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An identity involving Bernoulli numbers and the Stirling numbers of the second kind

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Abstract: Let B_n denote the Bernoulli numbers, and $S(n,k)$ denote the Stirling numbers of the second kind. We prove the following identity

$$B_{m+n} = \sum_{\substack{0\le k\le n\\ 0\le l\le m}} \frac{(-1)^{k+l}\, k!\, l!\, S(n,k)\, S(m,l)}{(k+l+1)\binom{k+l}{l}}.$$

To the best of our knowledge, the identity is new.

Keywords: Bernoulli numbers, Stirling numbers of the second kind, Riemann zeta function, Polylogarithm function.
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1 Introduction

Definition 1.1. The *Bernoulli numbers* B_n can be defined by the following generating function:

$$\frac{t}{e^t-1} = \sum_{n\ge 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 1.2. The *Stirling number of the second kind*, denoted by $S(n,m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

The following formula expresses the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind [3, 5]:

$$B_n = \sum_{k=0}^{n} \frac{(-1)^k\, k!\, S(n,k)}{k+1}. \tag{1}$$

In the following section, we prove a new identity for the Bernoulli numbers in terms of Stirling numbers of the second kind, of which the above formula is a special case.

2 Main result

Our main result is the following.

Theorem 2.1. *For all non-negative integers m, n we have*

$$B_{m+n} = \sum_{\substack{0\le k\le n\\ 0\le l\le m}} \frac{(-1)^{k+l}\, k!\, l!\, S(n,k)\, S(m,l)}{(k+l+1)\binom{k+l}{l}}.$$

Remark 2.2. *Letting $m = 0$ in the above equation gives us equation* (1).

Proof. We start with the following integral from [2]

$$(\alpha+\beta)\zeta(\alpha+\beta+1) = \int_0^\infty \frac{\mathrm{Li}_\alpha(-1/t)\ \mathrm{Li}_\beta(-t)}{t}\, dt, \tag{2}$$

where $\zeta(\cdot)$ is the Riemann zeta function, and $\mathrm{Li}_\alpha(t)$ is the polylogarithm function.

Letting $\alpha = -m$, and $\beta = -n$ (non-negative integers) in the preceding equation, we get

$$-(m+n)\zeta(1-m-n) = \int_0^\infty \frac{\mathrm{Li}_{-m}(-1/t)\ \mathrm{Li}_{-n}(-t)}{t}\, dt.$$

The following representation from the note [4]

$$\mathrm{Li}_{-n}(-t) = \sum_{k=0}^{n} k!\, S(n,k) \left(\frac{1}{1+t}\right)^{k+1} (-t)^k \tag{3}$$

allows us to evaluate the integral as

$$\begin{aligned}\int_0^\infty \frac{\mathrm{Li}_{-m}(-1/t)\ \mathrm{Li}_{-n}(-t)}{t}\, dt &= \int_0^\infty \sum_{\substack{0\le k\le n\\ 0\le l\le m}} \frac{(-1)^{k+l}\, k!\, l!\, S(n,k)\, S(m,l)\, t^k}{(1+t)^{k+l+2}}\, dt \\ &= \sum_{\substack{0\le k\le n\\ 0\le l\le m}} (-1)^{k+l}\, k!\, l!\, S(n,k)\, S(m,l) \int_0^\infty \frac{t^k}{(1+t)^{k+l+2}}\, dt \\ &= \sum_{\substack{0\le k\le n\\ 0\le l\le m}} (-1)^{k+l}\, k!\, l!\, S(n,k)\, S(m,l) \frac{\Gamma(k+1)\Gamma(l+1)}{\Gamma(k+l+2)}.\end{aligned}$$

Here $\Gamma(\cdot)$ is the gamma function. This completes the proof after noting the fact [1] that

$$-(m+n)\cdot\zeta(1-m-n) = B_{m+n}.$$ □

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