Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 3, 224–233
DOI: 10.7546/nntdm.2020.26.3.224-233

On the Padovan p-circulant numbers

Güzel İpek[1], Ömür Deveci[2] and Anthony G. Shannon[3]

[1] Department of Mathematics, Faculty of Science and Letters
Kafkas University, 36100 Kars, Turkey
e-mail: guzelipek_36@hotmail.com

[2] Department of Mathematics, Faculty of Science and Letters
Kafkas University, 36100 Kars, Turkey
e-mail: odeveci36@hotmail.com

[3] Warrane College, University of New South Wales
Kensington, NSW, 2033,Australia
e-mail: tshannon38@gmail.com

Received: 18 February 2019 **Revised:** 20 July 2020 **Accepted:** 24 July 2020

Abstract: In this paper, we define Padovan p-circulant numbers by using circulant matrices which are obtained from the characteristic polynomials of the Padovan p-numbers. Then, we derive the permanental and the determinantal representations of the Padovan p-circulant numbers by using certain matrices which are obtained from the generating matrix of Padovan p-circulant sequence. Also, we obtain the combinatorial representation, the exponential representation and the sums of the Padovan p-circulant numbers by the aid of the generating function and the generating matrix of the Padovan p-circulant sequence.
Keywords: Padovan p-circulant Sequence, Matrix, Representation.
2010 Mathematics Subject Classification: 11B50, 11C20, 20D60.

1 Introduction

It is well-known that Padovan sequence is defined by the following equation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$, where $P(0) = P(1) = P(2) = 1$.

Deveci and Karaduman defined [8] the Padovan p-numbers as shown:

$$pap\,(n+p+2) = pap\,(n+p) + pap\,(n)$$

for any given $p\,(p = 2, 3, 4, \ldots)$ and $n \geq 1$ with initial conditions $pap\,(1) = pap\,(2) = \cdots = pap\,(p) = 0$, $pap\,(p+1) = 1$ and $pap\,(p+2) = 0$.

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-3} & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geqslant 0$.

Many of the numbers obtained by using homogeneous linear recurrence relations and their miscellaneous properties have been studied by some authors; see for example [1,3,5–7,9–11,13, 18–20]. In this paper, we define Padovan p-circulant numbers and then we obtain their some properties such as the generating matrix, the Binet formula, the combinatorial representation, the generating function, exponential representation.

2 The Padovan p-circulant numbers

We define the Padovan p-circulant numbers for $n \geq 1$ as follows:

$$x_{n+p+3} = x_{n+p+2} - x_{n+p} - x_n \tag{1}$$

with initial constants $x_1 = \cdots = x_{p+2} = 0$ and $x_{p+3} = 1$, where $p \geq 2$.

It is important to note that equation (1) is a $(p+3)$-th order homogeneous linear recurrence relation example of the arbitrary-order equation (1.1) in [16].

By equation (1), we can write the following companion matrix:

$$C_p = [c_{ij}]_{(p+3)\times(p+3)} = \begin{bmatrix} 1 & 0 & -1 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix C_p is called the Padovan p-circulant matrix. It is easy to see that

$$(C_p)^\alpha \begin{bmatrix} x_{p+3} \\ x_{p+2} \\ \vdots \\ x_1 \end{bmatrix} = \begin{bmatrix} x_{\alpha+p+3} \\ x_{\alpha+p+2} \\ \vdots \\ x_{\alpha+1} \end{bmatrix}$$

