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A refinement of the $3x + 1$ conjecture

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Abstract: The $3x + 1$ conjecture pertains to iteration of the function T defined by $T(x) = x/2$ if x is even and $T(x) = (3x + 1)/2$ if x is odd. The conjecture asserts that the trajectory of every positive integer eventually reaches the cycle $(2, 1)$. We show that the essential dynamics of T-trajectories can be more clearly understood by restricting attention to numbers congruent to 2 (mod 3). This approach leads to an equivalent conjecture for an underlying function T_R whose iterates eliminate many extraneous features of T-trajectories. We show that the function T_R that governs the refined conjecture has particularly simple mapping properties in terms of partitions of the set of integers, properties that have no parallel in the classical formulation of the conjecture. We then use those properties to obtain a new characterization of T-trajectories and we show that the dynamics of the $3x + 1$ problem can be reduced to an iteration involving only numbers congruent to 2 or 8 (mod 9).
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1 Introduction

The $3x+1$ problem pertains to iteration of the following function defined on the set $\mathbb{Z}$ of integers:

$$T(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \equiv 0 \pmod 2, \\ \dfrac{3n+1}{2}, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

We adopt the notation $T^0(n) = n$ and $T^{k+1}(n) = T(T^k(n))$ for $k = 0, 1, 2, \ldots$. We refer to the sequence of iterates $(T^k(n))_{k \geq 0}$ as the T-trajectory of n, with similar terminology for other functions.

The $3x + 1$ conjecture states that, for every positive integer n, the T-trajectory of n eventually reaches the cycle $(2, 1)$. The $3x + 1$ problem, to either prove or disprove this conjecture, remains unsolved after decades of attention and continues to be a problem of great interest. Lagarias [1] provides a comprehensive survey of the problem, with extensive references and full text of some of the historically most important related articles.

In what follows, we show that the standard formulation of the conjecture in terms of T inherently involves extraneous features that obscure important properties of T-trajectories. We present a modified form of the conjecture that eliminates these unnecessary features and, as a consequence, reveals interesting new structure in the underlying dynamics.

Following some preliminary observations, we derive the refined conjecture for a modified function T_R in Section 2. In Section 3 we explore advantages of the new form of the conjecture. An immediate consequence is that the refined conjecture involves shorter, smoother trajectories. More significantly, however, the modified function has special mapping properties that are particularly easy to describe and which have no known equivalent in the traditional form of the conjecture. We extend those properties in Section 4 and then express the results back in terms of T, showing that T-trajectories are dominated by numbers congruent to 2 or 8 (mod 9). We then construct an accelerated iteration that effectively involves only those numbers.

2 Refining the $3x + 1$ iteration

We begin with two simple facts that follow from the definition of T. Our first proposition is not new, but is important as an initial illustration of how the classical $3x + 1$ conjecture incorporates spurious features that mask the essential dynamics of the problem.

Proposition 2.1. *The T-trajectory of any nonzero integer n consists of finitely many numbers congruent to* 0 (mod 3) *followed only by numbers congruent to* 1 *or* 2 (mod 3).

Proof. We may write any nonzero n in the form $n = m2^k$, with m odd and $k \geq 0$. The first k iterates of n are then obtained by divisions by 2, so these numbers are either all congruent to 0 (mod 3) or all not congruent to 0 (mod 3). The k^{th} iterate is m, which is odd. But no iterate of an odd number is divisible by 3, since $3n + 1$ is never divisible by 3 and such divisibility is not affected by subsequent divisions by 2. $\square$

Numbers divisible by 3 are therefore transient elements of the T-trajectory of any nonzero n, appearing at most as initial terms of such a trajectory. For the purpose of exploring long-term behavior of trajectories, it is then beneficial to ignore these transient numbers and restrict our attention only to numbers congruent to 1 or 2 (mod 3).

Proposition 2.2. *If* $n \equiv 1$ (mod 3)*, then* $T(n) \equiv 2$ (mod 3).

Proof. If $n \equiv 1$ (mod 3), then n is of the form $6j + 1$ or $6j + 4$. But $T(6j + 1) = 9j + 2$ and $T(6j + 4) = 3j + 2$, each of which is congruent to 2 (mod 3). $\square$

We refer to numbers congruent to 1 (mod 3) as *isolated* since no two such numbers appear consecutively in a T-trajectory. This is a term we will use again in subsequent sections.

