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Identities on generalized Fibonacci and Lucas numbers

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Abstract: In this article, the concepts of Fibonacci, Tribonacci, Lucas and Tetranacci numbers are generalized as continued sum. The generalized Fibonacci identity is proved by using induction and the binomial theorem. Further, it is proved that the generalized Fibonacci and Lucas sequences are logarithmically convex (concave) and some special identities are obtained.
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1 Introduction

A sequence is an arrangement of objects or a set of numbers in a particular order. Let $\{u_n\}$ be any sequence of terms $u_0, u_1, u_2, \ldots$ in which the n-th and $(n+1)$-st terms of the sequence are respectively denoted by u_n and u_{n+1}. If $u_{n+1} > (<)\ u_n$, then sequence $\{u_n\}$ is monotonically increasing (decreasing) [12, 16].

Fibonacci sequence: The sequence of numbers of the form $\{0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8, \ldots\}$ is called the Fibonacci sequence. The numbers 0, 1, 1, 2, $\ldots$ are called Fibonacci numbers. In 1202, Fibonacci used this concept to study the growth of rabbit population and some interesting results are found in [3,5–7,11]. The general term of the Fibonacci sequence is denoted by F_n and defined as follows.

Definition 1.1. *For all positive integers n,*

$$F_n = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2. \end{cases}$$

with $F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2, \dots$.

In general, any number is obtained by adding previous two terms.

Tribonacci sequence: A Tribonacci sequence is denoted by $\{T_n\}$, which is the generalized Fibonacci sequence. Here $T_0 = 0,\ T_1 = 1,\ T_2 = 1$, the terms T_3 onwards are obtained by adding previous three terms. That means, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$. The terms of the sequence are $\{0, 1, 1, 2, 4, 7, \dots\}$. The results and identities on Tribonacci numbers are discussed in [1,4,6,7].

Lucas sequence: The Lucas sequence, named after the mathematician François Édouard Anatole Lucas (1842–1891), is closely related to the Fibonacci sequence. The Lucas sequence has the same recursive relationship as the Fibonacci sequence, where each term is the sum of the previous two terms, but with different initial values. The sequence of the form {2, 1, 3, 4, 7, 11, 18, 29, 47, ...} is called Lucas sequence. The numbers 2, 1, 3, 4, ... are called Lucas numbers and some interesting results are found in [7, 11, 15]. The general term of the Lucas sequence is denoted by L_n and defined as follows.

Definition 1.2. *For all positive integers n,*

$$L_n = \begin{cases} 2, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ L_{n-1} + L_{n-2}, & \text{if } n \geq 2. \end{cases}$$

with $L_0 = 2,\ L_1 = 1,\ L_2 = 3,\ L_3 = 4, \dots$.

In general, any number is obtained by adding previous two numbers with the first two numbers are 2 and 1. The following are the identities involving Fibonacci and Lucas numbers [8,9].

$$(-1)^{n+1} = F_{n-1}F_{n+1} - F_n^2 \tag{1.1}$$

$$F_{m+n-1} = F_m F_n + F_{m-1}F_{n-1} \tag{1.2}$$

$$F_{m+n} = F_m F_{n+1} + F_{m-1}F_n = L_n F_n - (-1)^n F_{n-m} \tag{1.3}$$

$$L_{n+3} = F_n + F_{n+1} + F_{n+2} + F_{n+3} \tag{1.4}$$

$$L_n^2 = 5F_n^2 + 4(-1)^n \tag{1.5}$$

$$F_n = \frac{L_{n-1} + L_{n+1}}{5} = \frac{L_n + F_n}{2} \tag{1.6}$$

$$F_{n+m} = \frac{L_m F_n + L_n F_m}{2} \tag{1.7}$$

$$L_{n+m} = L_{m+1}F_n + L_m F_{n-1} = \frac{L_m L_n + 5F_n F_m}{2} = (-1)^m L_{n-m} + 5L_n F_m \tag{1.8}$$

D'Ocagne's identity:

$$F_m F_{n+1} - F_n F_{m+1} = (-1)^n F_{m-n} \tag{1.9}$$

$$F_{-n} = (-1)^{n+1} F_n, \quad \text{for} \quad n = 1, F_{-1} = 1 \tag{1.10}$$

In [2], Zvonko Cerin studied on factors of sums of consecutive Fibonacci and Lucas numbers. The author discovered that the sums $\sum_{j=0}^{4i+3} F_{k+j}$ have the Fibonacci number F_{2i+2} as a common factor, the alternating sums of 20 and 22 consecutive Fibonacci numbers are all respectively divisible by F_{10} and L_{11}. Also, obtained some interesting results on sums of consecutive products, and squares of consecutive numbers. Below are given few identities involving Fibonacci and Lucas numbers [2].

