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Hyperbolic k-Fibonacci and k-Lucas octonions

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Abstract: In this paper, we introduce the hyperbolic k-Fibonacci and k-Lucas octonions. We present Binet's formulas, Catalan's identity, Cassini's identity, d'Ocagne's identity and generating functions for the k-Fibonacci and k-Lucas hyperbolic octonions.
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1 Introduction

Fibonacci sequence is a well-known sequence which satisfy the second order recurrence relation. The Fibonacci sequence is generalized in different ways by changing initial conditions or recurrence relation. The k-Fibonacci sequence is one generalization of Fibonacci sequence which is first introduced by Falcon and Plaza [10]. For properties and applications of k-Fibonacci and k-Lucas numbers one can refer the articles [2, 6–9, 11, 12, 27]. The quaternions were first introduced by the Irish mathematician William Rowan Hamilton in 1843. Hamilton [18] introduced the set of quaternions which form a 4-dimensional real vector space with a multiplicative operation. The quaternions have many applications in applied sciences such as physics, computer science and Clifford algebras in mathematics. They are important in mechanics [16], chemistry [13], kinematics [1], quantum mechanics [24], differential geometry and pure algebra. In [19], Horadam defined the n-th Fibonacci and n-th Lucas quaternions.

In [23], Ramirez defined and studied the k-Fibonacci and k-Lucas quaternions. Quaternions of sequences have been studied by many researchers. Such as Iyer [20, 21] obtained various relations containing the Fibonacci and Lucas quaternions. Halici [17] studied combinatorial properties of Fibonacci quaternions. Akyigit et al. [25, 26] established and investigated the

Fibonacci generalized quaternions and split Fibonacci quaternions. Catarino [5] obtained properties of the $h(x)$-Fibonacci quaternion polynomials. Polatli and Kesim [22] have introduced quaternions with generalized Fibonacci and Lucas number components.

In [15], the hyperbolic k-Fibonacci and k-Lucas quaternions $\mathcal{Q}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{Q}^{\mathcal{L}}{}_{k,n}$ are defined as

$$\begin{aligned}\mathcal{Q}^{\mathcal{F}}{}_{k,n} &= F_{k,n}i_1 + F_{k,n+1}i_2 + F_{k,n+2}i_3 + F_{k,n+3}i_4 \\ &= \big\langle F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}\big\rangle\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}^{\mathcal{L}}{}_{k,n} &= L_{k,n}i_1 + L_{k,n+1}i_2 + L_{k,n+2}i_3 + L_{k,n+3}i_4 \\ &= \big\langle L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}\big\rangle,\end{aligned}$$

respectively, where $F_{k,n}$ is the n-th k-Fibonacci sequence and $L_{k,n}$ is n-th k-Lucas sequence. Here, i_1, i_2, i_3, i_4 are hyperbolic quaternion units which satisfy the multiplication rule

$$\begin{aligned}&{i_2}^2 = {i_3}^2 = {i_4}^2 = i_2i_3i_4 = +1, \quad i_1 = 1, \\ &i_2i_3 = i_4 = -i_3i_2, \quad i_3i_4 = i_2 = -i_4i_3, \quad i_4i_2 = i_3 = -i_2i_4.\end{aligned}$$

In [3, 4], A. Cariow, G. Cariow and J. Knapiski defined hyperbolic octonions. A hyperbolic octonion $\mathcal{O}$ is an expression of the form

$$\begin{aligned}\mathcal{O} &= h_0 + h_1i_1 + h_2i_2 + h_3i_3 + h_4e_4 + h_5e_5 + h_6e_6 + h_7e_7 \\ &= \big\langle h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7\big\rangle,\end{aligned}$$

with real components h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7 and i_1, i_2, i_3 are quaternion imaginary units, $e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions are defined as follows:

$$i_1e_4 = e_5, \quad i_2e_4 = e_6, \quad i_3e_4 = e_7, \quad {e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1.$$

The bases of hyperbolic octonion $\mathcal{O}$ appeared in [3, 4] have multiplication rules as in Table 1.

$\cdot$	i_1	i_2	i_3	e_4	e_5	e_6	e_7
i_1	-1	i_3	$-i_2$	e_5	e_4	$-e_7$	e_6
i_2	$-e_3$	-1	i_1	e_6	e_7	e_4	$-e_5$
i_3	i_2	$-i_1$	-1	e_7	$-e_6$	e_5	e_4
e_4	$-e_5$	$-e_6$	$-e_7$	1	i_1	i_2	i_3
e_5	$-e_4$	$-e_7$	e_6	$-i_1$	1	i_3	$-i_2$
e_6	e_7	$-e_4$	$-e_5$	$-i_2$	$-i_3$	1	i_1
e_7	$-e_6$	e_5	$-e_4$	$-i_3$	i_2	$-i_1$	1

Table 1. Rules for multiplication of hyperbolic octonion bases

In [14], we derived some properties of the hyperbolic k-Fibonacci and k-Lucas octonions.

In the current paper, our main aim is to prove the well-known identities like Binet's formulas, Catalan's identity, Cassini's identity, d'Ocagne's identity and generating functions for the k-Fibonacci and k-Lucas hyperbolic octonions.

2 Hyperbolic k-Fibonacci and k-Lucas octonions

In this section, some elementary properties of the hyperbolic k-Fibonacci and k-Lucas octonions are obtained.

Definition 2.1. For $n \geq 0$, the hyperbolic k-Fibonacci and k-Lucas octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ are defined by

$$\begin{aligned}\mathcal{O^F}_{k,n} &= F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}e_4 + F_{k,n+5}e_5 + F_{k,n+6}e_6 + F_{k,n+7}e_7 \\ &= \Big\langle F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}, F_{k,n+4}, F_{k,n+5}, F_{k,n+6}, F_{k,n+7}\Big\rangle,\end{aligned}$$

and

$$\begin{aligned}\mathcal{O^L}_{k,n} &= L_{k,n} + L_{k,n+1}i_1 + L_{k,n+2}i_2 + L_{k,n+3}i_3 + L_{k,n+4}e_4 + L_{k,n+5}e_5 + L_{k,n+6}e_6 + L_{k,n+7}e_7 \\ &= \Big\langle L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}, L_{k,n+4}, L_{k,n+5}, L_{k,n+6}, L_{k,n+7}\Big\rangle,\end{aligned}$$

respectively, where $F_{k,n}$ is the n-th k-Fibonacci sequence and $L_{k,n}$ is the n-th k-Lucas sequence. Here, i_1, i_2, i_3 are quaternion imaginary units, $e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ are defined as $i_1e_4 = e_5, i_2e_4 = e_6$, $i_3e_4 = e_7, {e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1$. The bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ have multiplication rules as in Table 1.