Proposition 2.2 suggests that, in addition to disregarding the transient numbers divisible by 3, further simplification of the dynamics may be obtained by advancing the iteration one step past numbers congruent to 1 (mod 3). This requires the additional observation that the only predecessor of $3n+1$ in a T-trajectory is the even number $6n+2$, since any odd number $2j+1$ iterates to $3j+2 \notin \{3n+1\}$. For brevity, we also introduce the notation $[j]_m$ to denote the set of integers congruent to j (mod m).

Definition 2.3. *Let $F : [2]_3 \to [2]_3$ be defined by*

$$F(n) = \begin{cases} T(n), & \textit{if } n \equiv 5 \pmod 6 \\ T^2(n), & \textit{if } n \equiv 2 \pmod 6. \end{cases} \tag{1}$$

For $n \in [2]_3$, the F-trajectory of n is then the same as the T-trajectory of n with numbers congruent to 1 (mod 3) removed.

Lemma 2.4. *The $3x+1$ conjecture is true if and only if the F-trajectory of any positive integer in $[2]_3$ converges to 2.*

Proof. Suppose the $3x+1$ conjecture holds, so that the T-trajectory of any positive $n \in [2]_3$ eventually reaches 2. The F-trajectory of n therefore also reaches 2, which is a fixed point of F.

Conversely, suppose the F-trajectory of any positive integer in $[2]_3$ reaches 2. If n is an arbitrary positive integer, then by Propositions 2.1 and 2.2, there is some k such that $T^k(n) \in [2]_3$. Since the F-trajectory of $T^k(n)$ consists of further iterations by T, the T-trajectory of $T^k(n)$ eventually reaches 2, which says that the T-trajectory of n reaches 2. $\square$

Having expressed the conjecture in terms only of integers congruent to 2 (mod 3), it is now possible to convert once again to an iteration on all of $\mathbb{Z}$ by means of a conjugacy with the function $S(x) = 3x+2$. As we show below, this leads to the following reformulation of the $3x+1$ conjecture.

Conjecture 2.5 (The refined $3x+1$ conjecture)**.** *Let $T_R : \mathbb{Z} \to \mathbb{Z}$ be defined by*

$$T_R(n) = \begin{cases} \dfrac{3n}{4}, & \textit{if } n \equiv 0 \pmod 4, \\ \dfrac{n-2}{4}, & \textit{if } n \equiv 2 \pmod 4, \\ \dfrac{3n+1}{2}, & \textit{if } n \equiv 1 \pmod 2. \end{cases} \tag{2}$$

Then for every integer $n \geq 0$, the T_R-trajectory of n converges to 0.

Theorem 2.6. *The $3x+1$ conjecture is true if and only if the refined $3x+1$ conjecture is true.*

Proof. Let $S : \mathbb{Z} \to [2]_3$ be defined by $S(n) = 3n+2$, so $S^{-1}(n) = \frac{n-2}{3}$. Define $G : \mathbb{Z} \to \mathbb{Z}$ by

$$G(n) = (S^{-1} \circ F \circ S)(n).$$

Then the G-trajectory of every $n \geq 0$ converges to 0 if and only if the F-trajectory of every positive $S(n) \in [2]_3$ converges to $S(0) = 2$. By Lemma 2.4, this, in turn, is equivalent to the $3x+1$ conjecture since every positive element of $[2]_3$ is equal to $S(n)$ for some $n \geq 0$. We will show that $G = T_R$ as defined above.