$$\sum_{j=0}^{4i+3} F_{k+j} = F_{2i+2} L_{k+2i+3} \quad \text{and} \quad \sum_{j=0}^{4i+3} (-1)^j F_{k+j} = F_{2i+2} L_{k+2i} \tag{1.11}$$

$$\sum_{j=0}^{4i+1} F_{k+j} = L_{2i+1} F_{k+2i+2} \quad \text{and} \quad \sum_{j=0}^{4i+1} (-1)^j F_{k+j} = L_{2i+1} F_{k+2i-1} \tag{1.12}$$

$$\sum_{j=0}^{4i} F_{k+j} = F_{2i} L_{k+2i} + L_{2i+1} F_{k+2i} \quad \text{and} \quad \sum_{j=0}^{4i} (-1)^j F_{k+j} = F_{k+2i} L_{2i+1} - L_{k+2i} F_{2i} \tag{1.13}$$

and for other identities interested readers may refer [2].

Golden ratio: Let u_n and u_{n+1} be any two consecutive terms of monotonically increasing sequence. If

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.6180339887\ldots,$$

then, the ratio $\frac{u_{n+1}}{u_n}$ is called a Golden ratio and is denoted by the Greek alphabet φ. The value of $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$ and $1 - \varphi = \phi = \frac{1-\sqrt{5}}{2} = -0.6180339887\ldots$. Both φ and ϕ are the solutions of the quadratic equation $x^2 - x - 1 = 0$, a good number of results on Golden ratio are found in [3–5, 11].

Definition 1.3 ([16]). *A sequence $\{u_n\}$ is said to be log-convex if $(u_{n+1})^2 < u_n u_{n+2}$ and is log-concave if $(u_{n+1})^2 > u_n u_{n+2}$ for all n, some interesting results on convexities are found in* [13, 14].

2 Generalization of Fibonacci numbers

From the motivation of the above literature survey, consider a set of new sequences of the form:

$$\begin{aligned}
D_j^0 &= \{0, 1, 1, 2, 3, \ldots\} = F_j, \ \text{ for all } j = 0, 1, 2, \ldots \\
D_j^1 &= \{0+1 = \mathbf{1}, 1+1 = \mathbf{2}, 1+2 = \mathbf{3}, 2+3 = \mathbf{5}, \ldots\} \\
&= F_j + F_{j+1}, \ \text{for all } j = 0, 1, 2, \ldots \\
D_j^2 &= \{0+1+1 =, \mathbf{2}, 1+1+2 = \mathbf{4}, 1+2+3 = \mathbf{6}, \ldots\} \\
&= F_j + F_{j+1} + F_{j+2}, \ \text{for all } j = 0, 1, 2, \ldots \\
D_j^3 &= \{0+1+1+2 = \mathbf{4}, 1+1+2+3 = \mathbf{7}, 1+2+3+5 = \mathbf{11}, \ldots\} \\
&= F_j + F_{j+1} + F_{j+2} + F_{j+3} \\
&= L_{j+3}, \ \text{for all } j = 0, 1, 2, \ldots \quad \text{and so on.}
\end{aligned}$$

The family of sequences $\{D_j^k\}$, $k = 0, 1, 2, \ldots$ is generalized as follows.

Definition 2.1. *For any two positive integers j and k, the generalized Fibonacci sequence $\{D_j^k\}$ is defined as:*

$$D_j^k = F_j + F_{j+1} + F_{j+2} + \ldots + F_{j+k-1} + F_{j+k} = \sum_{i=j}^{j+k} F_i \tag{2.1}$$

where $j = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$.

Obviously the sequence $\{D_j^k\}$ is monotonically increasing sequence and the following identities are obtained.

$$D_{j+2}^k = D_j^k + D_{j+1}^k, \quad \text{for all} \quad j, k = 0, 1, 2, 3, \ldots \tag{2.2}$$

$$D_j^{k+1} - D_j^k = D_{j+k+1}^0, \quad \text{for all} \quad j, k = 0, 1, 2, 3, \ldots \tag{2.3}$$

$$D_j^{k+1} - D_{j+1}^k = F_j, \quad \text{for all} \quad j, k = 0, 1, 2, 3, \ldots \tag{2.4}$$

$$D_j^2 = 2D_j^1, \quad \text{for all} \quad j = 0, 1, 2, 3, \ldots \tag{2.5}$$

The first 10 numbers of the sequence $\{D_j^k\}$ are represented in the form of the table as below.

