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A single parameter Hermite–Padé series representation for Apéry's constant

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Abstract: Inspired by the results of Rhin and Viola (2001), the purpose of this work is to elaborate on a series representation for $\zeta(3)$ which only depends on one single integer parameter. This is accomplished by deducing a Hermite–Padé approximation problem using ideas of Sorokin (1998). As a consequence we get a new recurrence relation for the approximation of $\zeta(3)$ as well as a corresponding new continued fraction expansion for $\zeta(3)$, which do no reproduce Apéry's phenomenon, i.e., though the approaches are different, they lead to the same sequence of Diophantine approximations to $\zeta(3)$. Finally, the convergence rates of several series representations of $\zeta(3)$ are compared.

Keywords: Riemann zeta function, Apéry's theorem, Hermite–Padé approximation problem, Recurrence relation, Continued fraction expansion, Series representation.
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1 Introduction

The study of the arithmetical properties of the Riemann zeta function at integer arguments

$$\zeta(k) := \sum_{n\geq 1} \frac{1}{n^k} = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 \frac{\log^{k-1} x}{1-x}\, dx, \quad k = 1, 2, \ldots,$$

as well as the results related to its series representations, has fascinated quite a number of mathematicians since the first half of the XVII century [57, 58], both for its theoretical implications and practical applications [32, 67]. Indeed, everything began when in 1644 the Italian mathematician Pietro Mengoli proposed the famous Basel problem in mathematical analysis, which also has relevance to number theory. Nine decades later, this problem was solved by Leonhard Euler. In his famous book on Differential Calculus of 1755 he gave the general case [11,35]

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}, \quad k = 1, 2, \ldots,$$

which is Euler's celebrated formula, where B_{2k} are the so-called Bernoulli numbers [5,21], with $B_{2k} \in \mathbb{Q}$ for all $k \in \mathbb{N}$. The generalization of the so-called Basel problem by Euler was a very important step in number theory. Later on Euler proposed the following conjecture for the odd case,

$$\zeta(2k+1) = \frac{p_k}{q_k} \pi^{2k+1},$$

where p_k and q_k are integer numbers. However, Euler's efforts to validate his conjecture did not work out [57], and meanwhile the conjecture itself has been refuted [39].

Regardless of Euler's frustrated attempts, he was able to derive the following series representation

$$\zeta(3) = \lim_{n \to \infty} \zeta_n^E(3), \tag{1}$$

where

$$\zeta_n^E(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{n} \frac{\zeta(2k)}{(2k+1)(2k+2) 2^{2k}}. \tag{2}$$

This representation has inspired a large number of mathematicians and has been recently discovered by several authors in many different ways [18,63,64].

After these first investigations by Euler, nothing was known on the arithmetical nature of the Riemann zeta function for odd arguments, until, on a Thursday afternoon in June 1978, at 2 pm, at the Journés Arithmetiques held at Marseille–Luminy, Roger Apéry surprised the mathematical community with a talk about the irrationality of $\zeta(3)$, see for instance [9,20,50,57]. In this talk he claimed to have proofs that both $\zeta(2)$ and $\zeta(3)$ were irrational.

The rational approximants of Apéry p_n/q_n, which are also named Apéry's Diophantine approximations, approach $\zeta(3)$ as n increases, i.e. converge for sufficiently large n as

$$\lim_{n \to \infty} \left| \zeta(3) - \frac{p_n}{q_n} \right| = 0.$$

One of the most crucial ingredients in Apéry's proof is the existence of the recurrence relation [23,50,57]

$$(n+2)^3 y_{n+2} - (2n+3)(17n^2 + 51n + 39) y_{n+1} + (n+1)^3 y_n = 0, \quad n \geq 0, \tag{3}$$

which is satisfied simultaneously by both the numerators p_n and denominators q_n of the Diophantine approximations p_n/q_n to $\zeta(3)$ with the respective initial condition

$$p_0 = 0, \quad p_1 = 6, \quad q_0 = 1, \quad q_1 = 5.$$

The rational approximants p_n and q_n are also given by the explicit representation of the sequences in question [9, 20, 50]

$$q_n := \sum_{0\le k\le n} \binom{n+k}{k}^2 \binom{n}{k}^2 \quad \text{and} \quad p_n := \sum_{0\le k\le n} \binom{n+k}{k}^2 \binom{n}{k}^2 \gamma_{n,k}, \tag{4}$$

where

$$\gamma_{n,k} = \sum_{1\le j\le n} \frac{1}{j^3} + \sum_{1\le j\le k} \frac{(-1)^{j-1}}{2j^3} \binom{n+j}{j}^{-1} \binom{n}{j}^{-1}.$$

Observe that, from (3) we can deduce that

$$\det \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \frac{6}{n^3}, \tag{5}$$

see [20, 50] for more details.

In order to reformulate the recurrence relations in terms of a continued fraction representation let us recall its definition and a basic lemma. We say that a number α can be written as an infinite irregular continued fraction expansion, if it admits the following representation

$$\alpha = a_0 + \frac{b_1 \mid}{\mid a_1} + \frac{b_2 \mid}{\mid a_2} + \cdots + \frac{b_n \mid}{\mid a_n} + \cdots = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots \, a_{n-1} + \cfrac{b_n}{a_n + \ddots}}}}$$

Lemma 1.1. *[31, p. 31] Let $(p_n)_{n\ge -1}$ and $(q_n)_{n\ge -1}$ be two sequences of numbers such that $q_{-1} = 0$, $p_{-1} = q_0 = 1$ and $p_n q_{n-1} - p_{n-1} q_n \neq 0$ for $n = 0, 1, 2, \ldots$. Then there exists a unique irregular continued fraction*

$$a_0 + \frac{b_1 \mid}{\mid a_1} + \frac{b_2 \mid}{\mid a_2} + \frac{b_3 \mid}{\mid a_3} + \cdots + \frac{b_n \mid}{\mid a_n} + \cdots, \tag{6}$$

whose n-th numerator is p_n and n-th denominator is q_n, for each $n \ge 0$. More precisely,

$$a_0 = p_0, \quad a_1 = q_1, \quad b_1 = p_1 - p_0 q_1,$$

$$a_n = \frac{p_n q_{n-2} - p_{n-2} q_n}{p_{n-1} q_{n-2} - p_{n-2} q_{n-1}}, \quad b_n = \frac{p_{n-1} q_n - p_n q_{n-1}}{p_{n-1} q_{n-2} - p_{n-2} q_{n-1}}, \quad n = 0, 1, 2, \ldots$$

Using Lemma 1.1, from (5) we obtain

$$\zeta(3) = \frac{6 \mid}{\mid 5} - \frac{1 \mid}{\mid 117} - \frac{64 \mid}{\mid 535} - \cdots - \frac{n^6 \qquad \mid}{\mid (2n+1)\left(17n^2 + 17n + 5\right)} - \cdots \quad .$$

Then, recognizing that $\zeta(3) - p_0/q_0 = \zeta(3)$, it can be induced that (see [20, 50] for more details)

$$\left| \zeta(3) - \frac{p_n}{q_n} \right| = \sum_{k\ge n+1} \frac{6}{k^3 q_{k-1} q_k} = \mathcal{O}\left(q_n^{-2}\right).$$

Observe that, by the recurrence relation (3) and Poincaré's theorem [44,49] that $q_n = \mathcal{O}\left(\varpi^{4n}\right)$, where $\varpi = \sqrt{2}+1$ is the silver ratio. Moreover, by the prime number theorem, it can be shown that

$$\mathcal{L}_n := \prod_{p \leq n} p^{\left\lfloor \frac{\log n}{\log p} \right\rfloor} \leq \prod_{p \leq n} n = \mathcal{O}\left(e^{(1+\epsilon)n}\right), \quad \forall\, \epsilon > 0, \tag{7}$$

where the product is over the prime numbers p below or equal to n. Therefore, setting $v_n = 2p_n\mathcal{L}_n^3 \in \mathbb{Z}$ and $u_n = 2q_n\mathcal{L}_n^3 \in \mathbb{Z}$, we obtain $u_n = \mathcal{O}\left(\varpi^{4n}e^{3n}\right)$ and

$$\left|\zeta(3) - \frac{v_n}{u_n}\right| = \mathcal{O}\left(u_n^{-(1+\delta)}\right),$$

where

$$\delta = \frac{\log\alpha - 3}{\log\alpha + 3} = 0.080529\ldots > 0.$$

This proves Apéry's theorem by virtue of the criterion for irrationality.

Theorem 1.2. *If there is a $\delta > 0$ and a sequence $(v_n/u_n)_{n\geq 0}$ of rational numbers such that $v_n/u_n \neq x$ and*

$$\left|x - \frac{v_n}{u_n}\right| < \frac{1}{u_n^{1+\delta}}, \quad n = 0, 1, \ldots,$$

then x is irrational.

Apéry's irrationality proof of $\zeta(3)$ operates with the series representation

$$\zeta(3) = \frac{5}{2}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}},$$

which converges faster than (1), see [63,64] for more details. The same was first obtained by A. A. Markov in 1890 [38]. In addition, Apéry's recurrence relation (3) leads to the characteristic equation $\lambda^2 - 34\lambda + 1 = 0$, which is associated to the irrationality measure $\mu = 13.4178202\ldots$, see [68] for more details.

Although initially somewhat controversial, the aforementioned result inspired several mathematicians to construct different methods to explain the irrationality of $\zeta(3)$ [10,16,40,51, 60,61,66] as well as to obtain other results related with $\zeta(2k+1)$, $k \geq 1$, [6,19,24,30,34,36,59]. Apéry's phenomenon consists of the observation that some of these alternative methods leads to the same sequence of Diophantine approximations (4) to $\zeta(3)$, and therefore, to the same characteristic equation $\lambda^2 - 34\lambda + 1 = 0$, corresponding to the recurrence relation (3), which is associated to the irrationality measure [68] obtained from Apéry's results [9,10,20].

In Section 2 variants of Apéry's phenomenon are recalled in order to put the new contribution into context. Though the irrationality of $\zeta(3)$ has been shown with different approaches, always the same rational approximants of Apérys are obtained.