Definition 2.2. For $n \geq 0$, the conjugate of hyperbolic k-Fibonacci and k-Lucas octonions $\bar{\mathcal{O}}^{\mathcal{F}}_{k,n}$ and $\bar{\mathcal{O}}^{\mathcal{L}}_{k,n}$ are defined by

$$\begin{aligned}\bar{\mathcal{O}}^{\mathcal{F}}_{k,n} &= F_{k,n} - F_{k,n+1}i_1 - F_{k,n+2}i_2 - F_{k,n+3}i_3 - F_{k,n+4}e_4 - F_{k,n+5}e_5 - F_{k,n+6}e_6 - F_{k,n+7}e_7 \\ &= \Big\langle F_{k,n}, -F_{k,n+1}, -F_{k,n+2}, -F_{k,n+3}, -F_{k,n+4}, -F_{k,n+5}, -F_{k,n+6}, -F_{k,n+7}\Big\rangle,\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{O}}^{\mathcal{L}}_{k,n} &= L_{k,n} - L_{k,n+1}i_1 - L_{k,n+2}i_2 - L_{k,n+3}i_3 - L_{k,n+4}e_4 - L_{k,n+5}e_5 - L_{k,n+6}e_6 - L_{k,n+7}e_7 \\ &= \Big\langle L_{k,n}, -L_{k,n+1}, -L_{k,n+2}, -L_{k,n+3}, -L_{k,n+4}, -L_{k,n+5}, -L_{k,n+6}, -L_{k,n+7}\Big\rangle,\end{aligned}$$

respectively, where $F_{k,n}$ is the n-th k-Fibonacci sequence and $L_{k,n}$ is the n-th k-Lucas sequence. Here, i_1, i_2, i_3 are quaternion imaginary units, $e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ are defined as $i_1e_4 = e_5, i_2e_4 = e_6$, $i_3e_4 = e_7, {e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1$. The bases of hyperbolic octonions $\mathcal{O^F}_{k,n}$ and $\mathcal{O^L}_{k,n}$ have multiplication rules as in Table 1.

Theorem 2.3. *For all $n \geq 0$, we have*

$$\begin{aligned}&(i) \quad \mathcal{O^F}_{k,n+2} = k\mathcal{O^F}_{k,n+1} + \mathcal{O^F}_{k,n}, \\ &(ii) \quad \mathcal{O^L}_{k,n+2} = k\mathcal{O^L}_{k,n+1} + \mathcal{O^L}_{k,n} \\ &(iii) \quad \mathcal{O^L}_{k,n} = \mathcal{O^F}_{k,n+1} + \mathcal{O^F}_{k,n-1} \\ &(iv) \quad \bar{\mathcal{O}}^{\mathcal{F}}_{k,n+2} = k\bar{\mathcal{O}}^{\mathcal{F}}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{F}}_{k,n}, \\ &(v) \quad \bar{\mathcal{O}}^{\mathcal{L}}_{k,n+2} = k\bar{\mathcal{O}}^{\mathcal{L}}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{L}}_{k,n} \\ &(vi) \quad \bar{\mathcal{O}}^{\mathcal{L}}_{k,n} = \bar{\mathcal{O}}^{\mathcal{F}}_{k,n+1} + \bar{\mathcal{O}}^{\mathcal{F}}_{k,n-1}.\end{aligned}$$

Proof. (i). Using Definition 2.1, we have

$$\begin{aligned}
k\mathcal{O}^{\mathcal{F}}{}_{k,n+1} + \mathcal{O}^{\mathcal{F}}{}_{k,n} &= k\big(F_{k,n+1} + F_{k,n+2}i_1 + F_{k,n+3}i_2 + F_{k,n+4}i_3 \\
&\quad + F_{k,n+5}e_4 + F_{k,n+6}e_5 + F_{k,n+7}e_6 + F_{k,n+8}e_7\big) \\
&\quad + \big(F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}e_4 \\
&\quad + F_{k,n+5}e_5 + F_{k,n+6}e_6 + F_{k,n+7}e_7\big) \\
&= \big(kF_{k,n+1} + F_{k,n}\big) + \big(kF_{k,n+2} + F_{k,n+1}\big)i_1 \\
&\quad + \big(kF_{k,n+3} + F_{k,n+2}\big)i_2 + \big(kF_{k,n+4} + F_{k,n+3}\big)i_3 \\
&\quad + \big(kF_{k,n+5} + F_{k,n+4}\big)e_4 + \big(kF_{k,n+6} + F_{k,n+5}\big)e_5 \\
&\quad + \big(kF_{k,n+7} + F_{k,n+6}\big)e_6 + \big(kF_{k,n+8} + F_{k,n+7}\big)e_7 \\
&= F_{k,n+2} + F_{k,n+3}i_1 + F_{k,n+4}i_2 + F_{k,n+5}i_3 + F_{k,n+6}e_4 \\
&\quad + F_{k,n+7}e_5 + F_{k,n+8}e_6 + F_{k,n+9}e_7 \\
&= \mathcal{O}^{\mathcal{F}}{}_{k,n+2}.
\end{aligned}$$

The proofs of (ii), (iii), (iv), (v) and (vi) are similar to (i), using Definition 2.1. □

Theorem 2.4 (**Binet Formulas**). *For all $n \geq 0$, we have*

$$\begin{aligned}
&(i) \quad \mathcal{O}^{\mathcal{F}}{}_{k,n} = \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2} \\
&(ii) \quad \mathcal{O}^{\mathcal{L}}{}_{k,n} = \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n, \\
&(iii) \quad \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n} = \frac{\bar{r_3}{r_1}^n - \bar{r_4}{r_2}^n}{r_1 - r_2} \\
&(iv) \quad \bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} = \bar{r_3}{r_1}^n + \bar{r_4}{r_2}^n,
\end{aligned}$$

where

$$\begin{aligned}
\bar{r_1} &= 1 + r_1 i_1 + {r_1}^2 i_2 + {r_1}^3 i_3 + {r_1}^4 e_4 + {r_1}^5 e_5 + {r_1}^6 e_6 + {r_1}^7 e_7 = \big\langle 1, r_1, {r_1}^2, {r_1}^3, {r_1}^4, {r_1}^5, {r_1}^6, {r_1}^7 \big\rangle, \\
\bar{r_2} &= 1 + r_2 i_1 + {r_2}^2 i_2 + {r_2}^3 i_3 + {r_2}^4 e_4 + {r_2}^5 e_5 + {r_2}^6 e_6 + {r_2}^7 e_7 = \big\langle 1, r_2, {r_2}^2, {r_2}^3, {r_2}^4, {r_2}^5, {r_2}^6, {r_2}^7 \big\rangle, \\
\bar{r_3} &= 1 - r_1 i_1 - {r_1}^2 i_2 - {r_1}^3 i_3 - {r_1}^4 e_4 - {r_1}^5 e_5 - {r_1}^6 e_6 - {r_1}^7 e_7 \\
&= \big\langle 1, -r_1, -{r_1}^2, -{r_1}^3, -{r_1}^4, -{r_1}^5, -{r_1}^6, -{r_1}^7 \big\rangle, \\
\bar{r_4} &= 1 + r_2 i_1 - {r_2}^2 i_2 - {r_2}^3 i_3 - {r_2}^4 e_4 - {r_2}^5 e_5 - {r_2}^6 e_6 - {r_2}^7 e_7 \\
&= \big\langle 1, -r_2, -{r_2}^2, -{r_2}^3, -{r_2}^4, -{r_2}^5, -{r_2}^6, -{r_2}^7 \big\rangle.
\end{aligned}$$

Here, i_1, i_2, i_3 are quaternion imaginary units, $e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{O}^{\mathcal{L}}{}_{k,n}$ are defined as $i_1 e_4 = e_5, i_2 e_4 = e_6$, $i_3 e_4 = e_7, {e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1$. The bases of hyperbolic octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{O}^{\mathcal{L}}{}_{k,n}$ have multiplication rules as in Table 1.