C	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	$\ldots$
j	D_j^0	D_j^1	D_j^2	D_j^3	D_j^4	D_j^5	D_j^6	D_j^7	$\ldots$
0	0	1	2	4	7	12	20	33	$\ldots$
1	1	2	4	7	12	20	33	54	$\ldots$
2	1	3	6	11	19	32	53	87	$\ldots$
3	2	5	10	18	31	52	86	141	$\ldots$
4	3	8	16	29	50	84	139	228	$\ldots$
5	5	13	26	47	81	136	225	369	$\ldots$
6	8	21	42	76	131	220	364	597	$\ldots$
7	13	34	68	123	212	356	589	966	$\ldots$
8	21	55	110	199	343	576	953	1563	$\ldots$
9	34	89	178	322	555	932	1542	2529	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2.1. The first 10 numbers of the sequence $\{D_j^k\}$

Note: The results studied by Zvonko Cerin [2] correspond to the columns C_3, C_7, C_{11}, C_{15}, ..., C_{4i+3} where $i = 0, 1, 2, 3, ...$

The objective of this article is to develop some identities involving the generalized Fibonacci sequence and to prove the sequence is log-convex (concave) for the set of even (odd) numbers.

3 Main results on the generalized Fibonacci sequence

Theorem 3.1. *If $\{D_j^k\}$ is the generalized Fibonacci sequence, then $\lim_{j\to\infty} \frac{D_{j+1}^k}{D_j^k}$ converges to the Golden ratio.*

Proof. Consider the ratio of two consecutive terms of the generalized Fibonacci sequence and using the identity (2.2) gives

$$\frac{D^k_{j+1}}{D^k_j} = \frac{D^k_{j-1} + D^k_j}{D^k_j} = 1 + \frac{D^k_{j-1}}{D^k_j}. \tag{3.1}$$

Define a quantity $x = \lim_{j\to\infty} \frac{D^k_{j+1}}{D^k_j}$, and its reciprocal $\frac{1}{x} = \lim_{j\to\infty} \frac{D^k_j}{D^k_{j+1}}$.

Also, the limit can be written as $\lim_{j\to\infty} \frac{D^k_{j-1}}{D^k_j}$, therefore $\frac{1}{x} = \lim_{j\to\infty} \frac{D^k_j}{D^k_{j+1}} = \lim_{j\to\infty} \frac{D^k_{j-1}}{D^k_j}$.

As j tends to ∞, equation (3.1) gives,

$$\lim_{j\to\infty} \frac{D^k_{j+1}}{D^k_j} = \lim_{j\to\infty} \left[1 + \frac{D^k_{j-1}}{D^k_j}\right] = 1 + \lim_{j\to\infty} \left[\frac{D^k_{j-1}}{D^k_j}\right]$$

which is equivalently, $x = 1 + \frac{1}{x}$ or $x^2 = x + 1$ or $x^2 - x - 1 = 0$.

The roots of above equation are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, i.e., $x = \lim_{j\to\infty} \frac{D^k_{j+1}}{D^k_j} = \varphi = \frac{1+\sqrt{5}}{2}$, which is the value of the Golden ratio. □

Lemma 3.1. *Let φ be a Golden ratio and $\phi = 1 - \varphi$, then $\varphi^n + \phi^n = 1 - \sum_{k=1}^{n-1} {}^n c_k (\varphi)^{n-k} \phi^k$.*

Proof. The binomial theorem states that for all integer values of *n*,

$$(a+b)^n = \sum_{k=0}^{n} {}^n c_k a^{n-k} b^k = a^n + {}^n c_1 a^{n-1} b + {}^n c_2 a^{n-2} b^2 + {}^n c_3 a^{n-3} b^3 + \ldots + b^n.$$

Consider $a = \left(\frac{1+\sqrt{5}}{2}\right) = \varphi$ and $b = \left(\frac{1-\sqrt{5}}{2}\right) = \phi$, then $a^n = \left(\frac{1+\sqrt{5}}{2}\right)^n = \varphi^n$ and $b^n = \left(\frac{1-\sqrt{5}}{2}\right)^n = \phi^n$.