First, if $n \equiv 0 \pmod 4$, $n = 4j$, then

$$\begin{aligned} G(n) &= S^{-1}\left(F(3(4j)+2)\right) \\ &= S^{-1}\left(F(12j+2)\right) \\ &= S^{-1}\left(T^2(12j+2)\right) \\ &= S^{-1}(9j+2) \\ &= 3j \\ &= \frac{3n}{4}. \end{aligned}$$

Next, if $n \equiv 2 \pmod 4$, $n = 4j+2$, then

$$\begin{aligned} G(n) &= S^{-1}\left(F(3(4j+2)+2)\right) \\ &= S^{-1}\left(F(12j+8)\right) \\ &= S^{-1}\left(T^2(12j+8)\right) \\ &= S^{-1}(3j+2) \\ &= j \\ &= \frac{n-2}{4}. \end{aligned}$$

Finally, if $n \equiv 1 \pmod 2$, $n = 2j+1$, then

$$\begin{aligned} G(n) &= S^{-1}\left(F(3(2j+1)+2)\right) \\ &= S^{-1}\left(F(6j+5)\right) \\ &= S^{-1}\left(T(6j+5)\right) \\ &= S^{-1}(9j+8) \\ &= 3j+2 \\ &= \frac{3n+1}{2}. \end{aligned}$$

□

3 Features of the refined $3x+1$ conjecture

By expressing the $3x+1$ conjecture in terms of T_R instead of T, we have effectively filtered out from T-trajectories the transient numbers divisible by 3 and the isolated numbers that are congruent to 1 (mod 3). The following result for T_R will be used later, and provides an interesting analogue to Proposition 2.2 for T.

Proposition 3.1. *If* $n \equiv 1$ (mod 3) , *then* $T_R(n) \not\equiv 1$ (mod 3).

Proof. If $n \equiv 1$ (mod 3), then n has the form $6j+1$, $12j+4$, or $12j+10$. But $T_R(6j+1) = 9j+2 \equiv$ 2 (mod 3), $T_R(12j+4) = 9j+3 \equiv 0$ (mod 3), and $T_R(12j+10) = 3j+2 \equiv 2$ (mod 3). □

By Proposition 3.1, numbers congruent to 1 (mod 3) are isolated terms in T_R-trajectories just as they are in T-trajectories.

We now explore additional features of the refined mapping T_R. In the subsequent section, we will use these results to obtain new information about T-trajectories.

3.1 Smoothing of trajectories

From the proof of Theorem 2.6, we have

$$F(n) = \left(S \circ T_R \circ S^{-1}\right)(n), \text{ for } n \equiv 2 \text{ (mod 3)}.$$

By iterating, it follows that

$$S\left(T_R^k(n)\right) = F^k\left(S(n)\right), \text{ for } k = 0, 1, 2, \ldots, \text{ and } n \in \mathbb{Z}. \tag{3}$$

So elements of a T_R-trajectory $(n, T_R(n), \ldots)$ are mapped by S to $(S(n), F(S(n)), \ldots)$. By the definition of F, these are the elements of the T-trajectory of $S(n)$ that are congruent to 2 (mod 3).

Example 3.2. *Figure 1 illustrates graphically, for* $n = 65$, *both the standard* T*-trajectory and the corresponding* F*-trajectory that is related to* T_R *by conjugation with* S. *The terms of the* T*-trajectory, with the elements of the* F*-trajectory in bold, are*

$$\left(T^k(65)\right)_{k>0} = (\mathbf{65}, \mathbf{98}, 49, \mathbf{74}, 37, \mathbf{56}, 28, \mathbf{14}, 7, \mathbf{11}, \mathbf{17}, \mathbf{26}, 13, \mathbf{20}, 10, \mathbf{5}, \mathbf{8}, 4, \mathbf{2}, 1, \mathbf{2}, \ldots).$$

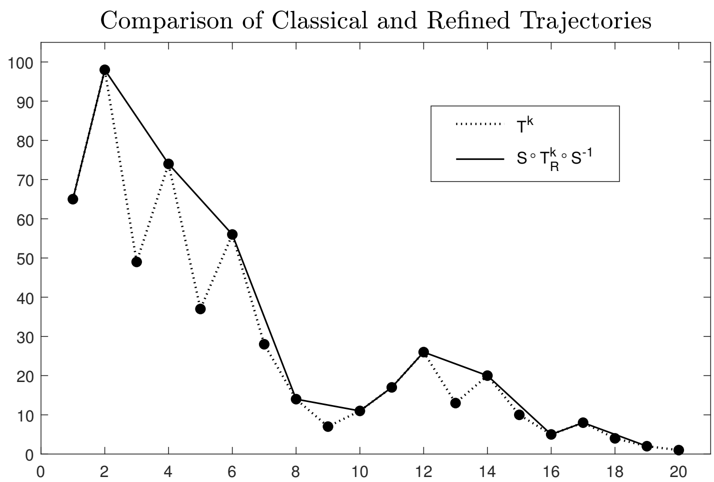

Fig. 1. The classical trajectory (dotted) and the refined trajectory (solid) for $n = 65$.