for $\alpha \geq 0$. Also, by an inductive argument, we may write

$$(C_p)^\alpha = \begin{bmatrix} x_{\alpha+p+3} & x_{\alpha+p+4} - x_{\alpha+p+3} & x_{\alpha+p+5} - x_{\alpha+p+4} & -x_{\alpha+3} & -x_{\alpha+4} & \cdots & -x_{\alpha+p+2} \\ x_{\alpha+p+2} & x_{\alpha+p+3} - x_{\alpha+p+2} & x_{\alpha+p+4} - x_{\alpha+p+3} & -x_{\alpha+2} & -x_{\alpha+3} & \cdots & -x_{\alpha+p+1} \\ x_{\alpha+p+1} & x_{\alpha+p+2} - x_{\alpha+p+1} & x_{\alpha+p+3} - x_{\alpha+p+2} & -x_{\alpha+1} & -x_{\alpha+2} & \cdots & -x_{\alpha+p} \\ x_{\alpha+p} & x_{\alpha+p+1} - x_{\alpha+p} & x_{\alpha+p+2} - x_{\alpha+p+1} & -x_{\alpha} & -x_{\alpha+1} & \cdots & -x_{\alpha+p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{\alpha+2} & x_{\alpha+3} - x_{\alpha+2} & x_{\alpha+4} - x_{\alpha+3} & -x_{\alpha-p+2} & -x_{\alpha-p+3} & \cdots & -x_{\alpha+1} \\ x_{\alpha+1} & x_{\alpha+2} - x_{\alpha+1} & x_{\alpha+3} - x_{\alpha+2} & -x_{\alpha-p+1} & -x_{\alpha-p+2} & \cdots & -x_{\alpha} \end{bmatrix} \quad (2)$$

for $\alpha \geq p$. we easily derive that $\det (C_p)^\alpha = (-1)^{p\alpha+\alpha}$.

Now we concentrate on the Binet formula for the Padovan p-circulant numbers by the aid of the determinantal representation.

Lemma 2.1. *The characteristic equation of the Padovan p-circulant sequence $x^{p+3} - x^{p+2} + x^p + 1 = 0$ does not have multiple roots.*

Proof. There is a similar proof in [8]. Let $f(x) = x^{p+3} - x^{p+2} + x^p + 1$ and suppose that z is a multiple root of $f(x)$. Then, since z is a multiple root, z is a root of $f'(x)$, that is, $f(z) = z^{p+3} - z^{p+2} + z^p + 1 = 0$ and $f'(z) = (p+3) z^{p+2} - (p+2) z^{p+1} + p z^{p-1} = z^{p-1} ((p+3) z^3 - (p+2) z^2 + p) = 0$. Since $f(0) \neq 0$, we consider the equations $(p+3) z^3 - (p+2) z^2 + p = 0$. Thus we obtain

$$z_1 = \frac{\left(\sqrt[3]{2}(-p-2)^2\right)}{3(p+3)\left(-25p^3-150p^2+3\sqrt{3}\sqrt{23p^6+276p^5+1230p^4+2380p^3+1563p^2-288p}-219p+16\right)^{\frac{1}{3}}} + \frac{\left(-25p^3-150p^2+3\sqrt{3}\sqrt{23p^6+276p^5+1230p^4+2380p^3+1563p^2-288p}-219p+16\right)^{\frac{1}{3}}}{3\sqrt[3]{2}(p+3)} - \frac{-p-2}{3(p+3)}$$

$$z_2 = -\frac{\left(1+i\sqrt{3}\right)(-p-2)^2}{3\times 2^{\frac{2}{3}}(p+3)\left(-25p^3-150p^2+3\sqrt{3}\sqrt{23p^6+276p^5+1230p^4+2380p^3+1563p^2-288p}-219p+16\right)^{\frac{1}{3}}} - \frac{\left(1-i\sqrt{3}\right)\left(-25p^3-150p^2+3\sqrt{3}\sqrt{23p^6+276p^5+1230p^4+2380p^3+1563p^2-288p}-219p+16\right)^{\frac{1}{3}}}{6\sqrt[3]{2}(p+3)} - \frac{-p-2}{3(p+3)}$$

and

$z_3 = -\frac{\left(1-i\sqrt{3}\right)(-p-2)^2}{3\times 2^{\frac{2}{3}}(p+3)\left(-25p^3-150p^2+3\sqrt{3}\sqrt{23p^6+276p^5+1230p^4+2380p^3+1563p^2-288p}-219p+16\right)^{\frac{1}{3}}} - \frac{\left(1+i\sqrt{3}\right)\left(-25p^3-150p^2+3\sqrt{3}\sqrt{23p^6+276p^5+1230p^4+2380p^3+1563p^2-288p}-219p+16\right)^{\frac{1}{3}}}{6\sqrt[3]{2}(p+3)} - \frac{-p-2}{3(p+3)}.$

For $p \geq 2$, $f(z_1) \neq 0$, $f(z_2) \neq 0$ and $f(z_3) \neq 0$, which is contradiction. Thus, the equation $f(x) = 0$ does not have multiple roots. □

