On a translated sum over primitive sequences related to a conjecture of Erdős

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Abstract: For \( x \) large enough, there exists a primitive sequence \( \mathcal{A} \), such that

\[
\sum_{a \in \mathcal{A}} \frac{1}{a (\log a + x)} \gg \sum_{p \in \mathcal{P}} \frac{1}{p (\log p + x)},
\]

where \( \mathcal{P} \) denotes the set of prime numbers.

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1 Introduction

A sequence \( \mathcal{A} \) of positive integers is said to be primitive if there is no element of \( \mathcal{A} \) which divides any other. We can see directly that the set of primes \( \mathcal{P} = (p_n)_{n \geq 1} \) is primitive, as well as the sequences of the form:

\[
\mathcal{A}_d^k = \left\{ p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} : \alpha_1, \ldots, \alpha_k \in \mathbb{N}, \alpha_1 + \ldots + \alpha_k = d \right\}.
\]
Erdős [2] showed that for a primitive sequence \( A \neq \{1\} \), the series \( \sum_{a \in A} \frac{1}{a \log a} \) converges. Later, in [3], he conjectured that if \( A \neq \{1\} \) is a primitive sequence, then \( \sum_{a \in A} \frac{1}{a \log a} \leq \sum_{p \in P} \frac{1}{p \log p} \).

Recently, in [4, 5], the authors studied a translated sums of the form \( S(A, x) = \sum_{a \in A} \frac{1}{a (\log a + x)} \), where \( x \in \mathbb{R}_+ \). In [5], the authors constructed a primitive sequence \( A \), such that for all \( x \geq 81 \), \( S(A, x) > S(P, x) \). In this note, we prove the following result.

**Theorem 1.1.** Let \( \lambda \geq 1 \) and \( t > 0 \), then for any \( x \geq 1656 \lambda^2 t (\log(\lambda^2 t + 2))^{3/2} \), there exists a primitive sequence \( A \) such that

\[
S(A, x) \geq \lambda^t S(P, x).
\]

For a real \( x \), the quantity \( \lfloor x \rfloor \) denotes the integer part of \( x \).

## 2 Lemmas

**Lemma 2.1 ([6, 8]).** We have:

\[
\begin{align*}
p_n & \geq n \log n \quad (\forall n \geq 2) \quad (1) \\
p_n & \leq n (\log n + \log \log n) \quad (\forall n \geq 6) \quad (2) \\
\sum_{p \in P, p \leq x} \frac{1}{p} & > \log \log x \quad (x > 1). \quad (3)
\end{align*}
\]

**Lemma 2.2 ([1]).** For \( k \geq 463 \),

\[
p_{k+1} \leq p_k \left( 1 + \frac{1}{2 \log^2 p_k} \right).
\]

**Lemma 2.3.** For any real number \( x > 0 \) and any integer \( k \geq 2 \) the following holds

\[
\sum_{n > k} \frac{1}{p_n (\log p_n + x)} \leq \frac{\log(1 + \frac{x}{\log k})}{x}.
\]

**Proof.** Let \( x > 0 \) be a real number and \( k \geq 2 \) be an integer. By (1) and since the function \( t \mapsto \frac{dt}{t \log t (\log t + x)} \) decreases on \([1, +\infty)\), then we obtain:

\[
\begin{align*}
\sum_{n > k} \frac{1}{p_n (\log p_n + x)} & \leq \sum_{n > k} \frac{1}{n \log n (\log n + \log \log n + x)} \\
& \leq \sum_{n > k} \frac{1}{n \log n (\log n + x)} \leq \int_k^{+\infty} \frac{dt}{t \log t (\log t + x)}.
\end{align*}
\]

We put \( u = \log t \), so

\[
\int_k^{+\infty} \frac{dt}{t \log t (\log t + x)} = \int_{\log k}^{+\infty} \frac{du}{u(u + x)} = \frac{1}{x} \int_{\log k}^{+\infty} \left( \frac{1}{u} - \frac{1}{u + x} \right) du = \frac{\log(1 + \frac{x}{\log k})}{x}.
\]

This ends the proof. \( \square \)
Lemma 2.4. For any integer \( n \neq 0 \), we have:

\[
n! \leq n^n e^{1-n}\sqrt{n}.
\]