Proof. (i). Using Definition 2.1 and the Binet formulas of k-Fibonacci and k-Lucas sequences, we have

$$\mathcal{O}^{\mathcal{F}}{}_{k,n} = F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3 + F_{k,n+4}e_4 + F_{k,n+5}e_5 + F_{k,n+6}e_6 + F_{k,n+7}e_7$$

$$= [\frac{{r_1}^n - {r_2}^n}{r_1 - r_2}] + [\frac{{r_1}^{n+1} - {r_2}^{n+1}}{r_1 - r_2}]i_1 + [\frac{{r_1}^{n+2} - {r_2}^{n+2}}{r_1 - r_2}]i_2 + [\frac{{r_1}^{n+3} - {r_2}^{n+3}}{r_1 - r_2}]i_3$$
$$+ [\frac{{r_1}^{n+4} - {r_2}^{n+4}}{r_1 - r_2}]e_4 + [\frac{{r_1}^{n+5} - {r_2}^{n+5}}{r_1 - r_2}]e_5 + [\frac{{r_1}^{n+6} - {r_2}^{n+6}}{r_1 - r_2}]e_6 + [\frac{{r_1}^{n+7} - {r_2}^{n+7}}{r_1 - r_2}]e_7$$
$$= \frac{{r_1}^n}{r_1 - r_2}\big(1 + r_1 i_1 + {r_1}^2 i_2 + {r_1}^3 i_3 + {r_1}^4 e_4 + {r_1}^5 e_5 + {r_1}^6 e_6 + {r_1}^7 e_7\big)$$
$$- \frac{{r_2}^n}{r_1 - r_2}\big(1 + r_2 i_1 + {r_2}^2 i_2 + {r_2}^3 i_3 + {r_2}^4 e_4 + {r_2}^5 e_5 + {r_2}^6 e_6 + {r_2}^7 e_7\big)$$
$$= \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2}$$

(ii). We have,

$$\mathcal{O^L}_{k,n} = L_{k,n} + L_{k,n+1}i_1 + L_{k,n+2}i_2 + L_{k,n+3}i_3 + L_{k,n+4}e_4 + L_{k,n+5}e_5 + L_{k,n+6}e_6 + L_{k,n+7}e_7$$
$$= \big({r_1}^n + {r_2}^n\big) + \big({r_1}^{n+1} + {r_2}^{n+1}\big)i_1 + \big({r_1}^{n+2} + {r_2}^{n+2}\big)i_2 + \big({r_1}^{n+3}{r_2}^{n+3}\big)i_3$$
$$+ \big({r_1}^{n+4}{r_2}^{n+4}\big)e_4 + \big({r_1}^{n+5}{r_2}^{n+5}\big)e_5 + \big({r_1}^{n+6}{r_2}^{n+6}\big)e_6 + \big({r_1}^{n+7}{r_2}^{n+7}\big)e_7$$
$$= {r_1}^n\big(1 + r_1 i_1 + {r_1}^2 i_2 + {r_1}^3 i_3 + {r_1}^4 e_4 + {r_1}^5 e_5 + {r_1}^6 e_6 + {r_1}^7 e_7\big)$$
$$+ {r_2}^n\big(1 + r_2 i_1 + {r_2}^2 i_2 + {r_2}^3 i_3 + {r_2}^4 e_4 + {r_2}^5 e_5 + {r_2}^6 e_6 + {r_2}^7 e_7\big)$$
$$= \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n.$$

The proofs of (iii) and (iv) are similar to these of (i) and (ii) using Definition 2.1. □

Some interesting properties of $\bar{r_1}$, $\bar{r_2}$, $\bar{r_3}$ and $\bar{r_4}$ are listed in [14]. Some of these are listed in Lemma 2.5.

Lemma 2.5. *For* $\bar{r_1} = \big\langle 1, r_1, {r_1}^2, {r_1}^3, {r_1}^4, {r_1}^5, {r_1}^6, {r_1}^7\big\rangle$, $\bar{r_2} = \big\langle 1, r_2, {r_2}^2, {r_2}^3, {r_2}^4, {r_2}^5, {r_2}^6, {r_2}^7\big\rangle$, $\bar{r_3} = \big\langle 1, -r_1, -{r_1}^2, -{r_1}^3, -{r_1}^4, -{r_1}^5, -{r_1}^6, -{r_1}^7\big\rangle$ *and* $\bar{r_4} = \big\langle 1, -r_2, -{r_2}^2, -{r_2}^3, -{r_2}^4, -{r_2}^5, -{r_2}^6, -{r_2}^7\big\rangle$, *we have*

(1) $\bar{r_1} - \bar{r_2} = \sqrt{\delta}\mathcal{O^F}_{k,0}$,

(2) $\bar{r_1} + \bar{r_2} = \mathcal{O^L}_{k,0}$,

(3) $\bar{r_1}\bar{r_2} = \big\langle 2, -2(r_1 - 2r_2), -2({r_1}^2 - 2{r_2}^2), 2(r_1 - r_2 + {r_2}^3), 2{r_1}^4, 2({r_1}^3 - {r_2}^3) + 2{r_1}^5, -2({r_1}^2 - {r_2}^2 + {r_1}^6), {r_1}^7 + {r_2}^7 - (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\big\rangle = \bar{u_1}$,

(4) $\bar{r_2}\bar{r_1} = \big\langle 2, -2(r_2 - 2r_1), -2({r_2}^2 - 2{r_1}^2), 2(r_2 - r_1 + {r_1}^3), 2{r_2}^4, 2({r_2}^3 - {r_1}^3) + 2{r_2}^5, -2({r_2}^2 - {r_1}^2 + {r_2}^6), {r_2}^7 + {r_1}^7 - (r_2 - r_1)({r_1}^4 + {r_2}^4 - 1)\big\rangle = \bar{u_2}$,

(5) $\bar{r_1}^2 = \big(-1 - {r_1}^2 - {r_1}^4 - {r_1}^6 + {r_1}^8 + {r_1}^{10} + {r_1}^{12} + {r_1}^{14}\big) + 2\bar{r_1} = \bar{u_3}$,

(6) $\bar{r_2}^2 = \big(-1 - {r_2}^2 - {r_2}^4 - {r_2}^6 + {r_2}^8 + {r_2}^{10} + {r_2}^{12} + {r_2}^{14}\big) + 2\bar{r_2} = \bar{u_4}$,