By binomial theorem,

$$\varphi^n = \frac{1}{2^n}[1 + {}^n c_1 \sqrt{5} + {}^n c_2 5 + {}^n c_3 5\sqrt{5} + \ldots]$$

and

$$\phi^n = \frac{1}{2^n}[1 - {}^n c_1 \sqrt{5} + {}^n c_2 5 - {}^n c_3 5\sqrt{5} + \ldots].$$

Adding gives

$$\varphi^n + \phi^n = \frac{1}{2^{n-1}}[1 + {}^n c_2 (\sqrt{5})^2 + {}^n c_4 (\sqrt{5})^4 + {}^n c_6 (\sqrt{5})^6 + \ldots] = \frac{1}{2^{n-1}} \sum_{r=0}^{n} {}^n c_{2r} (\sqrt{5})^{2r}.$$

Since the binomial theorem holds for all integer values of n, then

$$\varphi^{n+1} + \phi^{n+1} = \frac{1}{2^n}\left[\sum_{r=0}^{n+1} {}^{n+1} c_{2r} (\sqrt{5})^{2r}\right].$$ □

For $j = 0, 1, 2, ...$ and $k = 0, 1, 2, ...$, the main identity of this article is stated as below.

$$\Delta^k_j = (D^k_{j+1})^2 - D^k_j D^k_{j+2} = (-1)^j \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - 1\right] \tag{3.2}$$

By using Lemma 3.1,

$$\Delta_j^k = (D_{j+1}^k)^2 - D_j^k D_{j+2}^k = (-1)^j \left[(-1)^k - 1 + \frac{1}{2^k} \sum_{r=0}^{k+1} {}^{k+1}c_{2r} (\sqrt{5})^{2r} \right] \tag{3.3}$$

Theorem 3.2. *Let $\{D_j^k\}$ be the generalized Fibonacci sequence, then*

$$(D_{j+1}^k)^2 - D_j^k D_{j+2}^k = (-1)^j \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - 1 \right]$$

is log-convex for $j = 0, 2, 4, 6, \ldots$ and log-concave for $j = 1, 3, 5, \ldots$.

Proof. The theorem is proved by induction and the binomial theorem.

<u>Case 1.</u> Let k be fixed and j vary.

Put $k = 0,\ j = 0$, $(D_1^0)^2 - D_0^0 D_2^0 = (-1)^0 \left[(-1)^0 + \frac{1-\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} - 1 \right] = 1.$

Put $k = 0,\ j = 1$, $(D_2^0)^2 - D_1^0 D_3^0 = (-1)^1 \left[(-1)^1 + \frac{1-\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} - 1 \right] = 1.$

So, it holds for $k = 0, j = 0$ and 1.

Assume that the identity (3.2) holds true for fixed k and $j = m$, that is,

$$(D_{m+1}^k)^2 - D_m^k D_{m+2}^k = (-1)^m \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - 1 \right].$$

To prove the identity holds true for fixed k and $j = m+1$, by using the identity (2.2) gives

$$(D_{m+2}^k)^2 - D_{m+1}^k D_{m+3}^k = (D_{m+1}^k + D_m^k)^2 - D_{m+1}^k (D_{m+1}^k + D_{m+2}^k).$$

By simplification of the right-hand side, we have

$$= (D_{m+1}^k)^2 + (D_m^k)^2 + 2D_{m+1}^k D_m^k - (D_{m+1}^k)^2 - D_{m+1}^k D_{m+2}^k$$

$$= (D_m^k)^2 + D_{m+1}^k D_m^k - D_{m+1}^k D_{m+2}^k + D_{m+1}^k D_m^k$$

$$= D_m^k (D_m^k + D_{m+1}^k) - D_{m+1}^k (D_{m+2}^k - D_m^k)$$

$$= -[(D_{m+1}^k)^2 - D_m^k D_{m+2}^k].$$

From equation (3.2)

$$= -(-1)^j \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - 1 \right]$$

$$= (-1)^{j+1} \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - 1 \right].$$

This proves that the identity (3.2) holds true for all positive integer j by induction.

Case 2. Let j be fixed and k vary.

Put $j=0,\ k=0$, $(D_1^0)^2 - D_0^0 D_2^0 = (-1)^0 \left[(-1)^0 + \frac{1-\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} - 1\right] = 1.$

Put $j=0,\ k=1$, $(D_1^1)^2 - D_0^1 D_2^1 = (-1)^0 \left[(-1)^1 + \left(\frac{1-\sqrt{5}}{2}\right)^2 + \left(\frac{1+\sqrt{5}}{2}\right)^2 - 1\right] = 1.$

So, it holds for $j=0, k=0$ and 1.

Assume that the identity (3.2) holds true for fixed j and $k=m$, that is

$$\begin{aligned}(D_{j+1}^m)^2 - D_j^m D_{j+2}^m &= (-1)^j \left[(-1)^m + \left(\frac{1-\sqrt{5}}{2}\right)^{m+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - 1\right] \\ &= (-1)^j \left[(-1)^m - 1 + \frac{1}{2^m}\sum_{r=0}^{m+1} {}^{m+1}c_{2r}(\sqrt{5})^{2r}\right].\end{aligned}$$

To prove that the identity (3.2) holds true for fixed j and $k=m+1$.