Proof. For \( n = 1 \), the inequality is verified. For \( n \geq 2 \), the result follows from

\[
n! \leq n^n e^{-n}\sqrt{2\pi ne^{1/12n}}
\]

(see [7]). □

3 Proof of Theorem 1.1

Let \( \lambda \geq 1 \) and let \( t > 0 \). To prove this theorem, we need the parameters \( \alpha, c, \) and \( \beta \) which satisfy:

\[
\alpha \geq e^\beta + \log 1.008, \quad 0 < \alpha \leq \frac{5}{12} \tag{C1}
\]

\[
\beta \geq 1.950 \tag{C2}
\]

those parameters will be chosen later, the real \( c \) is chosen to be the smallest possible value so that; for any \( x \geq c\lambda^2 (\log(\lambda^2 + 2))^{3/2} \), there exists a primitive sequence \( A \neq \{1\} \) such that

\[
\sum_{a \in A} \frac{1}{a\log a + x} > \lambda \sum_{p \in P} \frac{1}{p(p \log p + x)}.
\]

Let \( p_k \) be the largest prime satisfying \( p_k \leq e^{\alpha x} \), then according to Lemma 2.2 and (1), we obtain

\[
p_k \leq e^{\alpha x} < p_{k+1} < 1.008p_k. \tag{4}
\]

Assume that \( d = \lfloor \beta + \log \lambda^2t + \frac{3}{2} \log \log (\lambda^2 + 2) \rfloor \), then from (C1) and (C2), we have

\[
x \geq \frac{1}{\alpha} (e^d + \log 1.008) \quad \text{and from (3) and (4), we obtain}
\]

\[
\sum_{n=1}^{k} \frac{1}{p_n} > \log \log p_k > \log \log \frac{p_{k+1}}{1.008} > \log \log \frac{e^{\alpha x}}{1.008} \geq d. \tag{5}
\]

Now, we define the following sets of positive integers:

\[
P^k = \{p_n \mid p_n \in P, p_n > p_k\}, \quad A = A_d^k \cup P^k.
\]

It is clear that \( A_d^k \cap P^k = \emptyset \) and the sets \( A_d^k, P^k, A \) are primitive sequences. Then, according to the multinomial formula and (5), we have

\[
\sum_{a \in A} \frac{1}{a} = \sum_{\alpha_1 + \ldots + \alpha_k = d} \frac{1}{P_{1}^{\alpha_1} P_{2}^{\alpha_2} \ldots P_{k}^{\alpha_k}} \geq \sum_{\alpha_1 + \ldots + \alpha_k = d} \frac{(1/p_1)^{\alpha_1} \ldots (1/p_k)^{\alpha_k}}{(\alpha_1)! \ldots (\alpha_k)!}.
\]

So,

\[
\frac{1}{d!} \left( \sum_{n=1}^{k} \frac{1}{p_n} \right)^d > \frac{1}{d!} \sum_{n=1}^{k} \frac{1}{p_n}.
\]