If $x_1, x_2, \ldots, x_{p+3}$ are roots of the equation $x^{p+3} - x^{p+2} + x^p + 1$, then by Lemma 2.1, it is known that $x_1, x_2, \ldots, x_{p+3}$ are distinct. Let V^p be a $(p+3) \times (p+3)$ Vandermonde matrix as follows:

$$V^p = \begin{bmatrix} (x_1)^{p+2} & (x_2)^{p+2} & \cdots & (x_{p+3})^{p+2} \\ (x_1)^{p+1} & (x_2)^{p+1} & \cdots & (x_{p+3})^{p+1} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_{p+3} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Let

$$W_i^p = \begin{bmatrix} (x_1)^{\alpha+p+3-i} \\ (x_2)^{\alpha+p+3-i} \\ \vdots \\ (x_{p+3})^{\alpha+p+3-i} \end{bmatrix}$$

and suppose that $V_{i,j}^p$ is a $(p+3) \times (p+3)$ matrix obtained from V^p by replacing the j-th column of V^p by W_i^p.

Theorem 2.2. *Let $(C_p)^\alpha = \left[c_{i,j}^{p,\alpha}\right]$, then $c_{i,j}^{p,\alpha} = \frac{\det V_{i,j}^p}{\det V^p}$, for $\alpha \geq p$ and $p \geq 2$.*

Proof. Since the eigenvalues of the matrix C_p, x_1, x_2, …, x_{p+3} are distinct, the matrix C_p is diagonalizable. Let $G_p = diag\,(x_1, x_2, \ldots x_{p+3})$, then it is easy to see that $C_p V^p = V^p G_p$. Since $\det V^p \neq 0$, the matrix V^p is invertible. Then we obtain $(V^p)^{-1} C_p V^p = G_p$. Thus, the matrix C_p is similar to G_p. So we get $(C_p)^\alpha V^p = V^p (G_p)^\alpha$ for $\alpha \geq p$ and $p \geq 2$. Then we can write the following linear system of equations:

$$\begin{cases} c_{i,1}^{p,\alpha}(x_1)^{p+2} + c_{i,2}^{p,\alpha}(x_1)^{p+1} + \cdots + c_{i,p+3}^{p,\alpha} = (x_1)^{\alpha+p+3-i} \\ c_{i,1}^{p,\alpha}(x_2)^{p+2} + c_{i,2}^{p,\alpha}(x_2)^{p+1} + \cdots + c_{i,p+3}^{p,\alpha} = (x_2)^{\alpha+p+3-i} \\ \vdots \\ c_{i,1}^{p,\alpha}(x_{p+3})^{p+2} + c_{i,2}^{p,\alpha}(x_{p+3})^{p+1} + \cdots + c_{i,p+3}^{p,\alpha} = (x_{p+3})^{\alpha+p+3-i} \end{cases}$$

for $\alpha \geq p$ and $p \geq 2$. Therefore, for each $i, j = 1, 2, \ldots, p+3$, we obtain $c_{i,j}^{p,\alpha}$ as follows

$$c_{i,j}^{p,\alpha} = \frac{\det V_{i,j}^p}{\det V^p}.$$ □

So we have the following useful results.

Corollary 2.3. *Let x_α be the αth the Padovan p-circulant number for $p \geq 2$. Then $x_\alpha = -\frac{\det V_{n,n}^p}{\det V^p}$ for $4 \leq n \leq p+3$.*

Now we consider the permanental representations of the Padovan p-circulant numbers.

Definition 2.1. *Let $M = [m_{i,j}]$ be $u \times v$ real matrix and let $r^1, r^2, \dots, r^u$ and $c^1, c^2, \dots, c^v$ be respectively, the row and column vectors of M. If r^α contains exactly two non-zero entries, then M is contractible on row α. Similarly, M is contractible on column β provided c^β contains exactly two non-zero entries.*

Let $x_1, x_2, \dots, x_u$ be row vectors of the matrix M and let M be contractible in the α-th column with $m_{i,\alpha} \neq 0, m_{j,\alpha} \neq 0$ and $i \neq j$. Then the $(u-1) \times (v-1)$ matrix $M_{ij:\alpha}$ obtained from M by replacing the i-th row with $m_{i,\alpha} x_j + m_{j,\alpha} x_i$ and deleting the j-th row and the α-th column is called the contraction in the α-th column relative to the i-th row and the j-th row.