In Section 3, we follow the aforementioned approaches of Rhin and Viola [52,53], and present a series representation for $\zeta(3)$, which only depends on one single integer parameter. As a modification of the approach of Nesterenko (1996) we propose to replace (11) by (18). This modification has a fundamental impact for the deduction of rational approximants to $\zeta(3)$ that leads to a series representation for Apéry's constant, which converges faster than some series proposed by several other authors. In order to complete this, we will deduce a Hermite–Padé approximation problem using Sorokin's ideas [60, 65]. By this approach we obtain new rational approximants to $\zeta(3)$ that prove its irrationality, but where Apéry's phenomenon does not appear.

In Section 4 we deduce a new recurrence relation as well as a new continued fraction expansion connected to $\zeta(3)$. Finally, in Section 5 the convergence rate of several series representations of $\zeta(3)$ are compared.

2 Apéry's phenomenon

During several years Apéry's phenomenon was interpreted by prestigious mathematicians from the point of view of different analytic methods. In 1979, a few months after the appearance of Apéry's celebrated proof of the irrationality of $\zeta(3)$, the Dutch mathematician Frits Beukers (1979) interpreted Apéry's phenomenon expressing the sequence of rational approximations to $\zeta(3)$ in terms of a triple integral [13, 14, 41]

$$\begin{aligned} q_n\zeta(3) - p_n &= -\int_0^1\int_0^1 \frac{\log xy}{1-xy} L_n(x)\, L_n(y)\; dxdy && (8)\\ &= \int_0^1\int_0^1\int_0^1 \frac{\left(xyz\,(1-x)\,(1-y)\,(1-z)\right)^n}{\left(1-(1-xy)\,z\right)^{n+1}}\, dxdydz, && (9)\end{aligned}$$

where $n \in \mathbb{N}$ and

$$L_n(x) \equiv \frac{1}{n!}\frac{d^n}{dx^n}x^n(1-x)^n = \sum_{0\le k\le n} (-1)^k \binom{n+k}{k}\binom{n}{k} x^k,$$

it is the Legendre-type polynomial, orthogonal with respect to the Lebesgue measure on $(0,1)$.

Moreover, Beukers (1979) showed that (8) behaves as $\mathcal{O}\left(\varpi^{-4n}\right)$, which proves Apéry's theorem. It is important to emphasize that this proof of irrationality of $\zeta(3)$ published by Beukers (1979) is much simpler and more comprehensible compared to the original proof given by Apéry. This approach was continued in later works [22, 27–29, 53, 72].

Indeed, for $\zeta(2)$ Rhin and Viola (1996) consider for $\zeta(3)$ a generalization of

$$\int_0^1\int_0^1 \frac{\left(xy\,(1-x)\,(1-y)\right)^n}{(1-xy)^{n+1}}\, dxdy \in \mathbb{Q} - \mathbb{Z}\zeta(2),$$

given by

$$\int_0^1\int_0^1 \frac{x^h\,(1-x)^i\,y^j\,(1-y)^k}{(1-xy)^{i+j-l+1}}\, dxdy \in \mathbb{Q} - \mathbb{Z}\zeta(2),$$

which depends on the five non-negative parameters h, i, j, k and l, see [52] for more details. Later, the same authors (2001) consider a family of integrals generalizing (9) by

$$\int_0^1\int_0^1\int_0^1 \frac{x^h\,(1-x)^l\,y^k\,(1-y)^s\,z^j\,(1-z)^q}{\left(1-(1-xy)\,z\right)^{q+h-r+1}}\, dxdydz \in \mathbb{Q}{-}2\mathbb{Z}\zeta(3),$$

which depends on eight non-negative parameters h, j, k, l, m, q, r and s, subject to the conditions $j+q=l+s$, and $m=k+r-h$, see [53] for more details. Indeed, these results combined with the group method improve the irrationality measures [71] for $\zeta(2)$ and $\zeta(3)$.

Two years after Apéry's result, Beukers (1981) considered the following rational approximation problem in an attempt to formulate Apéry's proof in a more natural way. It consisted in finding the polynomials $A_n(z)$, $B_n(z)$, $C_n(z)$ and $D_n(z)$ of degree n such that

$$\begin{aligned} A_n(z)\,\mathrm{Li}_1(z)+B_n(z)\,\mathrm{Li}_2(z)+C_n(z)&=\mathcal{O}\left(z^{2n+1}\right), \\ A_n(z)\,\mathrm{Li}_2(z)+2B_n(z)\,\mathrm{Li}_3(z)+D_n(z)&=\mathcal{O}\left(z^{2n+1}\right), \end{aligned}$$

where

$$\mathrm{Li}_n(z):=\sum_{k\geq 1}\frac{z^k}{k^n},$$

denotes the polylogarithm of order n. Thereafter, Beukers introduced the rational function

$$\mathcal{B}_n(z):=\frac{(n-z+1)_n^2}{(-z)_{n+1}^2}, \tag{10}$$

from which he deduced Apéry's rational approximants (4) by computing a partial fraction expansion, see [15] for more details. Here, $(\cdot)_n$ denotes the Pochhammer symbol defined by

$$\begin{aligned} (z)_k &:= \prod_{0\leq j\leq k-1}(z+j), \quad k\geq 1, \\ (z)_0 &= 1, \quad (-z)_k=0, \quad \text{if } z<k, \end{aligned}$$

which is also called the shifted factorial and in terms of the gamma function given by

$$(z)_k=\frac{\Gamma(z+k)}{\Gamma(z)}, \quad k=0,1,2,\ldots$$

On the other hand, in 1983, Apéry's phenomenon was interpreted by Gutnik (1983) in terms of Meijer's G-functions [37], i.e.,

$$q_n\zeta(3)-p_n = G_{4,4}^{4,2}\left(\begin{array}{c|c} -n,-n,n+1,n+1 & \\ & 1 \\ 0,\quad 0,\quad 0,\quad 0 & \end{array}\right).$$

This approach allowed him to prove several partial results on the irrationality of certain quantities involving $\zeta(2)$ and $\zeta(3)$, see [26] for more details.

Later, Sorokin (1993) obtained Apéry's rational approximants (4) in a similar way as Beukers (1979), by considering the approximation problem

$$\begin{aligned} A_n(z)\,f_1(z)+B_n(z)\,f_2(z)-C_n(z)&=\mathcal{O}\left(z^{-n-1}\right), \\ A_n(z)\,f_2(z)+2B_n(z)\,f_3(z)-D_n(z)&=\mathcal{O}\left(z^{-n-1}\right), \end{aligned}$$

where $A_n(z)$ and $B_n(z)$ are polynomials of degree n and

$$f_1(z)=\int_0^1\frac{dx}{z-x}\,dx, \quad f_2(z)=-\int_0^1\frac{\log x}{z-x}\,dx, \quad f_3(z)=\frac{1}{2}\int_0^1\frac{\log^2 x}{z-x}\,dx.$$

Thus, he proved that the solution of this problem is given by the orthogonality relations

$$\begin{aligned} \int_0^1 \left(A_n(x) - B_n(x)\log x\right) x^k\, dx = 0, & \\ & k = 0, \ldots, n-1, \\ \int_0^1 \left(\left(A_n(x) - B_n(x)\log x\right)\log x\right) x^k\, dx = 0, & \end{aligned}$$

together with the additional condition $A_n(1) = 0$. Then, using the Mellin convolution [56,65,66] he obtains

$$\begin{aligned} q_n\zeta(3) - p_n &= \int_0^1 \frac{\left(A_n(x) - B_n(x)\log x\right)\log x}{1-x}\, dx \\ &= -\int_0^1\int_0^1 \frac{\log xy}{1-xy} L_n(x)\, L_n(y)\; dxdy, \end{aligned}$$

which implies the irrationality of $\zeta(3)$ according to Beukers' estimation given by (8), see for instance [13, 60, 65].

After this, inspired by Gutnik (1983), Nesterenko (1996) proposed a new proof of Apéry's theorem. For this purpose, he considered the modification

$$\mathcal{N}_n(z) := \mathcal{B}_n(z+n+1) = \frac{(-z)_n^2}{(z+1)_{n+1}^2}, \tag{11}$$

of Beukers' rational function (10) and proved the expression

$$q_n\zeta(3) - p_n = -\sum_{k\geq 0} \frac{d}{dk}\mathcal{N}_n(k) = \frac{1}{2\pi \mathrm{i}}\int_L \mathcal{N}_n(\nu)\left(\frac{\pi}{\sin \pi\nu}\right)^2 d\nu, \tag{12}$$

for the error-term sequence, where L is the vertical line Re $z = C$, $0 < C < n+1$, oriented from top to bottom and

$$\frac{d}{dz}\mathcal{N}_n(z) = 2\mathcal{N}_n(z)\left(\sum_{0\leq k\leq n-1} \frac{1}{t-k} - \sum_{1\leq k\leq n+1}\frac{1}{t+k}\right).$$

Indeed, the use of Laplace's method allowed him to estimate the above contour integral (12) yielding the behavior $\mathcal{O}\left(\varpi^{-4n}\right)$, see [41] for more details. Moreover, he discovered a new continued fraction expansion for $\zeta(3)$ using the so-called Meijer functions [37], which have the form

$$2\zeta(3) = 2 + \frac{1\,|}{|\,2} + \frac{2\,|}{|\,4} + \frac{1\,|}{|\,3} + \frac{4\,|}{|\,2} + \frac{2\,|}{|\,4} + \frac{6\,|}{|\,6} + \frac{4\,|}{|\,5} + \cdots,$$

where the numerators a_n, $n \geq 2$, and denominators b_n, $n \geq 1$, are defined by

$$\begin{aligned} b_{4k+1} &= 2k+2, & a_{4k+1} &= k(k+1), \\ b_{4k+2} &= 2k+4, & a_{4k+2} &= (k+1)(k+2), \\ b_{4k+3} &= 2k+3, & a_{4k+3} &= (k+1)^2, \\ b_{4k} &= 2k, & a_{4k} &= (k+1)^2. \end{aligned}$$

In the same year, Prévost (1996) published a new way of interpreting Apéry's phenomenon by recovering Apéry's sequences using Padé approximations to the asymptotic expansion of the partial sum of $\zeta(3)$ and proving that

$$|q_n \zeta(3) - p_n| \leq \frac{4\pi^2}{(2n+1)^2} \binom{n+k}{k}^{-2} \binom{n}{k}^{-2}.$$

Based on the hypergeometric ideas of Nesterenko [40], Rivoal and Ball [12, 54], and on Zeilberger's algorithm of creative telescoping [45], Zudilin (2002) connected Apéry's rational aproximants with the following 'very-well-posed hypergeometric series' [25, 72]

$$\begin{aligned} q_n \zeta(3) - p_n = {} & \frac{n!^7 (3n+2)!}{(2n+1)!^5} \\ & \times {}_7F_6 \left(\left. \begin{array}{c} 3n+2, \dfrac{3n}{2}+2, n+1, \ldots, n+1 \\ \\ \dfrac{3n}{2}+1, 2n+2, \ldots, 2n+2 \end{array} \right| 1 \right) < 20 (n+1)^4 \varpi^{-4n}, \end{aligned}$$

which allowed him to prove the irrationality of $\zeta(3)$, see [70] for more details. Here, ${}_rF_s$ denotes the ordinary hypergeometric series [25, 33, 43] at the variable z defined by

$$ {}_rF_s \left(\left. \begin{array}{c} a_1, \ldots, a_r \\ \\ b_1, \ldots, b_s \end{array} \right| z \right) := \sum_{k \geq 0} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}. \tag{13}$$

Apéry's phenomenon is not a necessary feature in alternative proofs of Apéry's theorem. There are also proofs of Apéry's theorem, where Apéry's phenomenon does not appear.