The corresponding T_R-trajectory is

$$\left(T_R^k(21)\right)_{k>0} = (21, 32, 24, 18, 4, 3, 5, 8, 6, 1, 2, 0, 0, \ldots).$$

The refined F-trajectory is shorter and smoother than the classical T-trajectory, eliminating much oscillatory behavior. The removed numbers are congruent to 1 (mod 3) and so are either of the form $6j + 1$ with predecessor $12j + 2$ and successor $9j + 2$, or the form $6j + 4$ with predecessor $12j + 8$ and successor $3j + 2$. In either case the central number is less than the mean of its predecessor and successor, so the removed portions of the graph of the T-trajectory are always below the graph of the refined trajectory.

3.2 Mapping of a 5-partition to a 3-partition

We now show that the refined conjecture and associated function T_R reveal new information about $3x + 1$ dynamics. Consider partitioning the integers into sets of numbers congruent to 1 (mod 2), 0 (mod 4) or 2 (mod 4). The latter set can be further partitioned into sets of numbers congruent to $2, 6,$ or 10 (mod 12). This is useful since

$$\begin{aligned} T_R(4j) = T_R(12j+2) &= 3j, \\ T_R(12j+6) &= 3j+1, \\ T_R(12j+10) = T_R(2j+1) &= 3j+2. \end{aligned} \tag{4}$$

We will find it convenient to rewrite such relationships using the notation

$$[b]_a \xrightarrow{f} [d]_c$$

to indicate that $f(aj + b) = cj + d$ for every integer j. The label on the arrow may be dropped when the function f is clear from context.

With this notation, the results expressed in Equations (4) are represented in Figure 2. We see that each of five partition subsets of $\mathbb{Z}$ is mapped element-wise to one of the congruence classes of 3. This easily described structure has no analogue in the classical problem, and it holds promise of making the underlying dynamics of the $3x + 1$ conjecture amenable to further analysis. We describe one consequence of this structure in the next subsection.

$$\begin{tikzcd} \mathbb{Z} & = & {[0]_4} \arrow[dr] & \cup & {[2]_{12}} \arrow[dl] & \cup & {[6]_{12}} \arrow[d] & \cup & {[10]_{12}} \arrow[dr] & \cup & {[1]_2} \arrow[dl] & \\ & & & {[0]_3} & & \cup & {[1]_3} & \cup & & {[2]_3} & = & \mathbb{Z} \end{tikzcd}$$

Fig. 2. A schematic of the refined $3x + 1$ mapping T_R.

3.3 Symmetries in T_R-iterates

The mapping property of T_R as shown in Figure 2 suggests that further insights may be obtained by iterating the congruence classes of 3 to themselves. We summarize the result of such a process in this section. Our analysis rests upon the following useful partition:

$$
\begin{aligned}
\mathbb{Z} &= [1]_2 \cup [0]_2 \\
&= [1]_2 \cup [0]_4 \cup [2]_4 \\
&= [1]_2 \cup [0]_4 \cup [6]_8 \cup [2]_8 \\
&= [1]_2 \cup [0]_4 \cup [6]_8 \cup [2]_{16} \cup [10]_{16} \\
&= [1]_2 \cup [0]_4 \cup [6]_8 \cup [2]_{16} \cup [26]_{32} \cup [10]_{32} \\
&= [1]_2 \cup [0]_4 \cup [6]_8 \cup [2]_{16} \cup [26]_{32} \cup [10]_{64} \cup [42]_{64}. \qquad (5)
\end{aligned}
$$

We now advance the T_R iteration according to the congruence classes in Equation (5). This gives an accelerated iteration with special properties.

