Objects generated by an arbitrary natural number

Krassimir Atanassov

Department of Bioinformatics and Mathematical Modelling,
Institute of Biophysics and Biomedical Engineering,
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8000, Bulgaria
e-mail: krate@bas.bg

Received: 4 June 2020  Revised: 18 November 2020  Accepted: 20 November 2020

Abstract: The set \( \text{Set}(n) \), generated by an arbitrary natural number \( n \), is defined. Some arithmetic functions, defined over its elements are introduced. Some of the arithmetic, set-theoretical and algebraic properties of the new objects are studied.

Keywords: Algebraic objects, Arithmetic functions, Natural numbers, Sets.

2010 Mathematics Subject Classification: 11A25.

1 Introduction

In the present research, some new mathematical objects will be described. They are generated by a fixed arbitrary natural number \( n > 1 \). Let everywhere below it have the canonical form

\[
 n = \prod_{i=1}^{k} p_i^{\alpha_i},
\]

where \( k, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1 \) are natural numbers and \( p_1, p_2, \ldots, p_k \) are different prime numbers.

In [1], the following notations related to \( n \) that we will use below, are introduced:

\[
 \text{set}(n) = \{p_1, p_2, \ldots, p_k\},
\]

\[
 \text{mult}(n) = \prod_{i=1}^{k} p_i.
\]

We will show that these new objects have properties specific to algebra.
2 Main definitions

For the fixed $n \geq 2$, let us define the set

$$\text{Set}(n) = \{m| m = \prod_{i=1}^{k} p_i^{\beta_i} \& h(n) \leq \beta_i \leq H(n)\},$$

where

$$h(n) = \min(\alpha_1, \ldots, \alpha_k),$$

$$H(n) = \max(\alpha_1, \ldots, \alpha_k)$$

and let $\omega(n) = k$.

For example, $\text{Set}(12) = \text{Set}(2^2 \cdot 3) = \{6, 12, 18, 36\}$ and $h(12) = 1, H(12) = 2, \omega(12) = 2$.

$\text{Set}(72) = \text{Set}(2^3 \cdot 3^2) = \{36, 72, 108, 216\}$ and $h(72) = 2, H(72) = 3, \omega(72) = 2$.

It is suitable to define

$$\text{Set}(0) = \{0\}.$$

$$\text{Set}(1) = \{1\}.$$

Therefore, for each natural number $n$, $\text{Set}(n) \neq \emptyset$.

3 Properties of $\text{Set}(n)$

We see immediately that for $n$ being a prime number, and more generally, if $n = \text{mult}(n)$ and hence, $h(n) = H(n)$, then

$$\text{Set}(n) = \{n\}.$$

**Theorem 1.** For the natural number $n$ the cardinality $|\text{Set}(n)|$ of $\text{Set}(n)$ is equal to

$$|\text{Set}(n)| = (H(n) - h(n) + 1)^{\omega(n)}.$$

**Proof.** For $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, $\text{Set}(n)$ will contain all natural numbers $m$ with $\text{mult}(m) = \text{mult}(n)$ and with powers between $h(n)$ and $H(n)$. Therefore, each $p_i$ will be met with $H(n) - h(n) + 1$ different degrees and this is valid for each of the $\omega(n)$ in number divisors of $n$. Hence, the number of all elements of $\text{Set}(n)$ is exactly $(H(n) - h(n) + 1)^{\omega(n)}$. 

For example,

$$|\text{Set}(24)| = |\text{Set}(2^3 \cdot 3)|$$

$$= |\{2, 3, 2^2 \cdot 3, 2^3 \cdot 3, 2 \cdot 3^2, 2^2 \cdot 3^2, 2^3 \cdot 3^2, 2 \cdot 3^3, 2^2 \cdot 3^3, 2^3 \cdot 3^3\}|$$

$$= 9 = (3 - 1 + 1)^2,$$

$$|\text{Set}(36)| = |\text{Set}(2^2 \cdot 3^2)| = |\{2^2 \cdot 3^2\}| = 1 = (2 - 2 + 1)^2,$$

$$|\text{Set}(60)| = |\text{Set}(2^2 \cdot 3 \cdot 5)|$$

$$= |\{2 \cdot 3, 2^2 \cdot 3, 5, 2 \cdot 3^2, 5, 2^3 \cdot 3, 5, 2 \cdot 5, 2^2 \cdot 3 \cdot 5\}|$$

$$= 8 = (2 - 1 + 1)^3.$$
Theorem 2. For two natural numbers $m$ and $n$, if $m$ is a divisor of $n$, $h(m) = h(n)$ and $\text{set}(m) = \text{set}(n)$, then

$$\text{set}(m) \subseteq \text{set}(n).$$

Proof. Having in mind that $m$ is a divisor of $n$, we see that $H(m) \leq H(n)$.