Zudilin (2002) deduced a new sequence of rational approximants $\{\tilde{p}_n/\tilde{q}_n\}$ to $\zeta(3)$, whose numerator $\tilde{p}_n$ and denominator $\tilde{q}_n$ satisfy the recurrence relation

$$\begin{aligned} (n+1)^4 \varphi_0(n) y_{n+1} - \varphi_1(n) y_n + 4(2n-1) \varphi_2(n) y_{n-1} \\ - 4(n-1)^2 (2n-1)(2n-3) \varphi_0(n+1) y_{n-2} = 0, \end{aligned} \tag{14}$$

with initial conditions

$$\tilde{p}_0 = 0, \quad \tilde{p}_1 = 17, \quad \tilde{p}_2 = \frac{9405}{8}, \quad \tilde{q}_0 = 1, \quad \tilde{q}_1 = 14, \quad \tilde{q}_2 = 978,$$

where

$$\varphi_0(n) = 946n^2 - 731n + 153,$$

$$\varphi_1(n) = 2\big(104060n^6 + 127710n^5 + 12788n^4 - 34525n^3 - 8482n^2 + 3298n + 1071\big),$$

and

$$\varphi_2(n) = 3784n^5 - 1032n^4 - 1925n^3 + 853n^2 + 328n - 184.$$

Here, the approach does not show Apéry's phenomenon, since the characteristic equation of (14) does not coincide with that one obtained by Apéry and the rational approximants do not prove the irrationality $\zeta(3)$, see [69] for more details.

In addition, Nesterenko (2009) published a new proof of the irrationality of $\zeta(3)$. In this work, he proved that

$$(-1)^n \mathcal{L}_n^3 \sum_{k\geq 1} \frac{\partial}{\partial k} \left(k^{-2} \prod_{j=1}^{[(n-1)/2]} \frac{k-j}{k+j} \prod_{j=1}^{[n/2]} \frac{k-j}{k+j} \right) = \\ (-1)^{n-1} \mathcal{L}_n^3 \left(2\mathcal{D}_n \zeta(3) - \mathcal{J}_n\right) < (4/5)^n,$$

where $\mathcal{D}_n$ and $\mathcal{J}_n$ are defined in [42, eq. 5]. From this statement, the irrationality of $\zeta(3)$ can be proven. Here, Apéry's phenomenon does not appear, since neither the rational approximants nor the irrationality measure coincide with Apéry's results [42].

Most recently [58], from a modification of the rational function $\mathcal{N}_n(z)$, Soria-Lorente (2014) deduced the recurrence relation

$$\begin{aligned} &(n+2)^4 \left(24n^3 + 30n^2 + 16n + 3\right) y_{n+2} \\ &\qquad - 4(n+1)(204n^6 + 1173n^5 + 2668n^4 + 3065n^3 \\ &\qquad\qquad + 1905n^2 + 634n + 86) y_{n+1} \\ &\qquad\qquad\quad + n^4 \left(24n^3 + 102n^2 + 148n + 73\right) y_n = 0, \quad n \geq 1, \end{aligned} \tag{15}$$

which is satisfied by the numerators $\hat{p}_n$ and denominators $\hat{q}_n$ of the Diophantine approximations to $\zeta(3)$ given by

$$\hat{q}_n = \sum_{0\leq k\leq n} d_k^{(n)} \quad \text{and} \quad \hat{p}_n = \sum_{1\leq k\leq n} d_k^{(n)} H_k^{(3)} + 2^{-1} \sum_{1\leq k\leq n} c_k^{(n)} H_k^{(2)}, \tag{16}$$

where

$$\begin{aligned} d_k^{(n)} &= n^{-1} \binom{n+k-1}{k}^2 \binom{n}{k}^2 + n^{-1} \binom{n+k-1}{k}^2 \binom{n-1}{k-1} \binom{n}{k}, \\ c_k^{(n)} &= 2d_k^{(n)} \left[2H_k - H_{n+k-1} - H_{n-k} - 2^{-1} (n+k)^{-1}\right], \quad k = 0, \ldots, n, \end{aligned}$$

and $H_k^{(r)}$ denotes the harmonic number k of order r defined by

$$H_k^{(r)} = \sum_{1\leq j\leq k} \frac{1}{j^r}. \tag{17}$$

Hence, the irregular continued fraction expansion

$$\zeta(3) = \frac{7 \mid}{\mid 6} + \frac{-146 \mid}{\mid 827} + \frac{-38864 \mid}{\mid \mathcal{Q}_3} + \frac{\mathcal{P}_4 \mid}{\mid \mathcal{Q}_4} + \cdots + \frac{\mathcal{P}_n \mid}{\mid \mathcal{Q}_n} + \cdots,$$

could be derived, where

$$\mathcal{P}_n = -(n-2)^4 (n-1)^4 \left(24n^3 - 186n^2 + 484n - 423\right) \left(24n^3 - 42n^2 + 28n - 7\right),$$

and

$$\mathcal{Q}_n = 4(n-1) \left(204n^6 - 1275n^5 + 3178n^4 - 3999n^3 + 2667n^2 - 910n + 126\right),$$

as well as the following series expansion

$$\zeta(3)=\frac{7}{6}+\sum_{n\geq 1}\frac{24n^3+30n^2+16n+3}{2n^3(n+1)^3\Theta_n\Theta_{n+1}},$$

with

$$\Theta_n = {}_4F_3\left(\left.\begin{array}{c}-n,-n,n,n+1\\ \\ 1,1,1\end{array}\right| 1\right).$$

Observe that the characteristic equation of (3) coincides with that of (15), which is $\lambda^2-34\lambda+1=0$, and its zeros are $\lambda_1=\varpi^{4n}$ and $\lambda_2=\varpi^{-4n}$ respectively. Hence, from Poincaré's theorem [44, 49] it has the behavior $\hat{q}_n=\mathcal{O}(\varpi^{4n})$ and $\hat{q}_n\zeta(3)-\hat{p}_n=\mathcal{O}(\varpi^{-4n})$, as n goes to infinity, which proves Apéry's theorem. Moreover, in such an instance, the corresponding irrationality measure also coincides with the one obtained by Apéry, see also [10]. However, Apéry's phenomenon does not appear in this case, since the rational approximants (16) to $\zeta(3)$ do not coincide with (4).

3 Hermite–Padé approximation problem connected to $\zeta(3)$

Our interest in this Section is to get an Hermite–Padé approximation problem connected to $\zeta(3)$, from which in the following section we deduce a new continued fraction expansion as well as a new series representation to $\zeta(3)$. For this purpose, inspired by the results obtained by Sorokin (1998) ([60], see also [52, 53, 58, 65]), we introduce the following modification of the rational function $\mathcal{N}_n(z)$ defined by

$$\mathcal{F}_{n,1}^{(\rho)}(z):=\mathcal{N}_n(z)\left(\frac{z-\rho n}{z-n+1}\right)=\frac{(-z)_{n-1}^2(z-n+1)(z-\rho n)}{(z+1)_{n+1}^2}, \tag{18}$$

which consists in changing the simple zero $z=n-1$ of the rational function $\mathcal{N}_n(z)$, (11) by the zero $z=\rho n$, with $\rho\in\mathbb{N}$. Because of its specific form, we refer to $\mathcal{F}_{n,1}^{(\rho)}$ as the Nesterenko-like rational function. For abbreviation we denote

$$\mathcal{F}_{n,2}^{(\rho)}(z)=\frac{d}{dz}\mathcal{F}_{n,1}^{(\rho)}(z).$$

Lemma 3.1. *Let ρ be an integer number. Then, the following relation*

$$\mathcal{F}_{n,i}^{(\rho)}(z)=\int_0^1\psi_{n,i}^{(\rho)}(x)\,x^z\,dx,\quad i=1,2, \tag{19}$$

holds, where

$$\psi_{n,1}^{(\rho)}(x):=A_n^{(\rho)}(x)-B_n^{(\rho)}(x)\log x\quad\text{and}\quad\psi_{n,2}^{(\rho)}(x):=\psi_{n,1}^{(\rho)}(x)\log x, \tag{20}$$

being $A_n^{(\rho)}(x)$ and $B_n^{(\rho)}(x)$ polynomials of degree exactly n defined by

$$A_n^{(\rho)}(x):=\sum_{0\leq k\leq n}a_{k,n}^{(\rho)}x^k\quad\text{and}\quad B_n^{(\rho)}(x):=\sum_{0\leq k\leq n}b_{k,n}^{(\rho)}x^k, \tag{21}$$

with

$$\begin{aligned} a_{k,n}^{(\rho)} &= 2b_{k,n}^{(\rho)}\left[2H_k - H_{n+k-1} - H_{n-k} + \frac{(\rho+1)\,n+2k+1}{2\,(k+\rho n+1)\,(n+k)}\right], \\ b_{k,n}^{(\rho)} &= \binom{n+k}{k}^2\binom{n}{k}^2 (k+\rho n+1)\,(n+k)^{-1}, \quad k=0,\ldots,n, \end{aligned} \tag{22}$$

where $H_k^{(r)}$ denotes the harmonic number k of order r given by (17).

