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A note on the Fermat quartic $34x^4 + y^4 = z^4$

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Abstract: We show that the only primitive non-zero integer solutions to the Fermat quartic $34x^4 + y^4 = z^4$ are $(x, y, z) = (\pm 2, \pm 3, \pm 5)$. The proofs are based on a previously given complete solution to another Fermat quartic namely $x^4 + y^4 = 17z^4$.
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1 Introduction

A Fermat quartic can be regarded as a Diophantine equation of the form $ax^4 + by^4 = cz^4$ where a, b and c are fixed, non-zero and fourth power free integers [1,3]. We define a primitive non-zero solution to this equation as a solution (x_0, y_0, z_0) where ax_0, by_0 and cz_0 are pairwise relatively prime and $x_0 \cdot y_0 \cdot z_0 \neq 0$. From a well-known theorem of Darmon and Granville we conclude that a Fermat quartic has only a finite number of primitive non-zero solutions [2].

Hence under these conditions there are a finite number of solutions or no solution at all. When $a = b = c = 1$ we recognize Fermat's last theorem for $n = 4$.

2 Main results

Theorem 1. The only primitive non-zero integer solutions to the Fermat quartic $34x^4 + y^4 = z^4$ are $(x, y, z) = (\pm 2, \pm 3, \pm 5)$.

Proof. We have $34x^4 = z^4 - y^4 \implies 34x^4 = (z^2 + y^2) \cdot (z + y) \cdot (z - y)$ and after congruence considerations we realize that z and y are both odd and x is even.

Furthermore, according to prerequisites $34x$, y and z are pairwise relatively prime. Let $z = p + q$ and $y = p - q$ where $p \not\equiv q \mod 2$ and $(p, q) = 1$ since $(y, z) = 1$. Hence we get

$$34x^4 = 2 \cdot (p^2 + q^2) \cdot 2p \cdot 2q,$$
$$17x^4 = 2^2 \cdot (p^2 + q^2) \cdot p \cdot q.$$

Case I: p is even and q is odd.
Hence $p = 2^2 \cdot t$ since x is even and we get,

$$17x^4 = 2^4 \cdot (p^2 + q^2) \cdot t \cdot q. \tag{1}$$

Since $p^2 + q^2$, t and q are pairwise relatively prime, we can distinguish three different cases.

Subcase 1. $17 \mid q \Longrightarrow q = 17A$ and from (1) we have,

$$x^4 = 2^4 \cdot (p^2 + q^2) \cdot t \cdot A.$$

Hence,

$$p^2 + q^2 = B^4. \tag{2}$$

Since p and q are squared in equation (2) we may assume that p and q are positive. Hence, we have $p = 4t$ and $t = D^4 \Longrightarrow p = 4D^4$.

Thus, from (2) we have,

$$(4D^4)^2 + q^2 = B^4 \Longrightarrow q^2 = B^4 - (2D^2)^4,$$

which has no non-zero solutions according to [4].

Subcase 2. $17 \mid t \Longrightarrow t = 17E$ and if this t value is inserted in equation (1), we have,

$$x^4 = 2^4 \cdot (p^2 + q^2) \cdot E \cdot q.$$

We have,

$$p^2 + q^2 = G^4. \tag{3}$$

As in Subcase 1, we may assume that q is positive. Hence, $q = F^4$ and from (3) we get, $p^2 = G^4 - (F^2)^4$, which has no non-zero solutions according to [4].

Subcase 3. $17 \mid (p^2 + q^2)$. Thus we have,

$$p^2 + q^2 = 17H^4. \tag{4}$$

From (1) we get $x^4 = 2^4 \cdot H^4 \cdot t \cdot q$ and since the left-hand side is positive, we must have $t = K^4$ and $q = J^4$ or $t = -K^4$ and $q = -J^4$. If these substitutions are inserted in (4), we get since $p = 4t$,

$$(\pm 4K^4)^2 + (\pm J^4)^2 = 17H^4.$$

$$(2K^2)^4 + (J^2)^4 = 17H^4. \tag{5}$$

However, the equation $a^4 + b^4 = 17c^4$ has according to [3] the only primitive non-zero solutions $(a, b, c) = (\pm 1, \pm 2, \pm 1)$ and $(a, b, c) = (\pm 2, \pm 1, \pm 1)$. Hence, $(2K^2, J^2, H) = (\pm 1, \pm 2, \pm 1)$ and $(2K^2, J^2, H) = (\pm 2, \pm 1, \pm 1)$. Since $2K^2 \neq \pm 1$ and $J^2 \neq \pm 2$ only the second alternative must be applicable on (5). Hence, $2K^2 = 2 \Longrightarrow K = \pm 1$ and $J^2 = 1 \Longrightarrow J = \pm 1$. Thus, according to previous expressions of t, p and q we have $t = (\pm 1)^4 = 1$ and $q = (\pm 1)^4 = 1$ or $t = -(\pm 1)^4 = -1$ and $q = -(\pm 1)^4 = -1$. Since $p = 4t$, we get $p = 4$ and $q = 1$ or $p = -4$ and $q = -1$. Hence, since $z = p + q$ and $y = p - q$ we get $z = 4 + 1 = 5$ and $y = 4 - 1 = 3$ or $z = -4 + (-1) = -5$ and $y = -4 - (-1) = -3$. Thus, we have, $(z, y) = (5, 3)$ and $(z, y) = (-5, -3)$.

Case II. p is odd and q is even.
According to Case I we must have $p = 1$ and $q = 4$ or $p = -1$ and $q = -4$. Since $z = p + q$ and $y = p - q$, we get $(z, y) = (5, -3)$ and $(z, y) = (-5, 3)$.

Finally, we see that $34x^4 = (\pm 5)^4 - (\pm 3)^4 \Longrightarrow x = \pm 2$ and this completes the proof of Theorem 1. $\square$

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