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On Pythagorean triplet semigroups

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Abstract: In this note we explain the two pseudo-Frobenius numbers for $\langle m^2-n^2, m^2+n^2, 2mn\rangle$ where m and n are two coprime numbers of different parity. This lets us give an Apéry set for these numerical semigroups.
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1 Introduction and preliminaries

Let $a_1, \ldots, a_n$ be n positive integers with $\gcd(a_1, \ldots, a_n) = 1$, the set

$$S := \left\{ \sum_{i=1}^{s} \lambda_i a_i \big| s \in \mathbb{N}, \lambda_i \geq 0, \text{for all } i \right\}$$

be called the numerical semigroup S and the integers $a_1, \ldots, a_n$ be its generators. A numerical semigroup is minimally generated by $a_1, \ldots, a_n$ if we cannot remove a generator without changing the set S; in this case we denote S by $\langle a_1, a_2, \ldots, a_n\rangle$. Given $S \neq \mathbb{N}$, the number $F(S) := \max\{n \in \mathbb{N} | n \notin S\}$ (which exists, see [5, Theorem 1.0.1]) is the Frobenius number of S. For a numerical semigroup S let

$$T(S) := \{x \in \mathbb{N} | x \notin S, x + s \in S, \text{ for all } s \in S, s > 0\}.$$

The cardinality of $T(S)$ is called the type of S and a number in $T(S)$ is called a pseudo-Frobenius number. The Apéry set of S with respect to $n \in S$ is the set $\mathrm{Ap}(S, n) = \{s \in S | s - n \notin S\}$ and the genus of S denoted $g(S)$ is the cardinality of $\{\mathbb{N} \backslash S\}$.

Definition 1. *A numerical semigroup is said to be* Arf *if for all $s, r, t \in S$ with $s \geq r \geq t$, $s + r - t \in S$. For $S = \langle a_1, \ldots, a_n \rangle$ we define for every $i \in \{2, \ldots, n\}$:*

$$c_i = \min\{k \in \mathbb{N} \backslash \{0\} | k \cdot a_i \in \langle a_1, \ldots, a_{i-1} \rangle\},$$

S is then free *if $a_1 = c_2 \cdots c_n$.*

Remark 1 ([2,4,8])**.** *Let $a_1, a_2, \ldots, a_k$ be positive integers. If $\gcd(a_2, \ldots, a_n) = d$ and $a_j = d.a'_j$ for each $j > 1$, then*

- *The type of $\langle a_1, a_2, \ldots, a_n \rangle$ equals the type of $\langle a_1, a'_2, \ldots, a'_n \rangle$.*
- *The type of $S := \langle a_1, a_2, a_3 \rangle$ is at most two (see* [2, Theorem 11]*) and it equals two if S has pairwise coprime minimal generators (see* [7]*).*
- $\mathrm{Ap}(S, n)$ *has n elements and $g(S) = \frac{1}{n} \sum_{w \in \mathrm{Ap}(S,n)} w - \frac{n-1}{2}$ (see* [8, Chapter 1]).

A survey on finding Frobenius numbers for numerical semigroups can be found in [5].

A Pythagorean triplet is a positive integer triplet (x, y, z) verifying $x^2 + y^2 = z^2$. We say that this triplet is primitive if any two integers from x, y, z are coprime and we have: *Every primitive Pythagorean triplet can be expressed as $(m^2 - n^2, 2mn, m^2 + n^2)$ where m and n are coprime numbers of different parity.*

Proposition 1. *Let a and b be two coprime positive integers and let (x_0, y_0) denote the nonnegative couple (when it exists) verifying $ax_0 + by_0 = n$, $0 \leq y_0 < a$, ($by_0 = n \pmod a$) then the number of nonnegative integer solutions to the equation $ax + by = n$ equals $\left\lfloor \dfrac{n - by_0}{ab} \right\rfloor + 1$.*