Though the expression of the orthonogonality relation (19) resembles that of Sorokin (1993) the the approximants of Sorokin coincide with Apérys approximants, whereas the new approximants (21) do not.

Proof. In fact, let us expand the functions $\mathcal{F}_{n,1}^{(\rho)}(z)$ and $\mathcal{F}_{n,2}^{(\rho)}(z)$ on the sum of partial fractions

$$\mathcal{F}_{n,i}^{(\rho)}(z) = (-1)^{\delta_{i,2}} \sum_{0\le k\le n}\left(\frac{\tilde{a}_{k,n}^{(\rho)}}{(z+k+1)^{1+\delta_{i,2}}} + \frac{2^{\delta_{i,2}}\tilde{b}_{k,n}^{(\rho)}}{(z+k+1)^{\delta_{i,2}+2}}\right), \tag{23}$$

with $i=1,2$. Clearly

$$\tilde{b}_{k,n}^{(\rho)} = (z+k+1)^2\,\mathcal{F}_{n,1}^{(\rho)}(z)\Big|_{z=-k-1} \quad \text{and} \quad \tilde{a}_{k,n}^{(\rho)} = \operatorname*{Res}_{z=-k-1}\mathcal{F}_{n,1}^{(\rho)}(z),$$

which coincide with (22). Here, $\operatorname*{Res}_{z=z_0} f(z)$ denotes the residue of $f(z)$ at $z=z_0$. In addition, applying the identity

$$\frac{(-1)^j\,j!}{(i+1)^{j+1}} = \int_0^1 x^i \log^j x\,dx, \tag{24}$$

to (23) we have for $i=1,2$

$$\mathcal{F}_{n,i}^{(\rho)}(z) = \sum_{0\le k\le n}\int_0^1\left(a_{k,n}^{(\rho)}x^{z+k}\log^{\delta_{i,2}}x - b_{k,n}^{(\rho)}x^{z+k}\log^{1+\delta_{i,2}}x\right)dx.$$

Hence

$$\mathcal{F}_{n,i}^{(\rho)}(z) = \int_0^1\left(\sum_{0\le k\le n} a_{k,n}^{(\rho)}x^k - \log x\sum_{0\le k\le n} b_{k,n}^{(\rho)}x^k\right)x^z\,\log^{\delta_{i,2}}x\,dx,$$

which completes the proof. $\square$

For abbreviation we denote

$$\mathcal{R}_{n,i}^{(\rho)}(z) = \left(-\frac{1}{2}\right)^{\delta_{i,2}}\sum_{j\ge 0} z^{-j-1}\mathcal{F}_{n,i}^{(\rho)}(j).$$

Lemma 3.2. *Let $\mathbb{P}_n$ be the $(n+1)$-dimensional subspace of the linear space $\mathbb{P}$ of polynomials with complex coefficients. Then, the following relations hold*

$$\begin{aligned} \mathcal{R}_{n,i}^{(\rho)}(z) &= (-1)^{1+\delta_{i,2}}\,B_n^{(\rho)}(z)\,f_{\delta_{i,2}+2}(z) \\ &\quad + \left(-2^{-1}\right)^{\delta_{i,2}} A_n^{(\rho)}(z)\,f_{1+\delta_{i,2}}(z) - C_{n,i}^{(\rho)}(z) \\ &= \left(-2^{-1}\right)^{\delta_{i,2}}\int_0^1 \frac{\psi_{n,i}^{(\rho)}(x)}{z-x}\,dx, \quad i=1,2, \quad n=0,1,\ldots, \end{aligned}$$

where

$$f_j\left(z\right)=\frac{1}{(j-1)!}\int_0^1\frac{\log^{j-1}x}{z-x}\,dx,\quad j\in\mathbb{N},\tag{25}$$

as well as

$$C_{n,i}^{(\rho)}\left(z\right)=\left(-2^{-1}\right)^{\delta_{i,2}}\int_0^1\frac{\psi_{n,i}^{(\rho)}\left(z\right)-\psi_{n,i}^{(\rho)}\left(x\right)}{z-x}\,dx,\quad C_{n,i}^{(\rho)}\left(z\right)\in\mathbb{P}_n.\tag{26}$$

By $\delta_{i,j}$ we denote the Kronecker delta function.

Proof. In fact, from (23) we get

$$\mathcal{R}_{n,i}^{(\rho)}\left(z\right)=\sum_{j\geq 0}z^{-j-1}\sum_{0\leq k\leq n}\frac{b_{k,n}^{(\rho)}}{\left(j+k+1\right)^{\delta_{i,2}+2}}+2^{-\delta_{i,2}}\sum_{j\geq 0}z^{-j-1}\sum_{0\leq k\leq n}\frac{a_{k,n}^{(\rho)}}{\left(j+k+1\right)^{1+\delta_{i,2}}}.$$

Next, interchanging the sums we have

$$\begin{aligned}\mathcal{R}_{n,i}^{(\rho)}\left(z\right)&=\sum_{0\leq k\leq n}b_{k,n}^{(\rho)}z^k\sum_{j\geq 0}\frac{z^{-(j+k+1)}}{\left(j+k+1\right)^{2+\delta_{i,2}}}\\&\quad+2^{-\delta_{i,2}}\sum_{0\leq k\leq n}a_{k,n}^{(\rho)}z^k\sum_{j\geq 0}\frac{z^{-(j+k+1)}}{\left(j+k+1\right)^{1+\delta_{i,2}}}\\&=\sum_{0\leq k\leq n}b_{k,n}^{(\rho)}z^k\sum_{l\geq k+1}\frac{z^{-l}}{l^{2+\delta_{i,2}}}+2^{-\delta_{i,2}}\sum_{0\leq k\leq n}a_{k,n}^{(\rho)}z^k\sum_{l\geq k+1}\frac{z^{-l}}{l^{1+\delta_{i,2}}}.\end{aligned}$$

Splitting the sum over l as

$$\sum_{l\geq k+1}f\left(l\right)=\sum_{l\geq 1}f\left(l\right)-\sum_{1\leq l\leq k}f\left(l\right)=\left(\sum_{l\geq 1}-\sum_{1\leq l\leq k}\right)f\left(l\right),$$

we deduce

$$\mathcal{R}_{n,i}^{(\rho)}\left(z\right)=\sum_{0\leq k\leq n}b_{k,n}^{(\rho)}z^k\left(\sum_{l\geq 1}-\sum_{1\leq l\leq k}\right)\frac{z^{-l}}{l^{2+\delta_{i,2}}}+2^{-\delta_{i,2}}\sum_{0\leq k\leq n}a_{k,n}^{(\rho)}z^k\left(\sum_{l\geq 1}-\sum_{1\leq l\leq k}\right)\frac{z^{-l}}{l^{1+\delta_{i,2}}}.$$

Evidently

$$\begin{aligned}\mathcal{R}_{n}^{(\rho)}\left(z\right)=\sum_{0\leq k\leq n}b_{k,n}^{(\rho)}z^k\sum_{l\geq 1}\frac{z^{-l}}{l^{2+\delta_{i,2}}}+2^{-\delta_{i,2}}\sum_{0\leq k\leq n}a_{k,n}^{(\rho)}z^k\sum_{l\geq 1}\frac{z^{-l}}{l^{1+\delta_{i,2}}}\\-\sum_{1\leq k\leq n}b_{k,n}^{(\rho)}z^k\sum_{1\leq l\leq k}\frac{z^{-l}}{l^{2+\delta_{i,2}}}-2^{-\delta_{i,2}}\sum_{1\leq k\leq n}a_{k,n}^{(\rho)}z^k\sum_{1\leq l\leq k}\frac{z^{-l}}{l^{1+\delta_{i,2}}}.\end{aligned}$$

Clearly, from

$$\sum_{n\geq 1}\frac{z^{-n}}{n^j}=\left(-1\right)^{j-1}f_j\left(z\right),$$

we have

$$\begin{aligned}\mathcal{R}_{n,i}^{(\rho)}\left(z\right)=\left(-1\right)^{1+\delta_{i,2}}B_n^{(\rho)}\left(z\right)f_{2+\delta_{i,2}}\left(z\right)+\left(-2^{-1}\right)^{\delta_{i,2}}A_n^{(\rho)}\left(z\right)f_{1+\delta_{i,2}}\left(z\right)\\-\sum_{1\leq k\leq n}b_{k,n}^{(\rho)}z^k\sum_{1\leq l\leq k}\frac{z^{-l}}{l^{2+\delta_{i,2}}}-2^{-\delta_{i,2}}\sum_{1\leq k\leq n}a_{k,n}^{(\rho)}z^k\sum_{1\leq l\leq k}\frac{z^{-l}}{l^{1+\delta_{i,2}}}.\end{aligned}\tag{27}$$

Then, using (24) as well as

$$\int_0^1 \frac{A_n^{(\rho)}(z) - A_n^{(\rho)}(x)}{z-x} \log^{\delta_{i,2}} x \, dx = \sum_{\substack{0 \le k \le n \\ 0 \le j \le k-1}} a_{k,n}^{(\rho)} z^{k-j-1} \int_0^1 x^j \log^{\delta_{i,2}} x \, dx,$$

and

$$\int_0^1 \frac{B_n^{(\rho)}(z) - B_n^{(\rho)}(x)}{z-x} \log^{1+\delta_{i,2}} x \, dx = \sum_{\substack{0 \le k \le n \\ 0 \le j \le k-1}} b_{k,n}^{(\rho)} z^{k-j-1} \int_0^1 x^j \log^{1+\delta_{i,2}} x \, dx,$$

we deduce

$$\begin{aligned} &- \sum_{1 \le k \le n} b_{k,n}^{(\rho)} z^k \sum_{1 \le l \le k} \frac{z^{-l}}{l^{2+\delta_{i,2}}} - 2^{-\delta_{i,2}} \sum_{1 \le k \le n} a_{k,n}^{(\rho)} z^k \sum_{1 \le l \le k} \frac{z^{-l}}{l^{1+\delta_{i,2}}} \\ &\qquad = \left(-2^{-1}\right)^{\delta_{i,2}} \int_0^1 \frac{\psi_{n,i}^{(\rho)}(x) - \psi_{n,i}^{(\rho)}(z)}{z-x} dx. \qquad (28) \end{aligned}$$