In [2], Brualdi and Gibson obtained that $per\,(M) = per\,(N)$ if M is a real matrix of order $u > 1$ and N is a contraction of M.

Let $k \geq p+3$ be a positive integer and suppose that $G\,(k,p) = \left[g_{i,j}^{k,p}\right]$ is the $k \times k$ super-diagonal matrix, defined by

$$g_{i,j}^{k,p} = \begin{cases} 1, & \text{if } i = t \text{ and } j = t \text{ for } 1 \leq t \leq k \text{ and } i = t+1 \text{ and } j = t \text{ for } 1 \leq t \leq k-1, \\ -1, & \text{if } i = t \text{ and } j = t+2 \text{ for } 1 \leq t \leq k-2 \text{ and } i = t \\ & \qquad \text{and } j = t+p+2 \text{ for } 1 \leq t \leq k-p-2, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$G\,(k,p) = \begin{array}{c} \qquad\qquad (p+3)\,th \\ \qquad\qquad \downarrow \\ \begin{bmatrix} 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Theorem 2.4. *For $k \geqslant p+3$ and $p \geq 2$,*

$$perG\,(k,p) = x_{k+p+3}.$$

Proof. We will use the induction method on k. Suppose that the equation holds $k \geqslant p+3$, then we show that the equation holds for $k+1$. If we expand the $perG\,(k,p)$ by the Laplace expansion of permanent according to the first row, then we obtain

$$perG\,(k+1,p) = perG\,(k,p) - perG\,(k-2,p) - perG\,(k-p-2,p)\,.$$

Since $perG(k,p) = x_{k+p+3}$, $perG(k-2,p) = x_{k+p+1}$ and $perG(k-p-2,p) = x_{k+1}$, we easily obtain that $perG(k+1,p) = x_{k+p+4}$. $\square$

Let $k \geqslant p+3$ and let $Y(k,p) = \left[y_{i,j}^{k,p}\right]$ be the $k \times k$ matrix, defined by

$$y_{i,j}^{k,p} = \begin{cases} 1 & \begin{array}{c} \text{if } i=t \text{ and } j=t \text{ for } 1 \leq t \leq k-p-2 \\ \text{and} \\ i=t+1 \text{ and } j=t \text{ for } 1 \leq t \leq k-1, \end{array} \\ -1 & \begin{array}{c} \text{if } i=t \text{ and } j=t+2 \text{ for } 1 \leq t \leq k-p-2 \\ \text{and} \\ i=t \text{ and } j=t+p+2 \text{ for } 1 \leq t \leq k-p-2, \end{array} \\ 0 & \text{otherwise.} \end{cases}.$$

Now we define the $k \times k$ matrix $L(k,p) = \left[l_{i,j}^{k,p}\right]$ as follows:

$$\begin{array}{cc} & \begin{array}{c} (k-p-3)\,th \\ \downarrow \end{array} \\ L(k,p) = & \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & Y(k-1,p) & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix} \end{array}$$

where $k > p+3$.

Then we can give other permanental representations than the above.

Theorem 2.5. (i). *For* $k \geq p+3$,

$$perY(k,p) = -x_k.$$

(ii). *For* $k > p+3$,

$$perL(k,p) = -\sum_{i=1}^{k-1} x_i.$$

Proof. (i) .Suppose that the equation holds for $k \geq p+3$, then we show that the equation holds for $k+1$. If we expand the $perY(k,p)$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\begin{aligned} perY(k+1,p) &= perY(k,p) - perY(k-2,p) - perY(k-p-2,p) \\ &= -x_k + x_{k-2} + x_{k-p-2} = -x_{k+1}. \end{aligned}$$

The conclusion is obtained.

(ii). If we extend the $perL(k,p)$ with respect to the first row, we write

$$perL(k,p) = perL(k-1,p) + perY(k-1,p).$$

By the results of Theorem 2.4 and Theorem 2.5. (i) and the inductive argument, the proof is easily seen. $\square$

A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Let $k > p+3$ and let R be the $k \times k$ matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

It is easy to see that $perG(k,p) = \det(G(k,p) \circ R)$, $perY(k,p) = \det(Y(k,p) \circ R)$ and $perL(k,p) = \det(L(k,p) \circ R)$. Then we have the following useful results.