Define $T_R^* : \mathbb{Z} \to \mathbb{Z}$ by

$$
T_R^*(n) = \begin{cases} T_R(n), & \text{if } n \equiv 1 \pmod 2 \text{ or } n \equiv 0 \pmod 4 \\ T_R^2(n), & \text{if } n \equiv 6 \pmod 8 \text{ or } n \equiv 2 \pmod{16} \\ T_R^3(n), & \text{if } n \equiv 26 \pmod{32}, n \equiv 10 \pmod{64} \text{ or } n \equiv 42 \pmod{64}. \end{cases}
$$

In order to better describe the mapping properties of T_R^*, we first reorder the partition of Equation (5) as follows:

$$
\mathbb{Z} \ = \ [0]_4 \ \cup \ [2]_{16} \ \cup \ [10]_{64} \ \cup \ [42]_{64} \ \cup \ [26]_{32} \ \cup \ [6]_8 \ \cup \ [1]_2
$$

We then partition each of these seven subsets into congruence classes (mod 3), to obtain

$$
\begin{aligned}
[0]_3 &= [0]_{12} \cup [18]_{48} \cup [138]_{192} \cup [42]_{192} \cup [90]_{96} \cup [6]_{24} \cup [3]_6 \\
[1]_3 &= [4]_{12} \cup [34]_{48} \cup [10]_{192} \cup [106]_{192} \cup [58]_{96} \cup [22]_{24} \cup [1]_6 \\
[2]_3 &= [8]_{12} \cup [2]_{48} \cup [74]_{192} \cup [170]_{192} \cup [26]_{96} \cup [14]_{24} \cup [5]_6
\end{aligned}
$$

Theorem 3.3. *The function $T_R^* : \mathbb{Z} \to \mathbb{Z}$ maps the congruence classes of 3 as shown in the diagram below:*

$$
\begin{array}{ccccccccccccccc}
[0]_3 = & [0]_{12} & \cup & [18]_{48} & \cup & [138]_{192} & \cup & [42]_{192} & \cup & [90]_{96} & \cup & [6]_{24} & \cup & [3]_6 \\
 & T_R\big\downarrow & & T_R^2\big\downarrow & & T_R^3\big\downarrow & & T_R^3\big\downarrow & & T_R^3\big\downarrow & & T_R^2\big\downarrow & & T_R\big\downarrow \\
 & [0]_9 & \cup & [3]_9 & \cup & [6]_9 & \cup & [0]_3 & \cup & [8]_9 & \cup & [2]_9 & \cup & [5]_9
\end{array} \qquad (6a)
$$

$$
\underbrace{[0]_9 \cup [3]_9 \cup [6]_9}_{[0]_3} \cup [0]_3 \cup \underbrace{[8]_9 \cup [2]_9 \cup [5]_9}_{[2]_3}
$$

$$
\begin{array}{ccccccccccccccc}
[1]_3 = & [4]_{12} & \cup & [34]_{48} & \cup & [10]_{192} & \cup & [106]_{192} & \cup & [58]_{96} & \cup & [22]_{24} & \cup & [1]_6 \\
 & T_R\big\downarrow & & T_R^2\big\downarrow & & T_R^3\big\downarrow & & T_R^3\big\downarrow & & T_R^3\big\downarrow & & T_R^2\big\downarrow & & T_R\big\downarrow \\
 & [3]_9 & \cup & [6]_9 & \cup & [0]_9 & \cup & [1]_3 & \cup & [5]_9 & \cup & [8]_9 & \cup & [2]_9
\end{array} \qquad (6b)
$$

$$
\underbrace{[3]_9 \cup [6]_9 \cup [0]_9}_{[0]_3} \cup [1]_3 \cup \underbrace{[5]_9 \cup [8]_9 \cup [2]_9}_{[2]_3}
$$

$$
\begin{array}{cccccccccccccc}
[2]_3 = & [8]_{12} & \cup & [2]_{48} & \cup & [74]_{192} & \cup & [170]_{192} & \cup & [26]_{96} & \cup & [14]_{24} & \cup & [5]_6 \\
& T_R \downarrow & & T_R^2 \downarrow & & T_R^3 \downarrow & & T_R^3 \downarrow & & T_R^3 \downarrow & & T_R^2 \downarrow & & T_R \downarrow \\
& \multicolumn{5}{c}{\underbrace{[6]_9 \cup [0]_9 \cup [3]_9}_{[0]_3}} & \cup & [2]_3 & \cup & \multicolumn{5}{c}{\underbrace{[2]_9 \cup [5]_9 \cup [8]_9}_{[2]_3}}
\end{array}
\tag{6c}
$$