Therefore, substituting the above in (27) we arrived at the first equality. Next, let us prove the second equality. According to (20) and (25) we have

$$\begin{aligned} -2^{-1} \int_0^1 \frac{\psi_{n,2}^{(\rho)}(x)}{z-x} dx &= 2^{-1} \int_0^1 \frac{A_n^{(\rho)}(z) - A_n^{(\rho)}(x)}{z-x} \log x \, dx \\ &\quad - 2^{-1} \int_0^1 \frac{B_n^{(\rho)}(z) - B_n^{(\rho)}(x)}{z-x} \log^2 x \, dx \\ &\quad - 2^{-1} A_n^{(\rho)}(z) f_2(z) + B_n^{(\rho)}(z) f_3(z) \\ &= 2^{-1} \int_0^1 \frac{\psi_{n,2}^{(\rho)}(z) - \psi_{n,2}^{(\rho)}(x)}{z-x} dx - 2^{-1} A_n^{(\rho)}(z) f_2(z) + B_n^{(\rho)}(z) f_3(z), \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \frac{\psi_{n,1}^{(\rho)}(x)}{z-x} dx = - \int_0^1 &\frac{A_n^{(\rho)}(z) - A_n^{(\rho)}(x)}{z-x} dx \\ &+ \int_0^1 \frac{B_n^{(\rho)}(z) - B_n^{(\rho)}(x)}{z-x} \log x \, dx \\ &+ A_n^{(\rho)}(z) f_1(z) - B_n^{(\rho)}(z) f_2(z) \\ &= \int_0^1 \frac{\psi_{n,1}^{(\rho)}(x) - \psi_{n,1}^{(\rho)}(z)}{z-x} dx + A_n^{(\rho)}(z) f_1(z) - B_n^{(\rho)}(z) f_2(z). \end{aligned}$$

Thus, taking (27) and (28) into account, we obtain the desired result. □

Notice that, using the identity

$$\frac{1}{z-x} = \sum_{0 \le j \le n-1-\delta_{i,2}} \frac{x^j}{z^{j+1}} + \frac{x^{n-\delta_{i,2}}}{z^{n-\delta_{i,2}}} \frac{1}{z-x}, \quad i = 1, 2,$$

as well as the previous lemma we have for $i = 1, 2$ the following

$$
\begin{aligned}
&(-2)^{\delta_{i,2}} \mathcal{R}_{n,i}^{(\rho)}(z) \\
&\qquad = \sum_{0 \leq j \leq n-1-\delta_{i,2}} \frac{1}{z^{k+1}} \int_0^1 \psi_{n,i}^{(\rho)}(x)\, x^j\, dx + \frac{1}{z^{n-\delta_{i,2}}} \int_0^1 \frac{x^{n-\delta_{i,2}} \psi_{n,i}^{(\rho)}(x)}{z-x}\, dx \\
&\qquad\qquad = \sum_{0 \leq k \leq n-1-\delta_{i,2}} \frac{1}{z^{j+1}} \int_0^1 \psi_{n,i}^{(\rho)}(x)\, x^j\, dx + \mathcal{O}\left(z^{-n-\delta_{i,1}}\right).
\end{aligned}
$$

Next, taking Lemma 3.1 into account, as well as the zeros of the rational function (18), we infer the following orthogonal conditions for $i = 1, 2$,

$$
\int_0^1 \psi_{n,i}^{(\rho)}(x)\, x^j\, dx = 0, \quad j = 0, \ldots, n - \delta_{i,2} - 1, \tag{29}
$$

from which we see that $\mathcal{R}_{n,i}^{(\rho)}(z) = \mathcal{O}\left(z^{-n-\delta_{i,1}}\right)$ for $i = 1, 2$. Moreover, since $\mathcal{F}_{n,1}^{(\rho)}(z) = \mathcal{O}\left(z^{-2}\right)$ as $z \to \infty$, we deduce

$$
A_n^{(\rho)}(1) = \sum_{0 \leq k \leq n} \operatorname*{Res}_{z=-k-1} \mathcal{F}_{n,1}^{(\rho)}(z) = -\operatorname*{Res}_{z=\infty} \mathcal{F}_{n,1}^{(\rho)}(z) = 0. \tag{30}
$$

Having in mind all the above results, we observe that the functions (19) and the polynomials (21) and (26) are connected to the following Hermite–Padé approximation problem

$$
\begin{aligned}
\tilde{B}_n^{(\rho,i)}(z)\, f_{\delta_{i,2}+2}(z) + \tilde{A}_n^{(\rho,i)}(z)\, f_{1+\delta_{i,2}}(z) - C_{n,i}^{(\rho)}(z) &= \mathcal{O}\left(z^{-n-\delta_{i,1}}\right), \\
A_n^{(\rho)}(1) &= 0,
\end{aligned}
$$

where $i = 1, 2$, $\tilde{B}_n^{(\rho,i)}(z) = (-1)^{1+\delta_{i,2}} B_n^{(\rho)}(z)$ and $\tilde{A}_n^{(\rho,i)}(z) = \left(-2^{-1}\right)^{\delta_{i,2}} A_n^{(\rho)}(z)$.

From the Hermite–Padé approximation problem of the Lemma 3.2 we can deduce Corollary 3.2.1.

Corollary 3.2.1. *Let $n \geq 1$, then the following relation*

$$
\mathcal{R}_{n,2}^{(\rho)}(1) = B_n^{(\rho)}(1)\, \zeta(3) - C_{n,2}^{(\rho)}(1) = -2^{-1} \int_0^1 \frac{\psi_{n,2}^{(\rho)}(x)}{1-x} dx, \tag{31}
$$

holds, where

$$
C_{n,2}^{(\rho)}(1) = \sum_{1 \leq k \leq n} \left(b_{k,n}^{(\rho)} H_k^3 + 2^{-1} a_{k,n}^{(\rho)} H_k^2\right).
$$

This corollary is a specific case of the Hermite–Padé problem where $z = 1$, where $B_n^{(\rho)}(1)$ and $-C_{n,2}^{(\rho)}(1)$ are the denominators and numerators of the rational approximants of $\zeta(3)$, respectively, and the $\mathcal{R}_{n,2}^{(\rho)}(1)$ are the residuals.

4 Main results

In this Section the main results of this contribution are stated. We present a new recurrence relation as well as a new continued fraction expansion and a new series expansions for $\zeta(3)$, which depends on one single integer parameter.

With the abbreviations

$$\left(r_n^{(\rho)}\right)_{n\geq 1} = \left\{\mathcal{R}_{n,2}^{(\rho)}(1)\right\}_{n\geq 1}, \quad \left(q_n^{(\rho)}\right)_{n\geq 1} = \left\{B_n^{(\rho)}(1)\right\}_{n\geq 1},$$
$$\text{and} \quad \left(p_n^{(\rho)}\right)_{n\geq 1} = \left\{C_{n,2}^{(\rho)}(1)\right\}_{n\geq 1}, \tag{32}$$

equation (31) can be rewritten as

$$r_n^{(\rho)} = q_n^{(\rho)}\zeta(3) - p_n^{(\rho)}. \tag{33}$$

According to (33) we deduce that

$$p_n^{(\rho)} q_{n+1}^{(\rho)} - p_{n+1}^{(\rho)} q_n^{(\rho)} = q_n^{(\rho)} r_{n+1}^{(\rho)} - q_{n+1}^{(\rho)} r_n^{(\rho)}. \tag{34}$$

Notice that, as a consequence of Lemma 3.2 and the orthogonality conditions (29) we have

$$\int_0^1 \frac{P_{n-1}(x)\,\psi_{n,1}^{(\rho)}(x)}{1-x}dx = P_{n-1}(1)\int_0^1 \frac{\psi_{n,1}^{(\rho)}(x)}{1-x}dx,$$
$$P_{n-1}(1)\,r_n^{(\rho)} = -2^{-1}\int_0^1 \frac{P_{n-1}(x)\,\psi_{n,2}^{(\rho)}(x)}{1-x}dx, \tag{35}$$

where $P_{n-1}(x)$ is an arbitrary polynomial of degree at most $n-1$.

Lemma 4.1. *Let $\mathcal{F}_{n,1}^{(\rho)}(z)$ be the rational function defined by (18). Then, the following relations hold*

$$\mathcal{F}_{n,2}^{(\rho)}(n-1) = \frac{(\rho n - n + 1)(n-1)!^4}{(2n)!^2},$$

$$\mathcal{F}_{n,2}^{(\rho)}(n) = -\frac{2n(\rho-1)\,n!^2}{(n+1)_{n+1}^2}\left(2H_n - H_{2n+1} - \frac{\rho n - n + 1}{2n(\rho-1)}\right),$$

and

$$\mathcal{F}_{n,1}^{(\rho)}(n) = -\frac{n(\rho-1)\,n!^2}{(n+1)_{n+1}^2},$$

Proof. To prove the Lemma it is enough to evaluate $\mathcal{F}_{n,2}^{(\rho)}(n)$ at $n-1$ and n. Indeed, the desired result follows from a tedious but straightforward verification. □

Lemma 4.2. *The sequences $\left(p_n^{(\rho)}\right)_{n\geq 1}$ and $\left(q_n^{(\rho)}\right)_{n\geq 1}$ defined by (32) satisfy the following relation*

$$\det\begin{pmatrix} p_n^{(\rho)} & q_n^{(\rho)} \\ p_{n+1}^{(\rho)} & q_{n+1}^{(\rho)} \end{pmatrix} = -\frac{\Phi_n^{(\rho)}}{2n^4(n+1)^4}, \quad \rho\in\mathbb{N}, \quad n\geq 1, \tag{36}$$

where

$$\Phi_n^{(\rho)} = 24n^5\rho^2 - 12n^5 + 54n^4\rho^2 + 39n^4\rho - 33n^4 + 46n^3\rho^2 + 70n^3\rho + 19n^2\rho^2 + 56n^2\rho + 33n^2 + 3n\rho^2 + 21n\rho + 24n + 3\rho + 5. \quad (37)$$