(7) $\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1} = 2\mathcal{O^L}_{k,0}$,

(8) $\bar{r_1}\bar{r_2} - \bar{r_2}\bar{r_1} = 2\sqrt{\delta}\big\langle 0, -3, -3k, (1-k^2), k(k^2+2), k^4 + 5k^2 + 3, k^5 + 4k^3 + k, -(k^4 + 4k + 1)\big\rangle = \bar{u_5}$,

(9) $\bar{r_1}^2 - \bar{r_2}^2 = \sqrt{\delta}\big(k^{13} + 13k^{11} + 66k^9 + 165k^7 + 208k^5 + 116k^3 + 16k + 2\mathcal{O^F}_{k,0}\big) = \bar{u_6}$,

(10) $\bar{r_1}^2 + \bar{r_2}^2 = \big(k^{14} + 15k^{12} + 90k^{10} + 275k^8 + 448k^6 + 364k^4 + 112k^2 + 2\mathcal{O^L}_{k,0}\big) = \bar{u_7}$,

(11) $\bar{r_3} = 2 - \bar{r_1}$ *and* $\bar{r_4} = 2 - \bar{r_2}$,

(12) $\bar{r_1}\bar{r_3} = \bar{r_3}\bar{r_1} = \left(1 + {r_1}^2 + {r_1}^4 + {r_1}^6 - {r_1}^8 - {r_1}^{10} - {r_1}^{12} - {r_1}^{14}\right) = u_8$,

(13) $\bar{r_2}\bar{r_4} = \bar{r_4}\bar{r_2} = \left(1 + {r_2}^2 + {r_2}^4 + {r_2}^6 - {r_2}^8 - {r_2}^{10} - {r_2}^{12} - {r_2}^{14}\right) = u_9$,

(14) $\bar{r_1}\bar{r_4} = \langle 0, 4(r_1 - r_2), 4({r_1}^2 - {r_2}^2), -2(r_1 - r_2 - {r_1}^3 - {r_2}^3), 0,$
$- 2({r_1}^3 - {r_2}^3), 2({r_1}^2 - {r_2}^2 + 2{r_1}^6), {r_1}^7 - {r_2}^7 + (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\rangle = \bar{u_{10}}$,

(15) $\bar{r_4}\bar{r_1} = \langle 0, -2(r_1 - r_2), -2({r_1}^2 - {r_2}^2), 2(r_1 - r_2), -2({r_1}^4 - {r_2}^4), 2({r_1}^3 - {r_2}^3$
$- {r_2}^5 + {r_1}^5), -2({r_1}^2 - {r_2}^2 - {r_1}^6 - {r_2}^6), {r_1}^7 - {r_2}^7 - (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\rangle = \bar{u_{11}}$,

(16) $\bar{r_2}\bar{r_3} = \langle 0, -2(r_1 - r_2), -2({r_1}^2 - {r_2}^2), 2(r_1 - r_2 - {r_1}^3 + {r_2}^3), -2({r_1}^4 - {r_2}^4),$
$2({r_1}^3 - {r_2}^3), -2({r_1}^2 - {r_2}^2 - 2{r_2}^6), -{r_1}^7 + {r_2}^7 - (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\rangle = \bar{u_{12}}$,

(17) $\bar{r_3}\bar{r_2} = \langle 0, 2(r_1 - r_2), 2({r_1}^2 - {r_2}^2), -2(r_1 - r_2), -2({r_1}^4 - {r_2}^4), -2({r_1}^3 - {r_2}^3$
$+ {r_1}^5 - {r_2}^5), 2({r_1}^2 - {r_2}^2 + {r_1}^6 + {r_2}^6), -{r_1}^7 + {r_2}^7 + (r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\rangle = \bar{u_{13}}$,

(18) $\bar{r_3}\bar{r_4} = 4 - 2{\mathcal{O}^{\mathcal{L}}}_{k,0} + \bar{u_1}$,

(19) $\bar{r_4}\bar{r_3} = 4 - 2{\mathcal{O}^{\mathcal{L}}}_{k,0} + \bar{u_2}$,

(20) $\bar{r_3}\bar{r_4} - \bar{r_4}\bar{r_3} = \bar{u_1} - \bar{u_2}$,

(21) $\bar{r_3}\bar{r_2} = \langle 0, -2(r_1 - r_2), -2({r_1}^2 - {r_2}^2), -2({r_1}^3 - {r_2}^3), -4({r_1}^4 - {r_2}^4),$
$- 2({r_1}^5 - {r_2}^5), 2({r_1}^6 + 3{r_2}^6), -2({r_1}^7 - {r_2}^7)\rangle = \bar{u_{14}}$,

(22) $\bar{r_3}\bar{r_2} - \bar{r_2}\bar{r_3} = \langle 0, 6(r_1 - r_2), 6({r_1}^2 - {r_2}^2), -2(2r_1 - 2r_2 - {r_1}^3 + {r_2}^3), 0,$
$- 2(2{r_1}^3 - 2{r_2}^3 + {r_1}^5 - {r_2}^5), 2(2{r_1}^2 - 2{r_2}^2 + {r_1}^6 - {r_2}^6), 2(r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\rangle$
$= \bar{u_{15}}$,

(23) $\bar{r_1}\bar{r_4} - \bar{r_4}\bar{r_1} = \langle 0, 6(r_1 - r_2), 6({r_1}^2 - {r_2}^2), -2(2r_1 - 2r_2 - {r_1}^3 - {r_2}^3), 2({r_2}^4 - {r_1}^4),$
$- 2(2{r_1}^3 - 2{r_2}^3 + {r_1}^5 - {r_2}^5), 2(2{r_1}^2 - 2{r_2}^2 + {r_1}^6 - {r_2}^6), 2(r_1 - r_2)({r_1}^4 + {r_2}^4 - 1)\rangle$
$= \bar{u_{16}}$,

(24) $\bar{r_3}^2 = 4\bar{r_3} + \bar{u_3} - 4 = \bar{u_{17}}$,

(25) $\bar{r_4}^2 = 4\bar{r_4} + \bar{u_4} - 4 = \bar{u_{18}}$.