Corollary 2.6. *For $k > p+3$,*

$$\det(G(k,p) \circ R) = x_{k+p+3},$$

$$\det(Y(k,p) \circ R) = -x_k$$

and

$$\det(L(k,p) \circ R) = -\sum_{i=1}^{k-1} x_i.$$

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & k_3 & \cdots & k_v \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

See [14, 15] for more information about the companion matrix.

Theorem 2.7. *(Chen and Louck [4]).The (i,j) entry $k_{i,j}^{(u)}(k_1, k_2, \ldots, k_v)$ in the matrix $K^u(k_1, k_2, \ldots, k_v)$ is given by the following formula:*

$$k_{i,j}^{(u)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \tag{3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = u - i + j$, $\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$ is a multinomial coefficient, and the coefficients in (3) are defined to be 1 if $u = i - j$.

Then we have the following Corollary for the Padovan p-circulant numbers.

Corollary 2.8. *Let x_α be the αth Padovan p-circulant number. Then*

$$\begin{aligned} x_\alpha &= -\sum_{(t_1,t_2\ldots,t_{p+3})} \frac{t_4+t_5+\cdots+t_{p+3}}{t_1+t_2+\cdots+t_{p+3}} \times \binom{t_1+\cdots+t_{p+3}}{t_1,\ldots,t_{p+3}} (-1)^{t_3+t_{p+3}} \\ &= -\sum_{(t_1,t_2\ldots,t_{p+3})} \frac{t_5+t_6+\cdots+t_{p+3}}{t_1+t_2+\cdots+t_{p+3}} \times \binom{t_1+\cdots+t_{p+3}}{t_1,\ldots,t_{p+3}} (-1)^{t_3+t_{p+3}} \\ &= \cdots \\ &= -\sum_{(t_1,t_2\ldots,t_{p+3})} \frac{t_{p+3}}{t_1+t_2+\cdots+t_{p+3}} \times \binom{t_1+\cdots+t_{p+3}}{t_1,\ldots,t_{p+3}} (-1)^{t_3+t_{p+3}} \end{aligned}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+3)\,t_{p+3}=\alpha$.

Proof. In Theorem 2.7 , if we choose $v=p+3$ and $i=j$ such that $4\le i,j\le p+3$, then the proof is immediately seen from (2). □

The generating function of the Padovan p-circulant numbers is given by

$$g^p(y)=\frac{y^{p+2}}{1-y+y^3+y^{p+3}},$$

where $p\ge 2$.

Note that the generating function $g^p(y)$ is, in effect, a generalization of the main result in Section 2 of [17].

Now we give an exponential representation for the Padovan p-circulant numbers with the following Theorems.

Theorem 2.9. *The Padovan p-circulant numbers have the following exponential representation:*

$$g^p(y)=y^{p+2}\exp\left(\sum_{i=1}^{\infty}\frac{(y)^i}{i}\left(1-y^2+y^{p+2}\right)^i\right).$$

Proof. Since

$$\begin{aligned} \ln g^p(y) &= \ln\frac{y^{p+2}}{1-y+y^3+y^{p+3}} \\ &= \ln y^{p+2}-\ln\left(1-y+y^3+y^{p+3}\right) \end{aligned}$$

and

$$\begin{aligned} -\ln\left(1-y+y^3+y^{p+3}\right) &= -\Big[-y\left(1-y^2-y^{p+2}\right)-\frac{1}{2}y^2\left(1-y^2-y^{p+2}\right)^2-\cdots \\ &\quad -\frac{1}{n}y^n\left(1-y^2-y^{p+2}\right)^n-\cdots\Big], \end{aligned}$$

it is clear that

$$\ln\frac{g^p(y)}{y^{p+2}}=\sum_{i=1}^{\infty}\frac{(y)^i}{i}\left(1-y^2+y^{p+2}\right)^i.$$

Thus we have the conclusion. □

Now we give the sums of the Padovan p-circulant numbers.

