \frac{\pi}{k} \right)^{n-1} \frac{2^{n-1}}{n}
\]

by the well-known trigonometric identity \( \prod_{l=1}^{n-1} \sin \frac{\pi l}{n} = \frac{2^{n-1}}{n} \).

Now, relation (6) follows at once from the above. □

Proof of Theorem 2.2. By Theorem 2.1 and Corollary 2.1, the left-hand side of (7) can be written as

\[
\left( \frac{2\pi}{k} \right)^{\frac{1}{2} \sum_{d|n} d \mu \left( \frac{n}{d} \right) - \frac{1}{2} \sum_{d|n} \mu \left( \frac{n}{d} \right)} \sqrt{h(n)},
\]

where \( h(n) = \prod_{d|n} d^{\mu \left( \frac{n}{d} \right)} \).

Now, it is well-known that (see, e.g., [7]) \( \sum_{d|n} d \mu \left( \frac{n}{d} \right) = \sum_{d|n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n} \) and \( \sum_{d|n} \mu \left( \frac{n}{d} \right) = \sum_{d|n} \mu(d) = 0 \).

Also, \( \log h(n) = \Lambda(n) = \log p \) if \( n = p^m \), and it is equal to 0, if \( n \neq p^m \). Thus, identity (7) follows. □

Proof of Theorem 2.3. We will use the classical Riemann sum approach, based on the limit formula

\[
\int_0^1 f(x)dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{i}{n} \right).
\]

(15)

Let \( f(x) = \log \Gamma_k(kx) \). By Theorem 2.1, and relation (15) one has

\[
\int_0^1 \log \Gamma_k(kx)dx = \lim_{n \to \infty} \left( \frac{n-1}{2n} \log \left( \frac{2\pi}{k} \right) - \frac{1}{2n} \log n \right) = \frac{1}{2} \log \left( \frac{2\pi}{k} \right).
\]

Thus gives relation (8). □
Proof of Theorem 2.4. By Theorem 2.2 one can write
\[
\sum_{n \leq x} \log P_k(n) = \sum_{n \leq x} \left( \varphi(n) \cdot \frac{2}{k} \log \left( \frac{2\pi}{k} \right) - \frac{1}{2} \Lambda(n) \right)
\]
\[
= \frac{1}{2} \log \frac{2\pi}{k} \sum_{n \leq x} \varphi(n) - \frac{1}{2} \sum_{n \leq x} \Lambda(n)
\]
\[
= \frac{1}{2} \log \frac{2\pi}{k} \cdot \left( \frac{3}{\pi^2} x^2 + O(x \log x) \right) - \frac{1}{2} O(x)
\]
\[
= \frac{3}{2} \log \frac{2\pi}{k} x^2 + O(x \log x),
\]
where we have used the classical asymptotic relations (see, e.g., [7]):
\[
\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),
\]
and
\[
\sum_{n \leq x} \Lambda(n) = O(x).
\]
This completes the proof.

References