Theorem 2.6. *For all* $s, t \in Z^+, s \geq t$ *and* $n \in N$, *the generating functions for the hyperbolic* k*-Fibonacci and* k*-Lucas quaternions* ${\mathcal{O}^{\mathcal{F}}}_{k,tn}$ *and* ${\mathcal{O}^{\mathcal{L}}}_{k,tn}$ *are*

$$(i) \quad \sum_{n=0}^{\infty} {\mathcal{O}^{\mathcal{F}}}_{k,tn} x^n = \frac{{\mathcal{O}^{\mathcal{F}}}_{k,0} + \left({\mathcal{O}^{\mathcal{L}}}_{k,0} F_{k,t} - {\mathcal{O}^{\mathcal{F}}}_{k,t}\right)x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(ii) \quad \sum_{n=0}^{\infty} {\mathcal{O}^{\mathcal{L}}}_{k,tn} x^n = \frac{{\mathcal{O}^{\mathcal{L}}}_{k,0} - \left({\mathcal{O}^{\mathcal{L}}}_{k,0} L_{k,t} - {\mathcal{O}^{\mathcal{L}}}_{k,t}\right)x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(iii) \quad \sum_{n=0}^{\infty} {\mathcal{O}^{\mathcal{F}}}_{k,tn+s} x^n = \frac{{\mathcal{O}^{\mathcal{F}}}_{k,s} + (-1)^t x {\mathcal{O}^{\mathcal{F}}}_{s,s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(iv) \quad \sum_{n=0}^{\infty} {\mathcal{O}^{\mathcal{L}}}_{k,tn+s} x^n = \frac{{\mathcal{O}^{\mathcal{L}}}_{k,s} + (-1)^t x {\mathcal{O}^{\mathcal{L}}}_{s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

and the exponential generating functions for the hyperbolic k*-Fibonacci and* k*-Lucas quaternions*

$\mathcal{O^F}_{k,tn}$ *and* $\mathcal{O^L}_{k,tn}$ *are*

$$(v) \quad \sum_{n=0}^{\infty} \frac{\mathcal{O^F}_{k,tn}}{n!}x^n = \frac{\bar{r_1}e^{{r_1}^t x} - \bar{r_2}e^{{r_2}^t x}}{r_1 - r_2},$$

$$(vi) \quad \sum_{n=0}^{\infty} \frac{\mathcal{O^L}_{k,tn}}{n!}x^n = \bar{r_1}e^{{r_1}^t x} + \bar{r_2}e^{{r_2}^t x}.$$

Proof. (i). Using Theorem 2.4, we obtain

$$\sum_{n=0}^{\infty} \mathcal{O^F}_{k,tn}x^n = \sum_{n=0}^{\infty} \Big(\frac{\bar{r_1}{r_1}^{tn} - \bar{r_2}{r_2}^{tn}}{r_1 - r_2}\Big)x^n$$

$$= \frac{\bar{r_1}}{r_1 - r_2}\sum_{n=0}^{\infty}\left({r_1}^t\right)^n x^n - \frac{\bar{r_2}}{r_1 - r_2}\sum_{n=0}^{\infty}\left({r_2}^t\right)^n x^n$$

$$= \frac{\bar{r_1}}{r_1 - r_2}\Big(\frac{1}{1 - {r_1}^t x}\Big) - \frac{\bar{r_2}}{r_1 - r_2}\Big(\frac{1}{1 - {r_2}^t x}\Big)$$

$$= \frac{1}{r_1 - r_2}\Big[\frac{\left(\bar{r_1} - \bar{r_2}\right) + \left[\bar{r_2}{r_1}^t - \bar{r_1}{r_2}^t\right]x}{1 - \left({r_1}^t + {r_2}^t\right)x + x^2(r_1 r_2)^t}\Big]$$

$$= \frac{1}{r_1 - r_2}\Big[\frac{\left(\bar{r_1} - \bar{r_2}\right) + \left[\bar{r_2}{r_1}^t - \bar{r_2}{r_2}^t + \bar{r_2}{r_2}^t - \bar{r_1}{r_1}^t + \bar{r_1}{r_1}^t - \bar{r_1}{r_2}^t\right]x}{1 - \left({r_1}^t + {r_2}^t\right)x + x^2(r_1 r_2)^t}\Big]$$

$$= \frac{\Big(\frac{\bar{r_1} - \bar{r_2}}{r_1 - r_2}\Big) + \Big[\left(\bar{r_1} + \bar{r_2}\right)\Big(\frac{{r_1}^t - {r_2}^t}{r_1 - r_2}\Big) - \Big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1 - r_2}\Big)\Big]x}{1 - \left({r_1}^t + {r_2}^t\right)x + x^2(r_1 r_2)^t}.$$

Using Theorem 2.4 and Lemma 2.5, we obtain

$$= \frac{\mathcal{O^F}_{k,0} + \left(\mathcal{O^L}_{k,0}F_{k,t} - \mathcal{O^F}_{k,t}\right)x}{1 - xL_{k,t} + x^2(-1)^t}.$$

The proofs of (ii), (iii), (iv), (v) and (vi) are similar to (i), using Theorem 2.4. □

Theorem 2.7. *For all* $n \in N$*, we have*

$$(i) \quad \sum_{i=0}^{n}\binom{n}{i}k^i\mathcal{O^F}_{k,i} = \mathcal{O^F}_{k,2n},$$

$$(ii) \quad \sum_{i=0}^{n}\binom{n}{i}k^i\mathcal{O^L}_{k,i} = \mathcal{O^L}_{k,2n}.$$

Proof. (i). Using Theorem 2.4, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}k^i\mathcal{O^F}_{k,i} = \sum_{i=0}^{n}\binom{n}{i}k^i\Big(\frac{\bar{r_1}{r_1}^i - \bar{r_2}{r_2}^i}{r_1 - r_2}\Big)$$

$$= \frac{\bar{r_1}}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} (kr_1)^i - \frac{\bar{r_2}}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} (kr_2)^i$$

$$= \frac{\bar{r_1}}{r_1 - r_2} \big(1 + kr_1\big)^n - \frac{\bar{r_2}}{r_1 - r_2} \big(1 + kr_2\big)^n$$

$$= \frac{\bar{r_1} {r_1}^{2n} - \bar{r_2} {r_2}^{2n}}{r_1 - r_2}$$

$$= \mathcal{O}^{\mathcal{F}}{}_{k,2n}.$$

The proof of (ii) is similar to (i), using Theorem 2.4. □

Lemma 2.8. *For all* $t \geq 0$, *we have*

(i) $\dfrac{\bar{u_1} {r_2}^t - \bar{u_2} {r_1}^t}{r_1 - r_2} = \bar{\mathcal{U}}_{k,t},$

where

$$\bar{\mathcal{U}}_{k,t} = \big\langle -2F_{k,t}, -2F_{k,t-1} - 4F_{k,t+1}, 2F_{k,t-2} - 4F_{k,t+2}, 2F_{k,t-2} + 2F_{k,t+1} - 2F_{k,t+3}, -2F_{k,t-4}, 2F_{k,t-3} + 2F_{k,t+3} + 2F_{k,t-5}, 2F_{k,t-2} - 2F_{k,t+2} + 2F_{k,t-6}, F_{k,t-7} - F_{k,t+7} - L_{k,t+4} - L_{k,t-4} + L_{k,t}\big\rangle,$$