Let $t \in \text{set}(m)$. Therefore, $t = \prod_{i=1}^{k} p_i^\gamma_i$, where $h(m) \leq \gamma_i \leq H(m)$ for each $i = 1, \ldots, k$.

Hence,

$$h(n) = h(m) \leq \gamma_i \leq H(m) \leq H(n),$$

i.e., $t \in \text{set}(n)$. \hfill \Box

It is important to note that without one of the conditions $h(m) = h(n)$ and $\text{set}(m) = \text{set}(n)$, the Theorem is not valid. For example, 6 is a divisor of 72 and $\text{set}(6) = \text{set}(72) = \{2, 3\}$, but $\text{set}(6) = \{6\}$, while $\text{set}(72)$ mentioned above, does not contain the element 6.

On the other hand, 6 is a divisor of 30 and $h(6) = h(30) = 1$, but $\text{set}(30) = \{30\}$ and hence $\text{set}(6) \not\subseteq \text{set}(30)$.

For the well-known operations “Greatest Common Divisor” and “Least Common Multiple” over two natural numbers $m$ and $n$ that are marked by $(m, n)$ and $[m, n]$, respectively, the following equalities are valid.

Theorem 3. For two natural numbers $m$ and $n$ so that $\text{set}(m) = \text{set}(n)$:

$$\text{set}(m) \cap \text{set}(n) \subseteq \text{set}((m, n)), \quad (1)$$

$$\text{set}(m) \cup \text{set}(n) \supseteq \text{set}([m, n]). \quad (2)$$

Proof. Let $t \in \text{set}(m) \cap \text{set}(n)$. Therefore, $t = \prod_{i=1}^{k} p_i^\gamma_i$, where $\gamma_1, \ldots, \gamma_k \geq 1$ are natural numbers. From the fact that $t \in \text{set}(m)$ it follows that $h(m) \leq \gamma_i \leq H(m)$ and from the fact that $t \in \text{set}(n)$ it follows that $h(n) \leq \gamma_i \leq H(n)$ for $i = 1, \ldots, k$. Therefore

$$\max(h(m), h(n)) \leq \gamma_i \leq \min(H(m), H(n)).$$

Obviously, when $\max(h(m), h(n)) > \min(H(m), H(n))$, the number $t$ does not exist. For example,

$$\text{set}(6) \cap \text{set}(36) = \{6\} \cap \{36\} = \emptyset.$$

Therefore, (1) is valid.

Having in mind that

$$(m, n) = \prod_{i=1}^{k} p_i^\min(\alpha_i, \beta_i),$$

for $\text{set}((m, n))$ we see that

$$\text{set}((m, n)) = \{u | u = \prod_{i=1}^{k} p_i^{\delta_i} \& \min(h(m), \varepsilon(n)) \leq \delta_i \leq \min(H(m), H(n))\}.$$ 

Hence, when $\max(h(m), h(n)) \leq \min(H(m), H(n))$, for $t$ it is valid that

$$\min(h(m), h(n)) \leq \max(h(m), h(n)) \leq \gamma_i \leq \min(H(m), H(n)),$$

i.e., $t \in \text{set}((m, n))$. 

59
In the opposite case, if $t \in \text{Set}((m, n))$, then
\[
\min(h(m), h(n)) \leq \gamma_i \leq \min(H(m), H(n)).
\]

If $h(m) \leq h(n)$, then it will be certain that $t \in \text{Set}(m)$, but only if $h(n) \leq \gamma_i$ for each $i = 1, \ldots, k$, then $t \in \text{Set}(n)$ and therefore, $t \in \text{Set}(m) \cap \text{Set}(n)$.

Hence (1) is valid. The validity of (2) is proved in the same manner.

\[\square\]

4 Algebraic objects generated by an arbitrary natural number

Let us define for the fixed $n$:
\[
\bullet n = (\text{mult}(n))^{h(n)},
\bullet n = (\text{mult}(n))^{H(n)},
\]
and for each $m \in \text{Set}(n)$:
\[
\neg m = \prod_{i=1}^{k} p_i^{H(n)+h(n)-\beta_i}.
\]

We see immediately, that $\bullet n, \bullet n \in \text{Set}(n)$, and for each $m \in \text{Set}(n)$: $\neg m \in \text{Set}(n)$. Moreover,
\[
\neg m = \frac{\text{mult}(n)^{H(n)+h(n)}}{m} = \frac{\bullet n}{\bullet n}.
\]

Therefore
\[
\neg \bullet n = \bullet n,
\neg \bullet n = \bullet n.
\]

Following [3], we will mention that if $S$ is a fixed set with unit element $e_S$ and if $*$ is an operation over $S$, then $\langle S, *, e_S \rangle$ is a commutative monoid, if:

1. $(\forall u, v \in S)(u * v \in S)$,
2. $(\forall u, v, w \in S)(u * (v * w) = (u * v) * w)$,
3. $(\forall a \in S)(u * e_S = u = e_S * u)$,
4. $(\forall u, v \in S)(u * v = v * u)$.