Proof. In fact, using (30) as well as (35) we get

$$\begin{aligned} q_n^{(\rho)} r_{n+1}^{(\rho)} &= 2^{-1} A_n^{(\rho)}(1) \int_0^1 \frac{\psi_{n+1,1}^{(\rho)}(x)}{1-x} dx - 2^{-1} \int_0^1 \frac{B_n^{(\rho)}(x)\, \psi_{n+1,2}^{(\rho)}(x)}{1-x} dx \\ &= 2^{-1} \int_0^1 \frac{\psi_{n,1}^{(\rho)}(x)\, \psi_{n+1,1}^{(\rho)}(x)}{1-x} dx. \end{aligned}$$

In addition

$$2^{-1} \int_0^1 \frac{\psi_{n,1}^{(\rho)}(x)\, \psi_{n+1,1}^{(\rho)}(x)}{1-x} dx = 2^{-1} \int_0^1 \frac{A_{n+1}^{(\rho)}(x)\, \psi_{n,1}^{(\rho)}(x)}{1-x} dx - 2^{-1} \int_0^1 \frac{B_{n+1}^{(\rho)}(x)\, \psi_{n,2}^{(\rho)}(x)}{1-x} dx,$$

where

$$\int_0^1 \frac{A_{n+1}^{(\rho)}(x)\, \psi_{n,1}^{(\rho)}(x)}{1-x} dx = - \sum_{0\le k\le n+1} a_{k,n+1}^{(\rho)} \sum_{1\le j\le k} \int_0^1 x^{j-1} \psi_{n,1}^{(\rho)}(x)\, dx = -a_{n+1,n+1}^{(\rho)} \mathcal{F}_{n,1}^{(\rho)}(n),$$

and

$$-2^{-1} \int_0^1 \frac{B_{n+1}^{(\rho)}(x)\, \psi_{n,2}^{(\rho)}(x)}{1-x} dx = 2^{-1} \sum_{0\le k\le n+1} b_{k,n+1}^{(\rho)} \sum_{1\le j\le k} \int_0^1 x^{j-1} \psi_{n,2}^{(\rho)}(x)\, dx + q_{n+1}^{(\rho)} r_n^{(\rho)}.$$

Thus, taking the relations (19) and (34) into account, as well as the orthogonality conditions (29), we deduce

$$p_n^{(\rho)} q_{n+1}^{(\rho)} - p_{n+1}^{(\rho)} q_n^{(\rho)} = 2^{-1} b_{n,n+1}^{(\rho)} \mathcal{F}_{n,2}^{(\rho)}(n-1) + 2^{-1} b_{n+1,n+1}^{(\rho)} \mathcal{F}_{n,2}^{(\rho)}(n-1) + 2^{-1} b_{n+1,n+1}^{(\rho)} \mathcal{F}_{n,2}^{(\rho)}(n) - 2^{-1} a_{n+1,n+1}^{(\rho)} \mathcal{F}_{n,1}^{(\rho)}(n). \quad (38)$$

By considering Lemma 4.1 we conclude that (38) coincides with (36), which is the desired conclusion. □

Next, we apply the so-called algorithm of creative telescoping due to Gosper and Zeilberger [1–4, 45], from which we deduce the first part of the proof. For cross-validation we implemented this algorithm in different computer algebra systems, in particular, in Maple, in Python and Mathematica.

Theorem 4.3. *Let* $\left(p_n^{(\rho)}\right)_{n\geq 1}$, $\left(q_n^{(\rho)}\right)_{n\geq 1}$ *and* $\left(r_n^{(\rho)}\right)_{n\geq 1}$ *be the sequences defined by* (32)*, where* $\left(p_n^{(\rho)}\right)_{n\geq 1}$ *and* $\left(q_n^{(\rho)}\right)_{n\geq 1}$ *satisfy the relation* (36)*. Then the following recurrence relation*

$$(n+2)^4\Phi_n^{(\rho)}y_{n+2}+\beta_n^{(\rho)}y_{n+1}+n^4\Phi_{n+1}^{(\rho)}y_n=0,\quad n\geq 1,\quad \rho\in\mathbb{N}, \tag{39}$$

holds, where

$$\begin{aligned}\beta_n^{(\rho)}=-2(n+1)(&408n^8\rho^2-204n^8+3162n^7\rho^2+663n^7\rho\\&-1683n^7+10028n^6\rho^2+4433n^6\rho-4899n^6+16802n^5\rho^2\\&+12409n^5\rho-5487n^5+16070n^4\rho^2+18955n^4\rho\\&+735n^4+8888n^3\rho^2+17212n^3\rho+7366n^3\\&+2708n^2\rho^2+9340n^2\rho+6870n^2+344n\rho^2\\&+2776n\rho+2748n+344\rho+412),\end{aligned} \tag{40}$$

and $\Phi_n^{(\rho)}$ *is given in* (37)*.*

This recurrence relation has the special property that it depends only on ρ as parameter.

Proof. The proof will be divided into three steps. In fact, firstly let us prove that the sequence $\left(q_n^{(\rho)}\right)_{n\geq 1}$ satisfies the recurrence relation (39). For such purpose, let us suppose that there exist other constants $\alpha_n^{(\rho)}$, $\hat\beta_n^{(\rho)}$ and $\gamma_n^{(\rho)}$, which are not all equal to zero, such that

$$\alpha_n^{(\rho)}q_{n+2}^{(\rho)}+\hat\beta_n^{(\rho)}q_{n+1}^{(\rho)}+\gamma_n^{(\rho)}q_n^{(\rho)}=0,\quad n\geq 0.$$

This is equivalent to

$$\sum_{0\leq k\leq n+2}\left(\alpha_n^{(\rho)}b_{k,n+2}^{(\rho)}+\hat\beta_n^{(\rho)}b_{k,n+1}^{(\rho)}+\gamma_n^{(\rho)}b_{k,n}^{(\rho)}\right)=0,$$

since $b_{j,k}^{(\rho)}=0$, for $j>k$. (Compare with (22) for the definition of the $b_{k,n}^{(\rho)}$ terms.) Therefore

$$\alpha_n^{(\rho)}b_{k,n+2}^{(\rho)}+\hat\beta_n^{(\rho)}b_{k,n+1}^{(\rho)}+\gamma_n^{(\rho)}b_{k,n}^{(\rho)}=f_n\left(k+1\right)-f_n\left(k\right), \tag{41}$$

such that $f_n\left(0\right)=f_n\left(n+3\right)=0$. According to the method of Zeilberger we can define

$$f_n\left(k\right)=\frac{k^4\pi_{3,n}\left(k\right)\binom{n+k}{k}^2\binom{n}{k}^2}{\left(n-k+1\right)^2\left(n-k+2\right)^2\left(n+k\right)}, \tag{42}$$

where $\pi_{3,n}\left(k\right)$ is a polynomial of degree 3 in k, with coefficients depending on n. From (41) and (42) the following equation

$$\begin{aligned}\alpha_n^{(\rho)}&\left(n+k\right)\left(n+k+1\right)^2\left(n+k+2\right)\left(k+\rho n+2\rho+1\right)\\&+\hat\beta_n^{(\rho)}\left(n-k+2\right)^2\left(n+k\right)\left(n+k+1\right)\left(k+\rho n+\rho+1\right)\\&+\gamma_n^{(\rho)}\left(n-k+1\right)^2\left(n-k+2\right)^2\left(k+\rho n+1\right)\\&=\left(n-k+2\right)^2\left(n+k\right)\left(n+k+1\right)\pi_{3,n}\left(k+1\right)-k^4\pi_{3,n}\left(k\right),\end{aligned}$$

holds. The above leads to a 6-equation linear system with 7-unknowns. A particular solution to this system can be obtained by computer algebra and is given by $\alpha_n^{(\rho)} = (n+2)^4 \Phi_n^{(\rho)}$, $\gamma_n^{(\rho)} = n^4 \Phi_{n+1}^{(\rho)}$ and $\hat{\beta}_n^{(\rho)} = \beta_n^{(\rho)}$, which proves that the sequence $\left(q_n^{(\rho)}\right)_{n\geq 1}$ satisfies the recurrence relation (39).

Our next goal is to prove that the sequence $\left(r_n^{(\rho)}\right)_{n\geq 1}$ satisfies the recurrence relation (39). For this purpose let us use the Lemma 4.2, from which we have

$$
\begin{aligned}
q_n^{(\rho)} r_{n+1}^{(\rho)} &= q_{n+1}^{(\rho)} r_n^{(\rho)} - \frac{\Phi_n^{(\rho)}}{2n^4 (n+1)^4}, \\
q_{n+1}^{(\rho)} r_{n+2}^{(\rho)} &= q_{n+2}^{(\rho)} r_{n+1}^{(\rho)} - \frac{\Phi_{n+1}^{(\rho)}}{2(n+1)^4 (n+2)^4},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\frac{q_n^{(\rho)}}{q_{n+1}^{(\rho)}} r_{n+1}^{(\rho)} &= r_n^{(\rho)} - \frac{\Phi_n^{(\rho)}}{2n^4 (n+1)^4 q_{n+1}^{(\rho)}}, \\
r_{n+2}^{(\rho)} &= \frac{q_{n+2}^{(\rho)}}{q_{n+1}^{(\rho)}} r_{n+1}^{(\rho)} - \frac{\Phi_{n+1}^{(\rho)}}{2(n+1)^4 (n+2)^4 q_{n+1}^{(\rho)}}.
\end{aligned}
$$

Thus, multiplying the first equation by $-n^4 \Phi_{n+1}^{(\rho)}$, the second one by $(n+2)^4 \Phi_n^{(\rho)}$, and adding both equations we deduce

$$
(n+2)^4 \Phi_n^{(\rho)} r_{n+2}^{(\rho)} + \tilde{\beta}_n^{(\rho)} r_{n+1}^{(\rho)} + n^4 \Phi_{n+1}^{(\rho)} r_n^{(\rho)} = 0,
$$

where

$$
\tilde{\beta}_n^{(\rho)} = -\frac{n^4 \Phi_{n+1}^{(\rho)} q_n^{(\rho)}}{q_{n+1}^{(\rho)}} - \frac{(n+2)^4 \Phi_n^{(\rho)} q_{n+2}^{(\rho)}}{q_{n+1}^{(\rho)}},
$$

which coincides with (40) since the sequence $\left(q_n^{(\rho)}\right)_{n\geq 1}$ satisfies the recurrence relation (39). Therefore, we conclude that $\left(r_n^{(\rho)}\right)_{n\geq 1}$ also satisfies (39).