(ii) $\dfrac{{r_1}^{m-n} \bar{u_1} - {r_2}^{m-n} \bar{u_2}}{r_1 - r_2} = \bar{\mathcal{V}}_{k,m-n},$

where

$$\bar{\mathcal{V}}_{k,m-n} = \big\langle 2F_{k,m-n}, -2F_{k,m-n+1} - 4F_{k,m-n-1}, -2F_{k,m-n+2} + 4F_{k,m-n-2}, 2F_{k,m-n+1} + 2F_{k,m-n-1} - 2F_{k,m-n-3}, 2F_{k,m-n+4}, 2F_{k,m-n+3} + 2F_{k,m-n-3} + F_{k,m-n+5}, -2F_{k,m-n+2} + 2F_{k,m-n-2} - 2F_{k,m-n+6}, F_{k,m-n+7} - F_{k,m-n-7} - L_{k,m-n+4} - L_{k,m-n-4} + L_{k,m-n}\big\rangle,$$

(iii) $\bar{u_3} {r_1}^t + \bar{u_4} {r_2}^t = \bar{\mathcal{W}}_{k,t},$

where

$$\bar{\mathcal{W}}_{k,t} = \big(-L_{k,t} - L_{k,t+2} - L_{k,t+4} - L_{k,t+6} + L_{k,t+8} + L_{k,t+10} + L_{k,t+12} + L_{k,t+14}\big) + 2\mathcal{O}^{\mathcal{L}}{}_{k,t},$$

(iv) $\dfrac{\bar{u_3} {r_1}^t - \bar{u_4} {r_2}^t}{r_1 - r_2} = \bar{\mathcal{X}}_{k,t},$

where

$$\bar{\mathcal{X}}_{k,t} = \big(-F_{k,t} - F_{k,t+2} - F_{k,t+4} - F_{k,t+6} + F_{k,t+8} + F_{k,t+10} + F_{k,t+12} + F_{k,t+14}\big) + 2\mathcal{O}^{\mathcal{F}}{}_{k,t},$$

(v) $\dfrac{\bar{u_8} {r_1}^t - \bar{u_9} {r_2}^t}{r_1 - r_2} = L_{k,t+1} + L_{k,t+3} - L_{k,t+9} - L_{k,t+13},$

(vi) $\dfrac{\bar{u_{17}} {r_1}^t - \bar{u_{18}} {r_2}^t}{r_1 - r_2} = \bar{\mathcal{Y}}_{k,t},$

where

$$\bar{\mathcal{Y}}_{k,t} = \big(-F_{k,t} - F_{k,t+2} - F_{k,t+4} - F_{k,t+6} + F_{k,t+8} + F_{k,t+10} + F_{k,t+12} + F_{k,t+14}\big) + 4 - 2\mathcal{O}^{\mathcal{F}}{}_{k,t}.$$

Theorem 2.9 (**Catalan's Identity**). *For any integer t and s, we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,n-t}{\mathcal{O}^{\mathcal{F}}}_{k,n+t} - {{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2 = (-1)^{n-t}F_{k,t}\bar{\mathcal{U}}_{k,t},$$
$$(ii) \quad {\mathcal{O}^{\mathcal{L}}}_{k,n-t}{\mathcal{O}^{\mathcal{L}}}_{k,n+t} - {{\mathcal{O}^{\mathcal{L}}}_{k,n}}^2 = \delta(-1)^{n-t+1}F_{k,t}\bar{\mathcal{V}}_{k,m-n}.$$

Proof. Using Theorem 2.4, we have

$$\begin{aligned}
{\mathcal{O}^{\mathcal{F}}}_{k,n-t}{\mathcal{O}^{\mathcal{F}}}_{k,n+t} - {{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2 &= \Big(\frac{\bar{r_1}{r_1}^{n-t} - \bar{r_2}{r_2}^{n-t}}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n+t} - \bar{r_2}{r_2}^{n+t}}{r_1 - r_2}\Big) - \Big(\frac{\bar{r_1}{r_1}^{n} - \bar{r_2}{r_2}^{n}}{r_1 - r_2}\Big)^2 \\
&= \frac{1}{(r_1 - r_2)^2}\big[\bar{r_1}^2{r_1}^{2n} - \bar{r_1}\bar{r_2}{r_1}^{n-t}{r_2}^{n+t} - \bar{r_2}\bar{r_1}{r_2}^{n-t}{r_1}^{n+t} + \bar{r_2}^2{r_2}^{2n} \\
&\quad - \bar{r_1}^2{r_1}^{2n} + \bar{r_1}\bar{r_2}(r_1r_2)^n + \bar{r_2}\bar{r_1}(r_1r_2)^n - \bar{r_2}^2{r_2}^{2n}\big] \\
&= \frac{(r_1r_2)^n}{(r_1 - r_2)^2}\big[\bar{r_1}\bar{r_2}{r_1}^t{r_1}^{-t} - \bar{r_2}\bar{r_1}{r_1}^t{r_2}^{-t} - \bar{r_1}\bar{r_2}{r_2}^t{r_2}^{-t} + \bar{r_2}\bar{r_1}{r_2}^t{r_2}^{-t}\big] \\
&= \frac{(r_1r_2)^n}{(r_1 - r_2)^2}\big[{r_1}^t[(\bar{r_1}\bar{r_2}){r_1}^{-t} - (\bar{r_2}\bar{r_1}){r_2}^{-t}] - {r_2}^t[(\bar{r_1}\bar{r_2}){r_1}^{-t} - (\bar{r_2}\bar{r_1}){r_2}^{-t}]\big] \\
&= (r_1r_2)^n\Big(\frac{{r_1}^t - {r_2}^t}{r_1 - r_2}\Big)\Big(\frac{(\bar{r_1}\bar{r_2}){r_1}^{-t} - (\bar{r_2}\bar{r_1}){r_2}^{-t}}{r_1 - r_2}\Big) \\
&= (r_1r_2)^{n-t}\Big(\frac{{r_1}^t - {r_2}^t}{r_1 - r_2}\Big)\Big(\frac{(\bar{r_1}\bar{r_2}){r_1}^{t} - (\bar{r_2}\bar{r_1}){r_2}^{t}}{r_1 - r_2}\Big).
\end{aligned}$$

Using Lemma 2.8 and $r_1r_2 = -1$, we obtain

$$= (-1)^{n-t}F_{k,t}\bar{\mathcal{U}}_{k,t}.$$

The proof of (ii) is similar to (i), using Theorem 2.4 and Lemma 2.8. □

Theorem 2.10 (**Cassini's Identity**). *For all $n \geq 1$, we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,n-1}{\mathcal{O}^{\mathcal{F}}}_{k,n+1} - {{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2 = (-1)^{n-1}F_{k,t}\bar{\mathcal{U}}_{k,1},$$
$$(ii) \quad {\mathcal{O}^{\mathcal{L}}}_{k,n-1}{\mathcal{O}^{\mathcal{L}}}_{k,n+1} - {{\mathcal{O}^{\mathcal{L}}}_{k,n}}^2 = \delta(-1)^{n}F_{k,t}\bar{\mathcal{V}}_{k,1}.$$