Consider the right-hand side of equation (3.3) when $k=m+1$,

$$\begin{aligned}&= (-1)^j \left[(-1)^{m+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{m+2} + \left(\frac{1+\sqrt{5}}{2}\right)^{m+2} - 1\right] \\ &= (-1)^j \left[(-1)^{m+1} - 1 + \frac{1}{2^{m+1}}\sum_{r=0}^{m+2} {}^{m+2}c_{2r}(\sqrt{5})^{2r}\right].\end{aligned}$$

By Lemma 3.1

$$\begin{aligned}&= (-1)^j \left[(-1)^{m+1} - 1 + \varphi^{k+2} + \phi^{k+2}\right] \\ &= (D_{j+1}^{m+1})^2 - D_j^{m+1} D_{j+2}^{m+1}.\end{aligned}$$

This proves that the identity (3.2) holds for all positive integers of k by induction. □

Theorem 3.2 is illustrated for the first 10 values of $j = 0,1,2,\ldots$, $k = 0,1,2,\ldots$ and the values of $(D_{j+1}^k)^2 - D_j^k D_{j+2}^k$ are represented in the form of the table below.

C	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	$\ldots$
j	D_j^0	D_j^1	D_j^2	D_j^3	D_j^4	Δ_j^0	Δ_j^1	Δ_j^2	Δ_j^3	Δ_j^4	$\ldots$
0	0	1	2	4	7	–	–	–	–	–	$\ldots$
1	1	2	4	7	12	−1	−1	−4	−5	−11	$\ldots$
2	1	3	6	11	19	1	1	4	5	11	$\ldots$
3	2	5	10	18	31	−1	−1	−4	−5	−11	$\ldots$
4	3	8	16	29	50	1	1	4	5	11	$\ldots$
5	5	13	26	47	81	−1	−1	−4	−5	−11	$\ldots$
6	8	21	42	76	131	1	1	4	5	11	$\ldots$
7	13	34	68	123	212	−1	−1	−4	−5	−11	$\ldots$
8	21	55	110	199	343	1	1	4	5	11	$\ldots$
9	34	89	178	322	555	−1	−1	−4	−5	−11	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3.1. The first 10 numbers of the sequence represents $\{(D_{j+1}^k)^2 - D_j^k D_{j+2}^k\}$

It is observed that from the above table the values of Δ_j^k are negative (log-convex) for $j = 1, 3, 5, 7, \ldots$ and positive (log-concave) for $j = 2, 4, 6, 8, \ldots$.

Remark 3.1. *If $\{L_n\}$ is the Lucas sequence, then*

1. *L_{n+3} is log-convex if n is even.*
2. *L_{n+3} is log-concave if n is odd.*

Proof. Let L_{n+3} be a Lucas number, then by using equation (1.4) in $L_{n+4}^2 - L_{n+3}L_{n+5}$ equal to:

$= (F_{n+1}+F_{n+2}+F_{n+3}+F_{n+4})^2 - (F_n+F_{n+1}+F_{n+2}+F_{n+3})(F_{n+2}+F_{n+3}+F_{n+4}+F_{n+5})$

$= (2F_{n+3}+F_{n+4})^2 - (2F_{n+2}+F_{n+3})(2F_{n+4}+F_{n+5})$

$= (4F_{n+3}^2 + F_{n+4}^2 + 4F_{n+3}F_{n+4}-) - (4F_{n+2}F_{n+4} + 2F_{n+3}F_{n+4} + 2F_{n+2}F_{n+5} + F_{n+3}F_{n+5})$.

By using identity (1.1),

$= 4(-1)^{n+4} + (-1)^{n+5} + 2(F_{n+3}F_{n+4} - F_{n+2}F_{n+5})$

$= 4(-1)^n - (-1)^n + 2(F_{n+4}F_{n+3} - F_{n+5}F_{n+2})$.

By using identity (1.9),

$= 3(-1)^n + 2(-1)^n$

$= 5(-1)^n$.