Finally, the sequence $(p_n^{(\rho)} = q_n^{(\rho)} \zeta(3) - r_n^{(\rho)})_{n\geq 0}$ satisfies the recurrence relation (39) as a linear combination of the sequences $\left(q_n^{(\rho)}\right)_{n\geq 0}$ and $\left(r_n^{(\rho)}\right)_{n\geq 0}$. This completes the proof. $\square$

Using the expressions

$$
q_n^{(\rho)} = \sum_{1\leq k\leq n} b_{k,n}^{(\rho)} \quad \text{and} \quad p_n^{(\rho)} = \sum_{1\leq k\leq n} \left(b_{k,n}^{(\rho)} H_k^3 + 2^{-1} a_{k,n}^{(\rho)} H_k^2 \right),
$$

where H_k^r is the harmonic number k of order r as defined in (17), as well as $n b_{k,n}^{(\rho)} \in \mathbb{Z}$, $n \mathcal{L}_n a_{k,n}^{(\rho)} \in \mathbb{Z}$, and taking into account that $\mathcal{L}_n^j H_k^{(j)} \in \mathbb{Z}$ for $k = 0, 1, \ldots, n$, with $j \in \mathbb{Z}^+$, we deduce that $n q_n^{(\rho)} \in \mathbb{Z}$ and $2n\mathcal{L}_n^3 p_n^{(\rho)} \in \mathbb{Z}$. Thus, from Theorem 4.3 we have that the characteristic equation for (39) is $t^2 - 34t + 1 = 0$ and its zeros are $t_1 = \varpi^4$ and $t_2 = \varpi^{-4}$ respectively. From Poincaré's theorem [44,49] the characteristic equation has the behavior $q_n^{(\rho)} = \mathcal{O}(\varpi^{4n})$ and $r_n^{(\rho)} = \mathcal{O}(\varpi^{-4n})$, as n goes to infinity, for the two linearly independent solutions, respectively. Then, assuming that $\zeta(3) = p/q$, where $p, q \in \mathbb{Z}^+$, we have that $2qn\mathcal{L}_n^3 r_n^{(\rho)} =$

$2pn\mathcal{L}_n^3 q_n^{(\rho)} - 2qn\mathcal{L}_n^3 p_n^{(\rho)}$, is an integer different from zero. Therefore, as a consequence of the prime numbers theorem we deduce that $1 \leq 2qn\mathcal{L}_n^3 \left| r_n^{(\rho)} \right| = \mathcal{O}\left(\mathcal{L}_n^3 \varpi^{-4n}\right)$, which is a contradiction, and moreover $e^3\varpi^{-4} = 0,591263\ldots < 1$. Clearly, the above proves Apéry's theorem.

Note that the characteristic equation $t^2 - 34t + 1 = 0$ of (39) can be determined by the following steps: The coefficients of equation (39) are polynomials of order the same order, namely order 9. Moreover, the polynomials have the same leading coefficients. Therefore it is sufficient to divide all the equation by any of these polynomial coefficients and then apply the limit $n \to \infty$, which gives the characteristic equation.

An important consequence of the Theorem 4.3 is the continued fraction representation of the number $\zeta(3)$. Below we present a new continued fraction expansion for $\zeta(3)$ from our results.

Theorem 4.4. *[31, p. 31] Two irregular continued fractions*

$$a_0 + \frac{b_1 \mid}{\mid a_1} + \frac{b_2 \mid}{\mid a_2} + \frac{b_3 \mid}{\mid a_3} + \cdots + \frac{b_n \mid}{\mid a_n} + \cdots, \quad a_0' + \frac{b_1' \mid}{\mid a_1'} + \frac{b_2' \mid}{\mid a_2'} + \frac{b_3' \mid}{\mid a_3'} + \cdots + \frac{b_n' \mid}{\mid a_n'} + \cdots,$$

are equivalent if and only if there exists a sequence of non-zero $(c_n)_{n\geq 0}$ with $c_0 = 1$ such that

$$a_n' = c_n a_n, \quad n = 0, 1, 2, \ldots, \quad b_n' = c_n c_{n-1} b_n, \quad n = 1, 2, \ldots \tag{43}$$

Using the previous theorems we deduce the following results.

Corollary 4.4.1. *Let $\rho \in \mathbb{N}$, then the following irregular continued fraction expansion for $\zeta(3)$*

$$\begin{aligned}\zeta(3) = \frac{7\rho + 12 \mid}{\mid 6\rho + 10} &+ \frac{2\left(146\rho^2 + 189\rho + 17\right) \mid}{\mid \quad 1654\rho + 1981} \\ &+ \frac{-16(7\rho + 12)\left(2082\rho^2 + 1453\rho - 727\right) \mid}{\mid \quad \mathcal{Q}_3^{(\rho)}} \\ &+ \frac{\mathcal{P}_4^{(\rho)} \mid}{\mid \mathcal{Q}_4^{(\rho)}} + \cdots + \frac{\mathcal{P}_n^{(\rho)} \mid}{\mid \mathcal{Q}_n^{(\rho)}} + \cdots,\end{aligned}$$

holds, where

$$\begin{aligned}\mathcal{P}_n^{(\rho)} = &-(n-2)^4(n-1)^4(24n^5\rho^2 - 12n^5 - 306n^4\rho^2 + 39n^4\rho + 147n^4 \\ &+ 1558n^3\rho^2 - 398n^3\rho - 684n^3 - 3959n^2\rho^2 + 1532n^2\rho \\ &+ 1491n^2 + 5019n\rho^2 - 2637n\rho - 1470n - 2538\rho^2 + 1713\rho \\ &+ 473)(24n^5\rho^2 - 12n^5 - 66n^4\rho^2 + 39n^4\rho + 27n^4 + 70n^3\rho^2 \\ &- 86n^3\rho + 12n^3 - 35n^2\rho^2 + 80n^2\rho - 45n^2 + 7n\rho^2 \\ &- 37n\rho + 30n + 7\rho - 7),\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}_n^{(\rho)} = &\,2(n-1)(408n^8\rho^2 - 204n^8 - 3366n^7\rho^2 + 663n^7\rho + 1581n^7 \\ &+ 11456n^6\rho^2 - 4849n^6\rho - 4185n^6 - 20710n^5\rho^2 + 14905n^5\rho \\ &+ 3321n^5 + 21330n^4\rho^2 - 24795n^4\rho + 4425n^4 - 12488n^3\rho^2 \\ &+ 23932n^3\rho - 11066n^3 + 3892n^2\rho^2 - 13348n^2\rho + 8922n^2 \\ &- 504n\rho^2 + 4008n\rho - 3300n - 504\rho + 476).\end{aligned}$$

Theorem 4.5. *Let $\rho \in \mathbb{N}$, then the following relation*

$$\zeta(3) = \frac{7\rho + 12}{6\rho + 10} + \sum_{n \geq 1} \frac{\Phi_n^{(\rho)}}{2n^4 (n+1)^4 \Theta_n^{(\rho)} \Theta_{n+1}^{(\rho)}}, \tag{44}$$

holds, where

$$\Theta_n^{(\rho)} = \frac{\rho n + 1}{n} {}_5F_4 \left(\left. \begin{array}{c} n+1, n, -n, -n, \rho n + 2 \\ \\ 1, 1, 1, \rho n + 1 \end{array} \right| 1 \right), \tag{45}$$

and $\Phi_n^{(\rho)}$ is given in (37).

Proof. In fact, from (22) and (32) we deduce

$$q_n^{(\rho)} = \frac{\rho n + 1}{n} \sum_{0 \leq k \leq n} \frac{(n+1)_k (n)_k (-n)_k^2 (\rho n + 2)_k}{(1)_k^2 (1)_k (\rho n + 1)_k} \frac{1}{k!},$$

which corresponds with (45) according to (13). In Addition, having in account

$$\frac{p_n^{(\rho)}}{q_n^{(\rho)}} = \frac{p_1^{(\rho)}}{q_1^{(\rho)}} - \sum_{1 \leq k \leq n-1} \left(\frac{p_k^{(\rho)}}{q_k^{(\rho)}} - \frac{p_{k+1}^{(\rho)}}{q_{k+1}^{(\rho)}} \right),$$

and using (36) conjointly with

$$\zeta(3) = \lim_{n \to \infty} \frac{p_n^{(\rho)}}{q_n^{(\rho)}} = \frac{p_1^{(\rho)}}{q_1^{(\rho)}} - \sum_{n \geq 1} \left(\frac{p_n^{(\rho)} q_{n+1}^{(\rho)} - p_{n+1}^{(\rho)} q_n^{(\rho)}}{q_n^{(\rho)} q_{n+1}^{(\rho)}} \right),$$

we deduce (44). This completes the proof. □

5 Convergence

5.1 Series representations

In this paragraph several series representations of $\zeta(3)$ are recalled. Many years after Euler's results, Chen and Srivastava (1998) obtained several series representations for $\zeta(3)$, which converge faster than (2), including

$$\zeta(3) = \lim_{n \to \infty} \zeta_n^{CS}(3),$$

where

$$\zeta_n^{CS}(3) = -\frac{8\pi^2}{5} \sum_{k=0}^{n} \frac{\zeta(2k)}{(2k+1)(2k+2)(2k+3) 2^{2k}}.$$

Then, Srivastava (2000) [62] deduced the following result

$$\zeta(3) = \lim_{n \to \infty} \zeta_n^{S}(3),$$

where

$$\zeta_n^{S}(3) = -\frac{6\pi^2}{23} \sum_{k=0}^{n} \frac{(98k+121)\zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5) 2^{2k}}.$$

In addition, Borwein et al. (2000) [17] derived the following series representation

$$\zeta(3) = \lim_{n\to\infty} \zeta_n^B(3),$$

where

$$\zeta_n^B(3) = \frac{2\pi^2}{7}\left[\log 2 - \frac{1}{2} + \sum_{k=1}^{n} \frac{\zeta(2k)}{4^k(k+1)}\right].$$