Theorem 2.11 (**d'Ocagne's Identity**). *Let n be any non-negative integer and t a natural number. If $t \geq n+1$, then we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,t}{\mathcal{O}^{\mathcal{F}}}_{k,n+1} - {\mathcal{O}^{\mathcal{F}}}_{k,t+1}{\mathcal{O}^{\mathcal{F}}}_{k,n} = (-1)^{n}\bar{\mathcal{V}}_{k,t-n},$$
$$(ii) \quad {\mathcal{O}^{\mathcal{L}}}_{k,t}{\mathcal{O}^{\mathcal{L}}}_{k,n+1} - {\mathcal{O}^{\mathcal{L}}}_{k,t+1}{\mathcal{O}^{\mathcal{L}}}_{k,n} = (-1)^{n+1}\delta\bar{\mathcal{V}}_{k,t-n}.$$

Proof. Using Theorem 2.4, we get

$$\begin{aligned}
{\mathcal{O}^{\mathcal{F}}}_{k,t}{\mathcal{O}^{\mathcal{F}}}_{k,n+1} - {\mathcal{O}^{\mathcal{F}}}_{k,t+1}{\mathcal{O}^{\mathcal{F}}}_{k,n} &= \Big(\frac{\bar{r_1}{r_1}^{t} - \bar{r_2}{r_2}^{t}}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n+1} - \bar{r_2}{r_2}^{n+1}}{r_1 - r_2}\Big) \\
&\quad - \Big(\frac{\bar{r_1}{r_1}^{t+1} - \bar{r_2}{r_2}^{t+1}}{r_1 - r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{n} - \bar{r_2}{r_2}^{n}}{r_1 - r_2}\Big) \\
&= \frac{1}{(r_1 - r_2)^2}\big[\bar{r_1}^2{r_1}^{n+t+1} - \bar{r_1}\bar{r_2}{r_1}^t{r_2}^{n+1} - \bar{r_2}\bar{r_1}{r_2}^t{r_1}^{n+1} + \bar{r_2}^2{r_2}^{n+t+1} \\
&\quad - \bar{r_1}^2{r_1}^{n+t+1} + \bar{r_1}\bar{r_2}({r_1}^{t+1}{r_2}^n) + \bar{r_2}\bar{r_1}({r_1}^n{r_2}^{t+1}) - \bar{r_2}^2{r_2}^{n+t+1}\big]
\end{aligned}$$

$$= \frac{(r_1r_2)^n}{(r_1-r_2)^2}\big[\bar{r_1}\bar{r_2}{r_1}^{t-n}(r_1-r_2) - \bar{r_2}\bar{r_1}{r_2}^{t-n}(r_1-r_2)\big]$$
$$= (r_1r_2)^n\big[\frac{\bar{r_1}\bar{r_2}{r_1}^{t-n} - \bar{r_2}\bar{r_1}{r_2}^{t-n}}{r_1-r_2}\big].$$

Using Lemma 2.8 and $r_1r_2 = -1$, we obtain

$$= (-1)^n\bar{\mathcal{V}}_{k,t-n}.$$

The proof of (ii) is similar to (i), using Theorem 2.4 and Lemma 2.8. □

Theorem 2.12. *For any integer t, we have*

$$(i)\quad {\mathcal{O}^{\mathcal{F}}}_{k,t}^{\ 2} + {\mathcal{O}^{\mathcal{L}}}_{k,t}^{\ 2} = \frac{1}{\delta}\big[(1+\delta)\bar{\mathcal{W}}_{k,2t} + (\delta-1)(-1)^t{\mathcal{O}^{\mathcal{L}}}_{k,0}\big],$$
$$(ii)\quad {\mathcal{O}^{\mathcal{F}}}_{k,t}^{\ 2} - {\mathcal{O}^{\mathcal{L}}}_{k,t}^{\ 2} = \frac{1}{\delta}\big[(1-\delta)\bar{\mathcal{W}}_{k,2t} - (1+\delta)(-1)^t{\mathcal{O}^{\mathcal{L}}}_{k,0}\big].$$

Proof. Using Theorem 2.4, we get

$${\mathcal{O}^{\mathcal{F}}}_{k,t}^{\ 2} + {\mathcal{O}^{\mathcal{L}}}_{k,t+1}^{\ 2} = \big(\frac{\bar{r_1}{r_1}^t - \bar{r_2}{r_2}^t}{r_1-r_2}\big)^2 + \big(\bar{r_1}{r_1}^t + \bar{r_2}{r_2}^t\big)^2$$
$$= \frac{1}{(r_1-r_2)^2}\big[(\bar{r_1})^2{r_1}^{2t} + (\bar{r_2})^2{r_2}^{2t} - \bar{r_1}\bar{r_2}r_1{r_2}^t - \bar{r_2}\bar{r_1}r_1{r_2}^t\big]$$
$$+ \big[(\bar{r_1})^2{r_1}^{2t} + (\bar{r_2})^2{r_2}^{2t} + \bar{r_1}\bar{r_2}r_1{r_2}^t + \bar{r_2}\bar{r_1}r_1{r_2}^t\big]$$
$$= \frac{(1+\delta)}{\delta}\big[(\bar{r_1})^2{r_1}^{2t} + \bar{r_2})^2{r_2}^{2t}\big) + \frac{(\delta-1)(-1)^t}{\delta}\big[\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1}\big].$$

Using Lemma 2.5, we obtain

$$= \frac{1}{\delta}\big[(1+\delta)\bar{\mathcal{W}}_{k,2t} + (\delta-1)(-1)^t{\mathcal{O}^{\mathcal{L}}}_{k,0}\big].$$

The proof of (ii) is similar to (i), using Theorem 2.4 and Lemma 2.5. □

Theorem 2.13. *For any integer r, $s \geq t$, we have*

$${\mathcal{O}^{\mathcal{F}}}_{k,r+s}{\mathcal{O}^{\mathcal{L}}}_{k,r+t} - {\mathcal{O}^{\mathcal{F}}}_{k,r+t}{\mathcal{O}^{\mathcal{L}}}_{k,r+s} = 2(-1)^{r+t}\big({\mathcal{O}^{\mathcal{L}}}_{k,0} - 2\big)F_{k,s-t}.$$

Proof. Using Theorem 2.4 and $r_1r_2 = -1$, we get

$${\mathcal{O}^{\mathcal{F}}}_{k,r+s}{\mathcal{O}^{\mathcal{L}}}_{k,r+t} - {\mathcal{O}^{\mathcal{F}}}_{k,r+t}{\mathcal{O}^{\mathcal{L}}}_{k,r+s} = \frac{1}{r_1-r_2}\big[\big(\bar{r_1}{r_1}^{r+s} - \bar{r_2}{r_2}^{r+s}\big)\cdot$$
$$\big(\bar{r_1}{r_1}^{r+t} + \bar{r_2}{r_2}^{r+t}\big) - \big(\bar{r_1}{r_1}^{r+t} - \bar{r_2}{r_2}^{r+t}\big)\big(\bar{r_1}{r_1}^{r+s} + \bar{r_2}{r_2}^{r+s}\big)\big]$$
$$= \frac{(r_1r_2)^r}{r_1-r_2}\big[(\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1}){r_1}^s{r_2}^t - (\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1}){r_1}^t{r_2}^s\big]$$
$$= \frac{(r_1r_2)^r}{r_1-r_2}\big(\bar{r_1}\bar{r_2} + \bar{r_2}\bar{r_1}\big)\big({r_1}^s{r_2}^t - {r_1}^t{r_2}^s\big).$$