$\therefore$ If n is even, then L_{n+3} is log-convex, otherwise it is log-concave. □

The above remark is illustrated by using system software for $1 \le i \le 37$.

n	L_n	$(L_n)^2 - L_{n-1}L_{n+1}$	Result
0	2	–	–
1	1	-5	convexity
2	3	5	concavity
3	4	-5	convexity
4	7	5	concavity
5	11	-5	convexity
6	18	5	concavity
7	29	-5	convexity
8	47	5	concavity
9	76	-5	convexity
⋮	⋮	⋮	⋮

Table 3.2. The first 10 Lucas numbers represent $\{(L_n)^2 - L_{n-1}L_{n+1}\}$

Theorem 3.3. *For all positive integers j and k,* $D_j^k = \dfrac{L_{j+1}F_{k+1} + F_{j+1}L_{k+1} - 2F_{j+1}}{2}$.

Proof. This identity is proved by induction.

Case 1. Put $k=0, j=0, D_0^0=\dfrac{L_1F_1+F_1L_1-2F_1}{2}=0.$

Put $k=0, j=1, D_1^0=\dfrac{L_2F_1+F_2L_1-2F_2}{2}=1.$

So, it holds for $k=0, j=0$ and 1. Assume that identity holds for fixed k and $j=m$.

That is, $D_m^k=\dfrac{L_{m+1}F_{k+1}+F_{m+1}L_{k+1}-2F_{m+1}}{2}.$

To prove the identity holds for fixed k and $j=m+1$, by Definition 2.1 and by using the identity (1.7) gives,

$$
\begin{aligned}
D_{m+1}^k &= D_m^k+F_{m+k+1}-F_m \\
&= \frac{L_{m+1}F_{k+1}+F_{m+1}L_{k+1}-2F_{m+1}}{2}+\frac{L_mF_{k+1}+F_mL_{k+1}}{2}-F_m \\
&= \frac{L_{m+1}F_{k+1}+F_{m+1}L_{k+1}-2F_{m+1}+L_mF_{k+1}+F_mL_{k+1}}{2}-\frac{2F_m}{2}.
\end{aligned}
$$

On simplification,

$$D_{m+1}^k=\frac{L_{m+2}F_{k+1}+F_{m+2}L_{k+1}-2F_{m+2}}{2}$$

It holds for all positive integer j.

Case 2. Put $k=0, j=0, D_0^0=\dfrac{L_1F_1+F_1L_1-2L_1}{2}=0.$

Put $k=1, j=0, D_0^1=\dfrac{L_1F_2+F_1L_2-2F_1}{2}=1.$

So, it holds true for $j=0$ and $k=0$ and 1.

Assume that it holds good for fixed j and $k=m$,

i.e., $D_j^m=\dfrac{L_{j+1}F_{m+1}+F_{j+1}L_{m+1}-2F_{j+1}}{2}.$

To prove the identity holds good for fixed j and $k=m+1$, by Definition 2.1 and by using the identity (1.7) gives,

$$
\begin{aligned}
D_j^{m+1} &= D_j^m+F_{j+m+1} \\
&= \frac{L_{j+1}F_{m+1}+F_{j+1}L_{m+1}-2F_{j+1}}{2}+\frac{L_{j+1}F_m+F_{j+1}L_m}{2} \\
&= \frac{L_{j+1}F_{m+1}+F_{j+1}L_{m+1}-2F_{j+1}+L_{j+1}F_m+F_{j+1}L_m}{2}.
\end{aligned}
$$

On simplification,

$$D_j^{m+1}=\frac{L_{j+1}F_{m+2}+F_{j+1}L_{m+2}-2F_{j+1}}{2}.$$

It holds true for positive integer *k*. Hence the theorem is proved. $\square$

Remark 3.2. *For particular values of* j, $D_j^k = \dfrac{L_{j+1}F_{k+1} + F_{j+1}L_{k+1} - 2F_{j+1}}{2}$ *satisfies the following identities:*

1. $D_0^k = \dfrac{F_{k+1} + L_{k+1} - 2}{2}$

2. $D_1^k = \dfrac{3F_{k+1} + L_{k+1} - 2}{2}$

3. $D_3^k = \dfrac{4F_{k+1} + 2L_{k+1} - 4}{2}$

Remark 3.3. *Let* $\{D_j^k\}$ *be the generalized Fibonacci sequence, then for any positive integer* a *the following identity holds.*

$$D_j^k - D_{j-a}^k D_{j+a}^k = (-1)^{j+a}\left[(-1)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - 1\right](D_a^0)^2.$$

4 Generalization of Lucas numbers

A set of new sequences is defined by using Lucas numbers. Consider a set of new sequences of the form.