Let $S_\alpha = \sum_{n=1}^{\alpha} x_n$ and suppose that M_p is the $(p+4) \times (p+4)$ matrix such that

$$M_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & C_p & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then it can be shown by induction that

$$(M_p)^\alpha = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{\alpha+p+2} & & & \\ S_{\alpha+p+1} & & (C_p)^\alpha & \\ \vdots & & & \\ S_\alpha & & & \end{bmatrix}.$$

3 Conclusion

We have given Padovan p-circulant numbers. These sequences had defined by using circulant matrices which had obtained from the characteristic polynomials of the Padovan p-numbers. Also, we have given relationships between Padovan p-circulant numbers and the generating matrices of these sequences. Then we have obtained some properties of the Padovan p-circulant numbers such as the Binet formula, permanental, determinantal, combinatorial, exponential representations and we have derived a formula for the sums of the Padovan p-circulant numbers.

Acknowledgements

This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2017-FM-63.

References

[1] Akuzum, Y., Deveci, O., & Shannon A. G. (2017). On the Pell p-circulant sequences, *Notes Number Theory Disc. Math.*, 23 (2), 91–103.

[2] Brualdi, R. A., & Gibson, P. M. (1997). Convex polyhedra of doubly stochastic matrices I: applications of permanent function, *J. Combin. Theory*, 22, 194–230.

[3] Bueno, A. C. F. (2015). On right circulant matrices with general number sequence, *Notes Number Theory Disc. Math.*, 21 (2), 55–58.

[4] Chen, W. Y. C., & Louck, J. C. (1996). The combinatorial power of the companion matrix, *Linear Algebra Appl.*, 232, 261–278.

[5] Deveci, O. (2016). The Pell-Circulant sequences and their applications, *Maejo Int. J. Sci. Technol.*, 10 (03), 284–293.

[6] Deveci, O. (2018). The Padovan-circulant sequences and their applications, *Math. Reports*, 20 (70), 401–416.

[7] Deveci, O., Akuzum, Y., & Karaduman, E. (2015). The Pell–Padovan p-sequences and its applications, *Util. Math.*, 98, 327–347.

[8] Deveci, O., & Karaduman, E. (2017). On the Padovan p-numbers, *Hacettepe J. Math. Stat.*, 46 (4), 579–592.

[9] Deveci, O., Karaduman, E., & Campbell, C. M. (2017). The Fibonacci-circulant sequences and their applications, *Iranian J. Sci. Technol. Trans. A-Sci.*, 1–6.

[10] Deveci, O., & Shannon, A. G. (2017). Pell–Padovan-circulant sequences and their applications, *Notes Number Theory Disc. Math.*, 23 (3), 100–114.

[11] Gogin, N. D., & Myllari, A. A. (2007). The Fibonacci–Padovan sequence and MacWilliams transform matrices, *Program. Comput. Softw., published in Programmirovanie*, 33 (2), 74–79.

[12] Kalman, D. (1982). Generalized Fibonacci numbers by matrix methods, *Fibonacci Quart.*, 20 (1), 73–76.

[13] Kilic, E. (2009). The generalized Pell (p, i)-numbers and their Binet formulas, combinatorial representations, sums, *Chaos, Solitons Fractals*, 40 (4), 2047–2063.

[14] Lancaster, P., & Tismenesky, M. (1985). *The Theory of Matrices*, Academic Press.

[15] Lidl, R., & Niederreiter, H. (1986). *Introduction to finite fields and their applications*, Cambridge U. P.

[16] Shannon, A. G. (1974). Some properties of a fundamental recursive sequence of arbitrary order, *Fibonacci Quart.*, 12 (4), 327–335.

[17] Shannon, A. G., & Horadam, A. F. (1971). Generating functions for powers of third order recurrence sequences, *Duke Math. J.*, 38 (4), 791–794.

[18] Stakhov, A. P., & Rozin, B. (2006). Theory of Binet formulas for Fibonacci and Lucas p-numbers, *Chaos Solitons Fractals*, 27, 1162–1167.

[19] Tasci, D., & Firengiz, M. C. (2010). Incomplete Fibonacci and Lucas p-numbers, *Math. Comput. Modelling.*, 52, 1763–1770.

[20] Tuglu, N., Kocer, E. G., & Stakhov, A. P. (2011). Bivariate Fibonacci-like p-polynomials, *Appl. Math. Comput.*, 217 (24), 10239–10246.