Therefore,

\[
\sum_{a \in A} \frac{1}{a} > \frac{d^{d-1}}{d!} \sum_{n=1}^{k} \frac{1}{p_n}. \tag{6}
\]
Since \( x \geq c \lambda^{2t} (\log(\lambda^{2t} + 2))^{3/2} \), then from (C1) and (C2) we obtain \( e^{\alpha x} \geq 3303 \geq p_{464} \). Hence using (4), we find \( p_{464} \leq p_k \leq e^{\alpha x} < p_{k+1} < 1.008 p_k \). By using (2), we get
\[
\log p_k \leq \alpha x \leq \log p_k + \log 1.008 \leq \log (k (\log k + \log \log k)) + \log 1.008.
\]
Now, since the function \( t \mapsto \frac{\log (t (\log t + \log \log t)) + \log 1.008}{\log t} \) decreases on \([464, + \infty)\), then we have
\[
\frac{\log (t (\log t + \log \log t)) + \log 1.008}{\log t} \leq \frac{\log (464 (\log 464 + \log \log 464)) + \log 1.008}{\log 464} \simeq 1.339
\]
that is,
\[
\alpha x \leq 1.339 \log k. \tag{7}
\]
By using inequality (7) and Lemma 2.3, we find
\[
\sum_{n>k} \frac{1}{p_n (\log p_n + x)} \leq \frac{\log(1 + \frac{x}{\log k})}{x} < \frac{\log(1 + \frac{1.339}{\alpha})}{x}. \tag{8}
\]
On the other hand, according to (4) and (5), we have for \( x \neq 0 \)
\[
\sum_{n=1}^{k} \frac{1}{p_n (\log p_n + x)} \geq \sum_{n=1}^{k} \frac{1}{p_n (\alpha x + x)} \geq \frac{1}{(\alpha + 1) x} \sum_{n=1}^{k} \frac{1}{p_n} \geq \frac{d}{(\alpha + 1) x}.
\]
and from (8) we obtain
\[
\sum_{n=1}^{k} \frac{1}{p_n (\log p_n + x)} \geq \frac{d}{(\alpha + 1) \log(1 + \frac{1.339}{\alpha})} \sum_{n>k} \frac{1}{p_n (\log p_n + x)}.
\]
Now we put \( h(\alpha) = (\alpha + 1) \log(1 + \frac{1.339}{\alpha}) \), then we obtain
\[
\sum_{n=1}^{k} \frac{1}{p_n (\log p_n + x)} \geq \frac{d}{h(\alpha)} \left( \sum_{n=1}^{+\infty} \frac{1}{p_n (\log p_n + x)} - \sum_{n=1}^{k} \frac{1}{p_n (\log p_n + x)} \right),
\]
therefore,
\[
\left( 1 + \frac{d}{h(\alpha)} \right) \sum_{n=1}^{k} \frac{1}{p_n (\log p_n + x)} \geq \frac{d}{h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n (\log p_n + x)}.
\]
Thus
\[
\sum_{n=1}^{k} \frac{1}{p_n (\log p_n + x)} \geq \frac{d}{d + h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n (\log p_n + x)}. \tag{9}
\]
Since \( p_k^d \) is the largest element in \( A^k_d \), then according to (4), we have for any \( a \in A^k_d \)
\[
\log a \leq d \log p_k \leq d \alpha x,
\]
hence, from (6), we obtain:
\[
\sum_{a \in A} \frac{1}{a(\log a + x)} = \sum_{a \in A \cup P^k} \frac{1}{a(\log a + x)} = \sum_{a \in A^k} \frac{1}{a(\log a + x)} + \sum_{a \in P^k} \frac{1}{a(\log a + x)}
\]

\[
\geq \frac{1}{(d\alpha x + x)} \sum_{a \in A^k} \frac{1}{a} + \sum_{n > k} \frac{1}{p_n(\log p_n + x)}
\]

\[
> \frac{d^{d-1}}{d!(d\alpha + 1)} \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} + \sum_{n > k} \frac{1}{p_n(\log p_n + x)}
\]

\[
= \left( \frac{d^{d-1}}{d!(d\alpha + 1)} - 1 \right) \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} + \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}.
\]

According to (C1), and since the sequence \((u_n)_{n \geq 2}\) where \(u_n = \frac{n^{n-1} - n!}{nn!}\) increases, then we have \(\frac{d^{d-1}}{d!(d\alpha + 1)} - 1 \geq 0\) for \(d \geq 4\). By using this last inequality and (9), we obtain

\[
\left( \frac{d^{d-1}}{d!(d\alpha + 1)} - 1 \right) \sum_{n=1}^{k} \frac{1}{p_n(\log p_n + x)} \geq \left( \frac{d^{d-1}}{d!(d\alpha + 1)} - 1 \right) \frac{d}{d + h(\alpha)} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}.
\]

Therefore,

\[
\sum_{a \in A} \frac{1}{a(\log a + x)} > \left( \left( \frac{d^{d-1}}{d!(d\alpha + 1)} - 1 \right) \frac{d}{d + h(\alpha)} + 1 \right) \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}
\]

\[
= \frac{d^d + d!(d\alpha + 1)h(\alpha)}{d!(d\alpha + 1)(d + h(\alpha))} \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)},
\]

by applying Lemma 2.4, we get

\[
\sum_{a \in A} \frac{1}{a(\log a + x)} > \left( \frac{e^{d-1} + \sqrt{d}(d\alpha + 1)h(\alpha)}{\sqrt{d}(d\alpha + 1)(d + h(\alpha))} \right) \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}.
\]

(10)

It follows from the expression of \(d\), that: \(d > \beta - 1 + \log \lambda^{2t} + \frac{3}{2} \log \log (\lambda^{2t} + 2)\), then \(e^{d-1} > e^{\beta - 2} \lambda^{2t} (\log (\lambda^{2t} + 2))^{3/2}\).