$$E_j^0 = \{2, 1, 3, 4, 7, \ldots\} = L_j \text{ for all } j = 0, 1, 2, \ldots .$$

$$\begin{aligned} E_j^1 &= \{2+1=\mathbf{3}, 1+3=\mathbf{4}, 3+4=\mathbf{7}, 4+7=\mathbf{11}, 7+11=\mathbf{18}, \ldots\} \\ &= L_j + L_{j+1} \text{ for all } j = 0, 1, 2, \ldots \end{aligned}$$

$$\begin{aligned} E_j^2 &= \{2{+}1{+}3=\mathbf{6}, 1{+}3{+}4=\mathbf{8}, 3{+}4{+}7=\mathbf{14}, 4{+}7{+}11=\mathbf{22}, 7{+}11{+}18=\mathbf{36}, \ldots\} \\ &= L_j + L_{j+1} + L_{j+2} \text{ for all } j = 0, 1, 2, \ldots \text{ and so on.} \end{aligned}$$

The family of sequences $\{E_j^k\}$, $k = 0, 1, 2, \ldots$ is generalized as follows.

Definition 4.1. *For two positive integers* j *and* k*, the generalized Lucas sequence* $\{E_j^k\}$ *is defined as;*

$$E_j^k = L_j + L_{j+1} + L_{j+2} + \ldots + L_{j+k-1} + L_{j+k} = \sum_{i=j}^{j+k} L_i,$$

where $j = 0, 1, 2, \ldots$ *and* $k = 0, 1, 2, \ldots$.

Obviously, the sequence $\{E_j^k\}$ is a monotonically increasing sequence and the following identities are obtained.

1. $E_{j+2}^k = E_j^k + E_{j+1}^k$

2. $E_j^2 = 2E_j^1$

The first 10 numbers of the sequence $\{E_j^k\}$ are represented in the table below.

C	C_0	C_1	C_2	C_3	C_4	C_5	C_6	$\dots$
j	E_j^0	E_j^1	E_j^2	E_j^3	E_j^4	E_j^5	E_j^6	$\dots$
0	2	3	6	10	17	28	46	$\dots$
1	1	4	8	15	26	44	73	$\dots$
2	3	7	14	25	43	72	119	$\dots$
3	4	11	22	40	69	116	192	$\dots$
4	7	18	36	65	112	188	311	$\dots$
5	11	29	58	105	181	304	503	$\dots$
6	18	47	94	170	293	492	814	$\dots$
7	29	76	152	275	474	796	1317	$\dots$
8	47	123	246	445	767	1288	2131	$\dots$
9	76	199	398	720	1241	2084	3448	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 4.1. The first 10 number of the sequence $\{E_j^k\}$

5 Main results on the generalized Lucas sequence

Theorem 5.1. *Let $\{E_j^k\}$ be the generalized Lucas sequence, then $\lim_{j\to\infty} \frac{E_{j+1}^k}{E_j^k}$ converging to Golden ratio.*

Proof. The proof of this theorem follows from the Theorem 3.2. □

Theorem 5.2. *Let $\{E_j^k\}$ be the generalized sequence of Lucas numbers, then*

$$(E_{j+1}^k)^2 - E_j^k E_{j+2}^k = 5(-1)^j \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - 1 \right]$$

is log-convex for $j = 0, 2, 4, 6, \dots$ and log-concave for $j = 1, 3, 5, \dots$.

Proof. The proof of this theorem follows from the Theorem 3.2. □

Theorem 5.3. *For all positive integer j and k, $E_j^k = \frac{5F_{j+1}F_{k+1} + L_{j+1}L_{k+1} - 2L_{j+1}}{2}$.*

Proof. The identity is proved by mathematical induction.

Case 1. Put $k = 0, j = 0, E_0^0 = \frac{5F_1F_1 + L_1L_1 - 2L_1}{2} = 2.$

Put $k = 0,\ j = 1, E_1^0 = \frac{5F_2F_1 + L_2L_1 - 2F_2}{2} = 1.$

Hence, it holds for $k = 0, j = 0$ and 1. Assume that identity is holds for fixed k and $j = m$, i.e.,

$$E_m^k = \frac{5F_{m+1}F_{k+1} + L_{m+1}L_{k+1} - 2L_{m+1}}{2}.$$

To prove the identity holds good for fixed k and $j = m + 1$, consider from Definition 4.1,

$$E_{m+1}^{k} = L_{m+1} + L_{m+2} + L_{m+3} + \cdots + L_{m+k} + L_{m+k+1}.$$

On adding and subtracting L_m and by using the identity (1.8) gives

$$\begin{aligned}
E_{m+1}^{k} &= E_{m}^{k} + L_{m+k+1} - L_m \\
&= \frac{5F_{m+1}F_{k+1} + L_{m+1}L_{k+1} - 2L_{m+1}}{2} + \frac{L_m L_{k+1} + 5F_m F_{k+1}}{2} - F_m \\
&= \frac{5F_{m+1}F_{k+1} + L_{m+1}L_{k+1} - 2L_{m+1} + L_m L_{k+1} + 5F_m F_{k+1} - 2F_m}{2}.
\end{aligned}$$

On simplification,

$$E_{m+1}^{k} = \frac{5F_{m+2}F_{k+1} + L_{m+2}L_{k+1} - 2L_{m+2}}{2}.$$

It holds true for a fixed k and for all j.