Proof. The proof follows by verification of the individual mappings using the definition of T_R from Equation (2). As an example, to verify the third entry in (6c), we have

$$
\begin{aligned}
T_R^3(192j + 74) &= T_R^2(48j + 18) \\
&= T_R(12j + 4) \\
&= 9j + 3.
\end{aligned}
$$
□

While we can express T_R^* as a piecewise function[1], the relationships displayed in (6a)–(6c) reveal interesting symmetries that are not apparent from a formula. For example, the mapping sends the first three subsets of each partition to the three sets $[0]_9$, $[3]_9$, and $[6]_9$, permuted cyclically. A similar pattern appears for the last three subsets in each group.

Mappings (6a) and (6c) also reflect a striking symmetry between the sets $[0]_3$ and $[2]_3$ that does not involve the set $[1]_3$. In fact, the next theorem shows that numbers congruent to $1 \pmod 3$ are transient in T_R^*-trajectories in the same way that numbers congruent to $0 \pmod 3$ are transient in T-trajectories, as expressed in Proposition 2.1.

Theorem 3.4. *Every T_R^*-trajectory consists of finitely many numbers congruent to* $1 \pmod 3$ *followed only by numbers congruent to either* 0 *or* $2 \pmod 3$.

Proof. By (6a) and (6c), T_R^* maps a number in $[0]_3$ or $[2]_3$ to a number in one of those same two sets. So we need only show that every number in $[1]_3$ eventually iterates out of $[1]_3$. From (6b), if $n \in [1]_3$ and $T_R^*(n) \in [1]_3$, then $n = 192j + 106$ for some integer j. In that case, $T_R^*(n) = T_R^3(n) = (n - 42)/64$, from which $|T_R^*(n)| \leq |\frac{43n}{64}| < |n|$. Any sequence of T_R^*-iterates contained in $[1]_3$ therefore has decreasing magnitudes and so must have an element m of least absolute value. Then $T_R^*(m)$ is in $[0]_3$ or $[2]_3$. □

4 Reversion to T-trajectories

We have shown that the refined $3x + 1$ conjecture (Conjecture 2.5) and the special properties of T_R reveal patterns that are not easily discerned from the standard formulation of the conjecture. While the fundamental dynamics are more clearly revealed by T_R instead of T, it may also be of interest to transform results about T_R-trajectories back into the original setting of T-trajectories. We do so in this section, beginning with a new characterization of T-trajectories.

[1]$T_R^*(n) = (3n + 1)/2$ if $n \equiv 1 \pmod 2$, $3n/4$ if $n \equiv 0 \pmod 4$, $(3n - 2)/8$, if $n \equiv 6 \pmod 8$, $(3n - 6)/16$ if $n \equiv 2 \pmod{16}$, $(3n - 14)/32$ if $n \equiv 26 \pmod{32}$, $(3n - 30)/64$ if $n \equiv 10 \pmod{64}$, $(n - 42)/64$ if $n \equiv 42 \pmod{64}$.

Theorem 4.1. *The T-trajectory of every nonzero integer n has the following structure: finitely many numbers congruent to* $0 \pmod 3$*, followed by numbers congruent to* 2 *or* $8 \pmod 9$ *except for isolated numbers (i.e., no two in succession) that are congruent to* $1 \pmod 3$ *or isolated numbers congruent to* $5 \pmod 9$*.*

Proof. From Proposition 2.1, numbers congruent to $0 \pmod 3$ occur in the T-trajectory of a nonzero integer only as initial values, and there are finitely many such numbers. From Proposition 2.2, numbers congruent to $1 \pmod 3$ are isolated in the subsequent portion of the trajectory. The remaining numbers are those congruent to $2 \pmod 3$. We then need only show that numbers congruent to $5 \pmod 9$ are isolated within that set. But by Proposition 3.1, numbers of the form $3j + 1$ never appear consecutively in a T_R-trajectory, which implies that numbers of the form $S(3j+1) = 9j+5$ never appear consecutively in a T-trajectory. This says that numbers congruent to $5 \pmod 9$ are isolated in the T-trajectory. $\square$

