An identity involving Bernoulli numbers and the Stirling numbers of the second kind

Sumit Kumar Jha
International Institute of Information Technology
Hyderabad-500 032, India
e-mail: kumarjha.sumit@research.iiit.ac.in

Received: 17 November 2019 Revised: 16 July 2020 Accepted: 16 July 2020

Abstract: Let $B_n$ denote the Bernoulli numbers, and $S(n,k)$ denote the Stirling numbers of the second kind. We prove the following identity

$$B_{m+n} = \sum_{0 \leq k \leq n \atop 0 \leq l \leq m} \frac{(-1)^{k+l} k! l! S(n,k) S(m,l)}{(k+l+1) \binom{k+l}{l}}.$$

To the best of our knowledge, the identity is new.

Keywords: Bernoulli numbers, Stirling numbers of the second kind, Riemann zeta function, Polylogarithm function.

2010 Mathematics Subject Classification: 11B68, 11B73.

1 Introduction

Definition 1.1. The Bernoulli numbers $B_n$ can be defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 1.2. The Stirling number of the second kind, denoted by $S(n,m)$, is the number of ways of partitioning a set of $n$ elements into $m$ nonempty sets.
The following formula expresses the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind \[3, 5\]:

\[ B_n = \sum_{k=0}^{n} \frac{(-1)^k k! S(n, k)}{k + 1}. \] (1)

In the following section, we prove a new identity for the Bernoulli numbers in terms of Stirling numbers of the second kind, of which the above formula is a special case.

## 2 Main result

Our main result is the following.

**Theorem 2.1.** For all non-negative integers \( m, n \) we have

\[ B_{m+n} = \sum_{0 \leq k \leq n} \sum_{0 \leq l \leq m} \frac{(-1)^{k+l} k! l! S(n, k) S(m, l)}{(k + l + 1) (k + l + 2)}. \]

**Remark 2.2.** Letting \( m = 0 \) in the above equation gives us equation (1).

**Proof.** We start with the following integral from [2]

\[(\alpha + \beta)\zeta(\alpha + \beta + 1) = \int_{0}^{\infty} \frac{\text{Li}_\alpha(-1/t) \text{Li}_\beta(-t)}{t} \, dt, \] (2)

where \( \zeta(\cdot) \) is the Riemann zeta function, and \( \text{Li}_\alpha(t) \) is the polylogarithm function.

Letting \( \alpha = -m \), and \( \beta = -n \) (non-negative integers) in the preceding equation, we get

\[-(m + n)\zeta(1 - m - n) = \int_{0}^{\infty} \frac{\text{Li}_{-m}(-1/t) \text{Li}_{-n}(-t)}{t} \, dt.\]

The following representation from the note [4]

\[ \text{Li}_{-n}(-t) = \sum_{k=0}^{n} k! S(n, k) \left( \frac{1}{1 + t} \right)^{k+1} (-t)^k \] (3)

allows us to evaluate the integral as

\[ \int_{0}^{\infty} \frac{\text{Li}_{-m}(-1/t) \, \text{Li}_{-n}(-t)}{t} \, dt = \int_{0}^{\infty} \sum_{0 \leq k \leq n} \sum_{0 \leq l \leq m} \frac{(-1)^{k+l} k! l! S(n, k) S(m, l) t^k}{(1 + t)^{k+l+2}} \, dt \]

\[ = \sum_{0 \leq k \leq n} \sum_{0 \leq l \leq m} (-1)^{k+l} k! l! S(n, k) S(m, l) \int_{0}^{\infty} \frac{t^k}{(1 + t)^{k+l+2}} \, dt \]

\[ = \sum_{0 \leq k \leq n} \sum_{0 \leq l \leq m} (-1)^{k+l} k! l! S(n, k) S(m, l) \frac{\Gamma(k+1) \Gamma(l+1)}{\Gamma(k+l+2)}. \]

Here \( \Gamma(\cdot) \) is the gamma function. This completes the proof after noting the fact [1] that

\[-(m + n) \cdot \zeta(1 - m - n) = B_{m+n}. \]
References