Case 2. Put $k = 0, j = 0, E_0^0 = \dfrac{5F_1F_1 + L_1L_1 - 2L_1}{2} = 2.$

Put $k = 1, j = 0, E_0^1 = \dfrac{5F_1F_2 + L_1L_2 - 2L_1}{2} = 3.$

So, it holds good for $j = 0$ and $k = 0$ and 1. Assume that it holds good for a fixed j and $k = m$.

$$E_j^m = \frac{5F_{j+1}F_{m+1} + L_{j+1}L_{m+1} - 2L_{j+1}}{2}.$$

To prove the identity holds good for a fixed j and $k = m + 1$, by Definition 4.1 and using the identity (1.8) gives

$$\begin{aligned}
E_j^{m+1} &= E_j^m + L_{j+m+1} \\
&= \frac{5F_{j+1}F_{m+1} + L_{j+1}L_{m+1} - 2L_{j+1}}{2} + \frac{L_{j+1}L_m + 5F_{j+1}F_m}{2} \\
&= \frac{5F_{j+1}F_{m+1} + L_{j+1}L_{m+1} - 2L_{j+1} + L_{j+1}L_m + 5F_{j+1}F_m}{2}.
\end{aligned}$$

On simplification,

$$E_j^{m+1} = \frac{5F_{j+1}F_{m+2} + L_{j+1}L_{m+2} - 2L_{j+1}}{2}.$$

It holds true for a fixed j for all k. Hence the theorem is proved. □

The result is illustrated for the first 10 values of k and j in the form of table below.

C	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	$\dots$
j	E_j^0	E_j^1	E_j^2	E_j^3	E_j^4	∇_j^0	∇_j^1	∇_j^2	∇_j^3	∇_j^4	$\dots$
0	2	3	6	10	17	$-$	$-$	$-$	$-$	$-$	$\dots$
1	1	4	8	15	26	-5	-5	-20	-25	-55	$\dots$
2	3	7	14	25	43	5	5	20	25	55	$\dots$
3	4	11	22	40	69	-5	-5	-20	-25	-55	$\dots$
4	7	18	36	65	112	5	5	20	25	55	$\dots$
5	11	29	58	105	181	-5	-5	-20	-25	-55	$\dots$
6	18	47	94	170	293	5	5	20	25	55	$\dots$
7	29	76	152	275	474	-5	-5	-20	-25	-55	$\dots$
8	47	123	246	445	767	5	5	20	25	55	$\dots$
9	76	199	398	720	1241	-5	-5	-20	-25	-55	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 5.1. The first 10 generalized Lucas numbers represent $(E^k_{j+1})^2 - E^k_j E^k_{j+2}$

Remark 5.1. *Let $\{E^k_j\}$ be the generalized Lucas sequence, then the following identity holds.*

$$\nabla^k_j = (E^k_{j+1})^2 - E^k_j E^k_{j+2} = 5(-1)^j \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} + \left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - 1 \right]$$

is log-concave for $j = 0, 2, 4, 6, \dots$ and log-convex for $j = 1, 3, 5, \dots$.

Remark 5.2. *Let $\{E^k_j\}$ be the generalized Lucas sequence, then*

1. $(E^k_{j+1})^2 - E^k_j E^k_{j+2} = -5[(D^k_{j+1})^2 - D^k_j D^k_{j+2}]$;
2. $E^k_{j+1} = E^{k+1}_j - L_j$;
3. $E^k_j - E^k_{j-a} E^k_{j+a} = 5(-1)^{j+a+1} \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} + \left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - 1 \right] (E^0_a)^2.$

6 Conclusion

This article provides the generalization of Fibonacci and Lucas numbers. Few identities involving them are proved and few are stated directly. The results are illustrated for numerical values. The results studied by Zvonko Cerin [2] correspond to the columns $C_3, C_7, C_{11}, C_{15}, ..., C_{4i+3}$ of Table 2.1. The results have a good number of applications in the field of medical sciences and encryption, generating OTP and develop musical nodes.

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