Example 4.2. *The structure described in Theorem 4.1 is illustrated in Figure 3, for the trajectory of* 156*.*

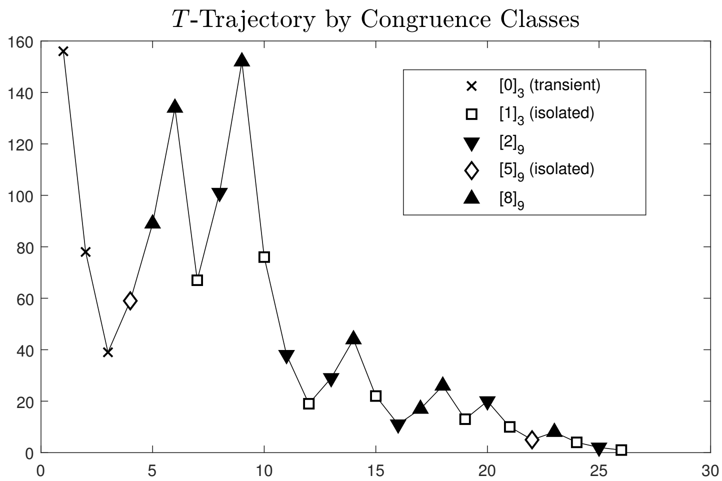

Fig. 3. The T-trajectory of 156, showing the properties stated in Theorem 4.1.

We can demonstrate even more clearly the distinctive role in T-trajectories of numbers congruent to 2 or $8 \pmod 9$ by the following result for an accelerated iteration whose long-term behavior involves only those numbers. As in Equation (5), we use a special partition, this time for the set $[2]_3$. We also express this partition with residues of least absolute value in order to more clearly show the pattern.

$$\begin{aligned}
[2]_3 &= [5]_6 \cup [2]_6 \\
&= [5]_6 \cup [2]_{12} \cup [8]_{12} \\
&= [5]_6 \cup [2]_{12} \cup [20]_{24} \cup [8]_{24} \\
&= [5]_6 \cup [2]_{12} \cup [20]_{24} \cup [8]_{48} \cup [32]_{48} \\
&= [5]_6 \cup [2]_{12} \cup [20]_{24} \cup [8]_{48} \cup [80]_{96} \cup [32]_{96} \\
&= [5]_6 \cup [2]_{12} \cup [20]_{24} \cup [8]_{48} \cup [80]_{96} \cup [32]_{192} \cup [128]_{192} \\
&= [-1]_6 \cup [2]_{12} \cup [-4]_{24} \cup [8]_{48} \cup [-16]_{96} \cup [32]_{192} \cup [-64]_{192}
\end{aligned}$$

Theorem 4.3. *Define* $T^* : [2]_3 \to \mathbb{Z}$ *by*

$$T^*(n) = \begin{cases} T(n) \;= (3n+1)/2, & \textit{if } n \equiv -1 \pmod 6 \\ T^2(n) = (3n+2)/4, & \textit{if } n \equiv 2 \pmod{12} \\ T^3(n) = (3n+4)/8, & \textit{if } n \equiv -4 \pmod{24} \\ T^4(n) = (3n+8)/16, & \textit{if } n \equiv 8 \pmod{48} \\ T^5(n) = (3n+16)/32, & \textit{if } n \equiv -16 \pmod{96} \\ T^6(n) = (3n+32)/64, & \textit{if } n \equiv 32 \pmod{192} \\ T^6(n) = \;n/64, & \textit{if } n \equiv -64 \pmod{192}. \end{cases}$$

Then

(i) the range of T^* *is* $[2]_3$*,*

(ii) every T^**-trajectory consists of finitely many numbers congruent to* $5 \pmod 9$ *followed only by numbers congruent to either* 2 *or* $8 \pmod 9$.