And since \(\log \lambda^{2t} < \log (\lambda^{2t} + 2)\), \(\log \log (\lambda^{2t} + 2) \leq \log (\lambda^{2t} + 2) - 1\) and \(\beta > 1.950\), we have \(d < (\beta + 1) \log (\lambda^{2t} + 2)\), then \(d\alpha + 1 < ((\beta + 1)\alpha + 1) \log (\lambda^{2t} + 2)\).

So, the formula (10) becomes

\[
\sum_{a \in A} \frac{1}{a(\log a + x)} > j_{\alpha,\beta}(\lambda) \sum_{n=1}^{+\infty} \frac{1}{p_n(\log p_n + x)}.
\]

(11)

where

\[
j_{\alpha,\beta}(\lambda) = \frac{e^{\beta - 2} \lambda^{2t} + \sqrt{\beta + 1}((\beta + 1)\alpha + 1)h(\alpha)}{\sqrt{\beta + 1}((\beta + 1)\alpha + 1)((\beta + 1) \log (\lambda^{2t} + 2) + h(\alpha))}.
\]

Now, we must choose \(\alpha\) and \(\beta\) so that, for any \(\lambda \geq 1\) and any \(t > 0\), \(j_{\alpha,\beta}(\lambda) \geq 1\) and \(\frac{e^{\beta} + \log 1.008}{\alpha}\) is the smallest possible. That is, for any \(\lambda \geq 1\) and for any \(t > 0\)

\[
\frac{e^{\beta - 2}}{\sqrt{\beta + 1}((\beta + 1)\alpha + 1)} \geq \frac{\log (\lambda^{2t} + 2)}{\lambda^{2t}}.
\]
Since, for any $t > 0$ the function $\lambda \mapsto \frac{\log (\lambda^{2t} + 2)}{\lambda^{2t}}$ decreases on $[1, + \infty)$, then

$$\frac{e^{\beta - 2}}{\sqrt{\beta + 1} (\beta + 1) (\beta + 1) \alpha + 1} \geq \log 3.$$  

Hence, $\frac{e^{\beta - 2} - (\beta + 1)^{\frac{3}{2}} \log 3}{(\beta + 1)^{\frac{3}{2}} \log 3} \geq \alpha$ and $\frac{e^{\beta} + \log 1.008}{\alpha} \geq \frac{e^{\beta} + \log 1.008 (\beta + 1)^{\frac{3}{2}} \log 3}{e^{\beta - 2} - (\beta + 1)^{\frac{3}{2}} \log 3}$.

Finally, we will choose $\beta$ so that the quantity $\frac{e^{\beta} + \log 1.008 (\beta + 1)^{\frac{3}{2}} \log 3}{e^{\beta - 2} - (\beta + 1)^{\frac{3}{2}} \log 3}$ is also the smallest possible.

A computer calculation gives $\beta \simeq 6.264$, $\alpha \simeq 0.317$ and $c \simeq 1655.234$. By replacing $\alpha$ and $\beta$ in (11), we get

$$\sum_{a \in A} \frac{1}{a (\log a + x)} > \frac{71.094 \lambda^{2t} + 19.381}{64.659 \ln (\lambda^{2t} + 2) + 19.381} \sum_{n=1}^{+\infty} \frac{1}{p_n (\log p_n + x)}.$$  

But, for every $\lambda \geq 1$ and every $t > 0$, we have

$$\frac{71.094 \lambda^{2t} + 19.381}{64.659 \ln (\lambda^{2t} + 2) + 19.381} > \lambda^t,$$

which leads to the inequality our main theorem. Thus, for $\lambda \geq 1$, $t > 0$ and for any $x \geq 1656.3 \lambda^{2t} (\log (\lambda^{2t} + 2))^{3/2}$, since $d = \left[6.264 + \log \lambda^{2t} + \frac{3}{2} \log \log (\lambda^{2t} + 2)\right]$ and $k$ is the greatest integer such that $p_k \leq e^{0.317x}$, the sequence $A$ is well defined. This ends the proof.  

\section*{References}