Proof. Set $x_i = x_0 - ib$ and $y_i = ia + y_0$, since a and b are coprime, for every i there is a unique y_i with $ia \leq y_i < (i + 1)a$ such that $ax_i + by_i = n$. Counting the nonnegative couples (x_i, y_i) we get the result. □

Corollary 1. *Let m and n be two coprime positive integers of different parity. If $t \in \mathbb{N}$, $t \geq 1$ and $m > (t + 1)n$, then (tn, tn) is the unique nonnegative solution to*

$$2tmn = (m - n)x + y(m + n). \tag{1}$$

Proof. We apply Proposition 1 $y_0 = tn < m - n$, notice that $tmn - tn^2 < m^2 - n^2 \iff tm(n - m) - n^2(t - 1) < 0$. □

Corollary 2 (Bézout). *The integer solutions of* (1) *are of the form:*

$$(tn + k(m + n), tn - k(m - n))$$

for some $k \in \mathbb{Z}$.

2 Main result

From the definition of a pseudo-Frobenius number F for a given $S := \langle a_1, a_2, a_3\rangle$, $z := F + a_3 \in S$ but since $F \notin S$, $z = \sum_{i=1}^{2} u_i a_i$, consequently any such number F can be written as $ua_1 + va_2 - a_3$ for some $u \geq 0$ and $v \geq 0$. It is known ([6]) that for any numerical semigroup $\langle a, b\rangle$ a positive integer $x \notin S$ if and only if $x = \alpha a - \beta b$ for some $0 < \alpha < b$ and $0 < \beta < a$.

We set $a_1 = 2mn$, $a_2 = m^2 + n^2$ and $a_3 = m^2 - n^2$ so $S := \langle m^2 - n^2, m^2 + n^2, 2mn\rangle$: when $m = n + 1$, $a_1 = 2n(n + 1)$, $a_2 = 2n^2 + 2n + 1 = (2n + 1)(2n + 1) - 2n(n + 1) = 2n(n+1)(2n) - (2n^2 - 1)(2n+1)$ and $a_3 = 2n+1$, using Theorem 11's proof [2], we can find the two pseudo-Frobenius numbers of this semigroup (we leave it as an exercise). This method does not easily settle the general case, however the Frobenius number $F(S)$ (as $g(S)$) was given in Example 3 of [1], see also [3]. Recently a complete (different) study of Pythagorean semigroups including finding $T(S)$, $F(S)$ and $g(S)$ was done by A. Tripathi and E. F. Elizeche [9]. We thank the authors for correspondences.

Remark 2. *We have* $m(2mn) = n(m^2 - n^2) + n(m^2 + n^2)$, $m(m^2 + n^2) = n(2mn) + m(m^2 - n^2)$ *and* $(m + n)(m^2 - n^2) = (m - n)(m^2 + n^2) + (m - n)(2mn)$.

Theorem 2.1. *Let* $S = \langle m^2 - n^2, m^2 + n^2, 2mn\rangle$, m *coprime with* n *and of distinct parity, then* $T(S) = \{PF(S), F(S)\}$ *where*

$$PF(S) = (m - 1)(m^2 + n^2) + (n - 1)(m^2 - n^2) - 2mn$$

and

$$F(S) = (m - 1)(m^2 - n^2) + (m - 1)2mn - (m^2 + n^2)$$

Proof. The proof is straight computationally, we verify that the two given numbers can not be in S and that $T(S) + a_i \in S$, $(i = 1, 2, 3)$.

$$\begin{aligned}
F(S) + a_2 &= (m - 1)(m^2 - n^2) + (m - 1)2mn\\
F(S) + a_3 &= (m - 1)(m^2 + n^2) + (m - n - 1)2mn\\
F(S) + a_1 &= (n - 1)(m^2 + n^2) + (m + n - 1)(m^2 - n^2)\\
PF(S) + a_3 &= (m - n - 1)(m^2 + n^2) + (m - 1)2mn\\
PF(S) + a_1 &= (m - 1)(m^2 + n^2) + (n - 1)(m^2 - n^2)\\
PF(S) + a_2 &= (n - 1)2mn + (m + n - 1)(m^2 - n^2)
\end{aligned}$$