Now, we prove the following theorem.

**Theorem 4.** For the fixed $n$:

(a) $\langle \text{Set}(n), (.), \bullet n \rangle$,

(b) $\langle \text{Set}(n), [,], \bullet n \rangle$

are commutative monoids.
Proof. Let \( n \) be fixed. To see the validity of (a), we check sequentially the following equalities.

Let \( u, v, w \in \text{Set}(n) \). Therefore,

\[
\begin{align*}
    u &= \prod_{i=1}^{k} p_i^{\beta_i}, \\
    v &= \prod_{i=1}^{k} p_i^{\gamma_i}, \\
    w &= \prod_{i=1}^{k} p_i^{\delta_i},
\end{align*}
\]

where for each \( i = 1, 2, \ldots, k \):

\( h(n) \leq \beta_i, \gamma_i, \delta_i \leq H(n) \). Hence,

\[
(u, v) = \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)}
\]

and from \( h(n) \leq \min(\beta_i, \gamma_i) \leq H(n) \) it follows that \( (u, v) \in \text{Set}(n) \).

\[
\begin{align*}
    (u, (v, w)) &= \left( \prod_{i=1}^{k} p_i^{\beta_i}, \left( \prod_{i=1}^{k} p_i^{\gamma_i}, \prod_{i=1}^{k} p_i^{\delta_i} \right) \right) \\
    &= \left( \prod_{i=1}^{k} p_i^{\beta_i}, \prod_{i=1}^{k} p_i^{\min(\gamma_i, \delta_i)} \right) \\
    &= \prod_{i=1}^{k} p_i^{\min(\beta_i, \min(\gamma_i, \delta_i))} = \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i, \delta_i)} = \prod_{i=1}^{k} p_i^{\min(\min(\beta_i, \gamma_i), \delta_i)} \\
    &= \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)} \prod_{i=1}^{k} p_i^{\delta_i} \\
    &= \left( \left( \prod_{i=1}^{k} p_i^{\beta_i}, \prod_{i=1}^{k} p_i^{\gamma_i} \right), \prod_{i=1}^{k} p_i^{\delta_i} \right) \\
    &= \left( (u, v), w \right).
\end{align*}
\]

\[
\begin{align*}
    (u, \boxdot(n)) &= \prod_{i=1}^{k} p_i^{\min(\beta_i, H(n))} = \prod_{i=1}^{k} p_i^{\beta_i} = u = \prod_{i=1}^{k} p_i^{\min(H(n), \beta_i)} = (\boxdot(n), u). \\
    (u, v) &= \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)} = \prod_{i=1}^{k} p_i^{\min(\gamma_i, \beta_i)} = (v, u).
\end{align*}
\]

The validity of the second assertion is proved in the same manner. \( \square \)

In [2], the author introduced the following concepts.

We call \( \langle M, *, e_*, e_o \rangle \) a “(commutative) multi unitary group” (shortly, \((c-)*\)-group) if and only if \( e_0 \in M \), \( \langle M, *, e_* \rangle \) is a (commutative) monoid and

\[
(\forall a \in M)(\exists e_o \in M)(a \ast a_o = e_o = a_o \ast a).
\]

Two \((c-)*\)-groups \( MG_1 \) and \( MG_2 \) are dual, if and only if they have the forms

\[
MG_1 = \langle M, *, e_*, e_o \rangle \quad \text{and} \quad MG_2 = \langle M, o, e_o, e_* \rangle
\]

for some given operations \( * \) and \( o \), and for the unitary elements \( e_*, e_o \in M \).

**Theorem 5.** For the fixed natural number \( n \)

\[
\langle \text{Set}(n), \langle ., \rangle, \square n, \square n \rangle \quad \text{and} \quad \langle \text{Set}(n), [], \square n, \square n \rangle
\]

are dual \((c-)*\)-groups.
Proof. From Theorem 4 we saw that \(\langle \text{Set}(n), (.), \boxdot n \rangle\) and \(\langle \text{Set}(n), [,], \boxdot n \rangle\) are commutative monoids. Now, we see that for arbitrary \(u \in \text{Set}(n)\):

\[
(u, \boxdot n) = \boxdot n = (\boxdot n, n)
\]

and

\[
[u, \boxdot n] = \boxdot n = [\boxdot n, n],
\]

i.e., condition (3) is satisfied and hence \(\langle \text{Set}(n), (.), \boxdot n, \boxdot n \rangle\) and \(\langle \text{Set}(n), [,], \boxdot n, \boxdot n \rangle\) are dual (c-)\(\mu\)-groups. □

5 Conclusion

In a next research, other properties of the introduced here objects will be discussed. In addition, we will show that these objects have properties specific for modal logic.

References