Using Lemma 2.5, we obtain

$$= 2(-1)^{r+t}\big({\mathcal{O}^{\mathcal{L}}}_{k,0} - 2\big)F_{k,s-t}.$$ □

Theorem 2.14. *For any integer s, and t, we have*

$$(i)\quad \mathcal{O}^{\mathcal{F}}{}_{k,s+t}+(-1)^t\mathcal{O}^{\mathcal{F}}{}_{k,s-t}=\mathcal{O}^{\mathcal{F}}{}_{k,s}L_{k,t},$$
$$(ii)\quad \mathcal{O}^{\mathcal{L}}{}_{k,s+t}+(-1)^t\mathcal{O}^{\mathcal{L}}{}_{k,s-t}=\mathcal{O}^{\mathcal{L}}{}_{k,s}L_{k,t}.$$

Proof. Using Theorem 2.4, we get

$$\mathcal{O}^{\mathcal{F}}{}_{k,s+t}+(-1)^t\mathcal{O}^{\mathcal{F}}{}_{k,s-t}=\frac{1}{r_1-r_2}\Big[\big(\bar{r_1}{r_1}^{s+t}-\bar{r_2}{r_2}^{s+t}\big)+(-1)^t\big(\bar{r_1}{r_1}^{s-t}+\bar{r_2}{r_2}^{s-t}\big)\Big]$$
$$=\frac{1}{r_1-r_2}\Big[\bar{r_1}{r_1}^{s+t}-\bar{r_2}{r_2}^{s+t}+\bar{r_1}{r_1}^{s}{r_2}^{t}-\bar{r_2}{r_1}^{t}{r_2}^{s}\Big]$$
$$=\frac{1}{r_1-r_2}\Big[\bar{r_1}{r_1}^{s}({r_1}^{t}+{r_2}^{t})-\bar{r_2}{r_2}^{s}({r_1}^{t}+{r_2}^{t})\Big]$$
$$=\Big(\frac{\bar{r_1}{r_1}^{s}-\bar{r_2}{r_2}^{s}}{r_1-r_2}\Big)\big({r_1}^{t}+{r_2}^{t}\big).$$

Using Theorem 2.4, we obtain

$$=\mathcal{O}^{\mathcal{F}}{}_{k,s}L_{k,t}.$$

The proof of (ii) is similar to (i), using Theorem 2.4. □

Theorem 2.15. *For any integer $s\leq t$, we have*

$$(i)\quad \mathcal{O}^{\mathcal{F}}{}_{k,s}\mathcal{O}^{\mathcal{F}}{}_{k,t}-\mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{F}}{}_{k,s}=(-1)^s\delta^{-\frac{1}{2}}\bar{u_5}F_{k,t-s},$$
$$(ii)\quad \mathcal{O}^{\mathcal{L}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t}-\mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s}=(-1)^t\delta^{\frac{1}{2}}\bar{u_5}F_{k,s-t},$$
$$(iii)\quad \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s}-\mathcal{O}^{\mathcal{F}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t}=2(-1)^sF_{k,t-s}\mathcal{O}^{\mathcal{L}}{}_{k,0},$$
$$(iv)\quad \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s}-\mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{F}}{}_{k,s}=2(-1)^s\bar{\mathcal{V}}_{k,t-s}.$$

Proof. (i). Using Theorem 2.4, we have

$$\mathcal{O}^{\mathcal{L}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t}-\mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s}=\Big(\frac{\bar{r_1}{r_1}^{s}-\bar{r_2}{r_2}^{s}}{r_1-r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{t}-\bar{r_2}{r_2}^{t}}{r_1-r_2}\Big)-\Big(\frac{\bar{r_1}{r_1}^{t}-\bar{r_2}{r_2}^{t}}{r_1-r_2}\Big)\Big(\frac{\bar{r_1}{r_1}^{s}-\bar{r_2}{r_2}^{s}}{r_1-r_2}\Big)$$
$$=\frac{1}{(r_1-r_2)^2}\big({r_1}^{t}{r_2}^{s}-{r_1}^{s}{r_2}^{t}\big)\big(\bar{r_1}\bar{r_2}-\bar{r_2}\bar{r_1}\big)$$
$$=(r_1r_2)^s(r_1-r_2)^{-1}\big(\bar{r_1}\bar{r_2}-\bar{r_2}\bar{r_1}\big)\Big(\frac{{r_1}^{t-s}-{r_2}^{t-s}}{r_1-r_2}\Big)$$

Using Lemma 2.5, we obtain

$$=(-1)^s\delta^{-\frac{1}{2}}\bar{u_5}F_{k,t-s}.$$

The proof of (ii), (iii) and (iv) is similar to (i), using Theorem 2.4 and Lemma 2.5. □

Theorem 2.16. *For any integer $n\geq 0$, we have*

$$(i)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n}-\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n}=(-1)^n\delta^{-\frac{1}{2}}(2\bar{u_{10}}-\bar{u_{14}}),$$
$$(ii)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n}+\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n}=2\delta^{-\frac{1}{2}}(L_{k,t+1}+L_{k,t+3}-L_{k,t+9}-L_{k,t+13})$$
$$+(-1)^n\delta^{-\frac{1}{2}}(2\bar{u_{15}}-\bar{u_{16}}),$$
$$(iii)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{L}}{}_{k,n}-\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}=\bar{\mathcal{X}}_{k,2n}+\delta^{-\frac{1}{2}}\big((-1)^n\bar{u_5}-\bar{u_1}+\bar{u_2}\big)-\bar{\mathcal{Y}}_{k,2n}.$$

Proof. (i). Using Theorem 2.4, we have

$$\mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}^{\mathcal{L}}}_{k,n} - \bar{\mathcal{O}^{\mathcal{F}}}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n} = \big(\frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2}\big)\big(\bar{r_3}{r_1}^n + \bar{r_4}{r_2}^n\big) - \big(\frac{\bar{r_3}{r_1}^n - \bar{r_4}{r_2}^n}{r_1 - r_2}\big)\big(\bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n\big)$$

$$= \frac{(r_1 r_2)^n}{(r_1 - r_2)}\big(2\bar{r_1}\bar{r_4} - \bar{r_2}\bar{r_3} - \bar{r_3}\bar{r_2}\big)$$

Using Lemma 2.5, we obtain

$$= (-1)^n \delta^{-\frac{1}{2}}(2\bar{u_{10}} - \bar{u_{14}}).$$

The proof of (ii) and (iii) is similar to (i), using Theorem 2.4 and Lemma 2.5. □

3 Conclusion

This study introduce hyperbolic k-Fibonacci and k-Lucas octonions. Moreover, we obtain Binet‘s formulas, Catalan‘s identity, Cassini‘s identity and d‘Ocagne‘s identity for hyperbolic k-Fibonacci and k-Lucas octonions.

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