Suppose $F(S) = \alpha(m^2 + n^2) + \beta(m^2 - n^2) + \gamma(2mn)$ where α, β, γ are nonnegative and we can assume that $\gamma < m$ by Remark 2 with $\alpha < 2m - 3$. If $\gamma = \alpha + v \geq \alpha$, then $F(S) = \alpha(m+n)^2 + \beta(m-n)(m+n) + v(2mn)$ implying that $(m+n)$ divides $2mn(v-m)$, a contradiction. If otherwise $\gamma < \alpha = \gamma + v < 2m-3$, we get $F(S) = \gamma(m+n)^2 + \beta(m-n)(m+n) + v(m^2+n^2)$, which implies that $(m + n)$ divides $2mn(v + m)$, so $v = n$ or $v = m + 2n$. In case $v = n$, respectively $v = m + 2n$, after simplifying by $(m+n)$, we need to solve $2mn = (\gamma+1+n)(m+n) + (\beta - m + 1)(m - n)$, respectively, $4mn = (\gamma + 1 + m + 2n)(m + n) + (\beta - m + 1)(m - n)$, from

Corollary 2 we see that supposing $n+1+\gamma = n+k(m-n)$, $(k \geq 1)$ (so $\beta-m+1 = n-k(m+n)$), $\beta = m-1+n-k(m+n) < 0$, a contradiction. The same contradiction is true for the respective case.

For the other number $PF(S) = (m-1)(m^2+n^2)+(n-1)(m^2-n^2)-2mn$ the same arguments hold: Suppose $PF(S) = \alpha(m^2+n^2)+\beta(m^2-n^2)+\gamma(2mn)$ where α, β, γ are nonnegative and we can assume that $\gamma < m$ by Remark 2 with $\alpha < m+2n-3$. If $\gamma \leq \alpha = \gamma+v$, then $PF(S) = \gamma(m+n)^2+\beta(m-n)(m+n)+v(m^2+n^2+2mn-2mn)$ implying that $(m+n)$ divides $2mn(v-m)$, so v must equal m, simplifying by $(m+n)$ we have to solve $(\gamma+1)(m+n) = (n-1-\beta)(m-n)$, a contradiction. If otherwise $\alpha < \gamma = \alpha+v < m$, we get $PF(S) = \alpha(m+n)^2+\beta(m-n)(m+n)+v(2mn)$, which implies that $(m+n)$ divides $2mn(v+m)$, so $v = n$. After simplifying by $(m+n)$, we need to solve $2mn = (m-\alpha-1)(m+n)+(n-1-\beta)(m-n)$ from Corollary 2 we see that α and β cannot be both nonnegative, a contradiction. $\square$

Now from Remark 1 and Theorem 2.1 we can give the Apéry set for $\langle m^2-n^2, m^2+n^2, 2mn\rangle$.

Lemma 2.2. *Let* $S = \langle m^2-n^2, m^2+n^2, 2mn\rangle$*, then* $\mathrm{Ap}(S, 2mn) = \{a(m^2+n^2)+b(m^2-n^2),$ $0 \leq a \leq (m-1)$ *and* $0 \leq b \leq (n-1)$ *or* $0 \leq a \leq n-1$ *and* $n \leq b \leq m+n-1\}$ *and*

$$g(S) = \frac{m^3-n^3+1}{2} + m^2n - m^2 - mn.$$

A numerical semigroup S is symmetric, respectively pseudo-symmetric, if $T(S) = \{F(S)\}$, respectively $T(S) = \{F(S), \frac{F(S)}{2}\}$. For $\langle m^2-n^2, m^2+n^2, 2mn\rangle$

$$2 \cdot PF(S) - F(S) > (m-3)\cdot(m^2+n^2),$$

by Theorem 2.1's expressions, a Pythagorean triplet semigroup is not free nor symmetric and it is Arf and pseudo-symmetric if $m = 2 = n+1$.

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