Proof. By applying $S(x) = 3x + 2$ to the sets shown in (6a)–(6c) and making use of Equations (3) and (1), we obtain the following mapping of numbers congruent to $2 \pmod 3$, listed by congruence class (mod 9). The mappings shown in (7a)–(7c) coincide with the definition of T^*, as may be verified directly from the definition of T. Result *(i)* is then a consequence of (7a)–(7c), and *(ii)* follows by applying S to the numbers described in Theorem 3.4.

$$\begin{gathered}
\begin{array}{rccccccccccccc}
[2]_9 = & [2]_{36} & \cup & [56]_{144} & \cup & [416]_{576} & \cup & [128]_{576} & \cup & [272]_{288} & \cup & [20]_{72} & \cup & [11]_{18} \\
& T^2\big\downarrow & & T^4\big\downarrow & & T^6\big\downarrow & & T^6\big\downarrow & & T^5\big\downarrow & & T^3\big\downarrow & & T\big\downarrow
\end{array} \\
\underbrace{[2]_{27} \cup [11]_{27} \cup [20]_{27}}_{[2]_9} \cup [2]_9 \cup \underbrace{[26]_{27} \cup [8]_{27} \cup [17]_{27}}_{[8]_9}
\end{gathered} \tag{7a}$$

$$\begin{gathered}
\begin{array}{rccccccccccccc}
[5]_9 = & [14]_{36} & \cup & [104]_{144} & \cup & [32]_{576} & \cup & [320]_{576} & \cup & [176]_{288} & \cup & [68]_{72} & \cup & [5]_{18} \\
& T^2\big\downarrow & & T^4\big\downarrow & & T^6\big\downarrow & & T^6\big\downarrow & & T^5\big\downarrow & & T^3\big\downarrow & & T\big\downarrow
\end{array} \\
\underbrace{[11]_{27} \cup [20]_{27} \cup [2]_{27}}_{[2]_9} \cup [5]_9 \cup \underbrace{[17]_{27} \cup [26]_{27} \cup [8]_{27}}_{[8]_9}
\end{gathered} \tag{7b}$$

$$\begin{gathered}
\begin{array}{rccccccccccccc}
[8]_9 = & [26]_{36} & \cup & [8]_{144} & \cup & [224]_{576} & \cup & [512]_{576} & \cup & [80]_{288} & \cup & [44]_{72} & \cup & [17]_{18} \\
& T^2\big\downarrow & & T^4\big\downarrow & & T^6\big\downarrow & & T^6\big\downarrow & & T^5\big\downarrow & & T^3\big\downarrow & & T\big\downarrow
\end{array} \\
\underbrace{[20]_{27} \cup [2]_{27} \cup [11]_{27}}_{[2]_9} \cup [8]_9 \cup \underbrace{[8]_{27} \cup [17]_{27} \cup [26]_{27}}_{[8]_9}
\end{gathered} \tag{7c}$$

$\square$

Example 4.4. *To illustrate Theorem (4.3), we consider again the* T*-trajectory of* 156*, which was shown in Figure 3. That trajectory is*

$$(156, 78, 39, 59, 89, 134, 67, 101, 152, 76, 38, 19, 29, 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, \ldots)\,.$$

The first three terms are elements of the transient set $[0]_3$*, so the* T^**-trajectory begins with* 59 *and is as follows:*

$$(59, 89, 134, 101, 152, 29, 44, 17, 26, 20, 8, 2, \ldots)\,.$$

Note that the first term is congruent to 5 *(mod* 9*) and all subsequent terms are congruent to either* 2 *or* 8 *(mod* 9*). However, not all terms of a* T*-trajectory that are congruent to* 2 *or* 8 *(mod* 9*) appear in the accelerated* T^**-trajectory. In this example,* 38 *and* 11 *are two such numbers that are bypassed in the iteration by* T^**.*

5 Summary

The classical formulation of the $3x + 1$ problem states that the trajectory of any positive integer under iteration by a particular function T eventually reaches the limit cycle $(2, 1)$. We have constructed a streamlined version of the conjecture that involves only numbers congruent to 2 (mod 3). This eliminates many extraneous features of trajectories. We have then recast the $3x + 1$ problem as a conjecture involving a modified function T_R for which the trajectory of any positive integer converges to 0 instead of to a limit cycle. By iterating the action of T_R and expressing the result back in terms of T, we have shown that the essential dynamics of the $3x + 1$ problem can effectively be reduced to a process involving only numbers congruent to 2 or 8 (mod 9).

References

[1] J. C. Lagarias, Editor, *The Ultimate Challenge: The* $3x + 1$ *Problem*, Amer. Math. Soc., 2010.