Later, Pilehrood and Pilehrood (2008) [46] deduced the expression

$$\zeta(3) = \lim_{n\to\infty} \zeta_n^A(3),$$

where

$$\zeta_n^A(3) = \frac{1}{4}\sum_{k\geq 1} (-1)^{k-1} \frac{56k^2 - 32k + 5}{k^3(2k-1)^2\binom{2k}{k}\binom{3k}{k}},$$

which is known as Amdeberhan's formula for $\zeta(3)$, see [7] for more details. Then, Pilehrood and Pilehrood (2010) [47] arrived at the following expression

$$\zeta(3) = \lim_{n\to\infty} \zeta_n^{PP08}(3),$$

where

$$\zeta_n^{PP08}(3) = \sum_{k=0}^{n} (-1)^k \frac{k!^{10}\left(205k^2 + 250k + 77\right)}{64(2k+1)!^5},$$

obtained initially by Amdeberhan and Zeilberger (1997), see [8] for more details. Analogously, Pilehrood and Pilehrood (2010) [48] deduced the following formula

$$\zeta(3) = \lim_{n\to\infty} \zeta_n^{PP10}(3),$$

where

$$\zeta_n^{PP10}(3) = \frac{1}{2}\sum_{k\geq 1} (-1)^{k-1} \frac{205k^2 - 160k + 32}{k^5\binom{2k}{k}^5}.$$

More recently, Scheufens (2013) [55] obtained

$$\zeta(3) = \lim_{n\to\infty} \zeta_n^{Sch}(3),$$

where

$$\zeta_n^{Sch}(3) = -\frac{2\pi^2}{7}\sum_{k=0}^{n} \frac{\zeta(2k)}{4^k(k+1)(2k+1)},$$

and Soria-Lorente (2014) [58] deduced

$$\zeta(3) = \lim_{n\to\infty} \zeta_n^{SL}(3),$$

where

$$\zeta_n^{SL}(3) = \frac{7}{6} + \sum_{k=0}^{n} \frac{24n^3 + 30n^2 + 16n + 3}{2n^3(n+1)^3\Theta_n\Theta_{n+1}}.$$

Clearly, there are other series representations for $\zeta(3)$, and there are ongoing investigations in this direction. It is important to point out that the main result obtained in this work improves the convergence in comparison with the aforementioned results.

5.2 Convergence rates

If $\zeta_n(3)$ is the approximation at the n-th iteration and $\zeta(3)$ the exact value then the absolute error can be defined as

$$\varepsilon_n = |\zeta_n(3) - \zeta(3)|. \tag{46}$$

In Figure 1, the absolute error ε_n is visualized as a function of the index n for several iteration methods. Here, 20 iterations are realized in the index span from $n = 51$ to $n = 70$.

ζ^{SL}	ζ^{CS}	ζ^{Sr}	ζ^{B}	ζ^{PP08}	ζ^{A}	ζ^{PP10}	ζ^{Sch}
6,00E-159	8,56E-37	6,15E-38	3,48E-33	1,70E-77	6,98E-158	7,22E-155	3,29E-35
5,20E-162	2,02E-37	1,43E-38	8,53E-34	6,07E-79	6,75E-161	6,98E-158	7,93E-36
4,50E-165	4,79E-38	3,32E-39	2,09E-34	2,16E-80	6,54E-164	6,75E-161	1,91E-36
3,90E-168	1,14E-38	7,74E-40	5,14E-35	7,72E-82	6,32E-167	6,54E-164	4,61E-37
3,38E-171	2,69E-39	1,80E-40	1,26E-35	2,76E-83	6,12E-170	6,32E-167	1,11E-37
2,93E-174	6,39E-40	4,21E-41	3,10E-36	9,86E-85	5,93E-173	6,12E-170	2,68E-38
2,54E-177	1,52E-40	9,85E-42	7,63E-37	3,53E-86	5,74E-176	5,93E-173	6,48E-39
2,20E-180	3,61E-41	2,30E-42	1,88E-37	1,26E-87	5,56E-179	5,74E-176	1,57E-39
1,90E-183	8,59E-42	5,40E-43	4,61E-38	4,52E-89	5,38E-182	5,56E-179	3,79E-40
1,65E-186	2,05E-42	1,26E-43	1,13E-38	1,62E-90	5,22E-185	5,38E-182	9,18E-41
1,43E-189	4,87E-43	2,97E-44	2,79E-39	5,80E-92	5,05E-188	5,22E-185	2,22E-41
1,24E-192	1,16E-43	6,98E-45	6,87E-40	2,08E-93	4,90E-191	5,05E-188	5,38E-42
1,07E-195	2,78E-44	1,64E-45	1,69E-40	7,47E-95	4,74E-194	4,90E-191	1,30E-42
9,30E-199	6,63E-45	3,86E-46	4,16E-41	2,68E-96	4,60E-197	4,74E-194	3,16E-43
8,06E-202	1,58E-45	9,10E-47	1,03E-41	9,63E-98	4,46E-200	4,60E-197	7,68E-44
6,98E-205	3,79E-46	2,14E-47	2,53E-42	3,46E-99	4,32E-203	4,46E-200	1,86E-44
6,05E-208	9,07E-47	5,06E-48	6,23E-43	1,24E-100	4,19E-206	4,32E-203	4,52E-45
5,24E-211	2,17E-47	1,20E-48	1,53E-43	4,47E-102	4,06E-209	4,19E-206	1,10E-45
4,54E-214	5,21E-48	2,83E-49	3,78E-44	1,61E-103	3,94E-212	4,06E-209	2,67E-46
3,93E-217	1,25E-48	6,68E-50	9,32E-45	5,79E-105	3,82E-215	3,94E-212	6,49E-47

Table 1. Convergence of several iterations.

Table 1 shows the convergence of several iteration methods. On a logarithmic y-scale the error plot is a straight line, allowing a linear curve fit by the exponential model

$$\varepsilon_n = qe^{\beta n}. \tag{47}$$

Taking the logarithm on both sides gives the linear model

$$\ln \varepsilon_n = \ln q + \beta n,$$

where the parameters from the $\varepsilon_n,\ n = 1, \ldots, N$ can be calculated by solving the overdetermined

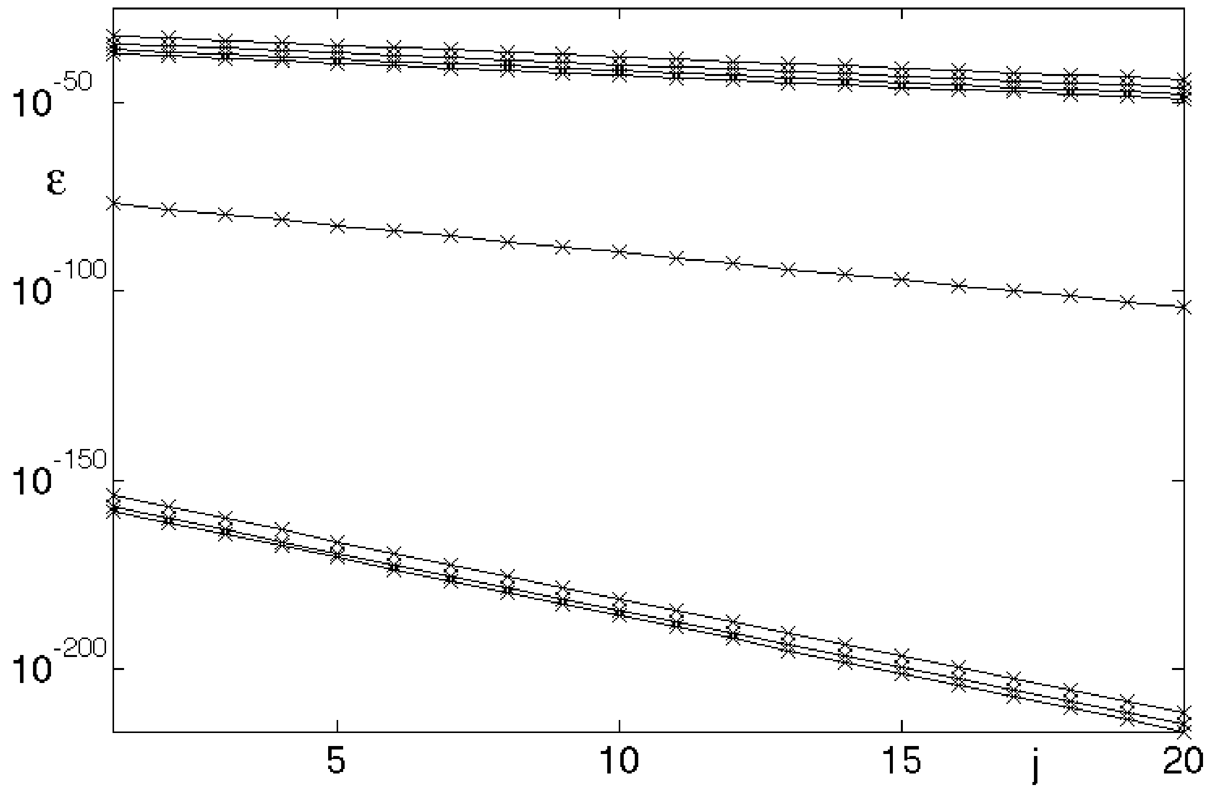

Fig. 1. Error reduction rates

system of linear equations

$$\begin{pmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & N \end{pmatrix} \begin{pmatrix} \ln q \\ \beta \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{pmatrix},$$

by minimal squares. A variant of (47) with arbitrary basis (e.g. $b = 1/10$ for a decimal number system) is

$$\varepsilon = q(b^n)^r. \tag{48}$$

The parameter r in model (48) can be deduced from model (47) by $r = \beta/\ln b$. The basis $b = 1/10$ gives the number of digits obtained by one iteration; increasing the index by one corresponds to reducing the error by the factor $(1/10)^r$. In Table 2 the convergence parameters according to several authors are compared.

	ζ^{SL}	ζ^{CS}	ζ^{Sr}	ζ^{B}	ζ^{PP08}	ζ^{A}	ζ^{PP10}	ζ^{Sch}
$\ln q$	-357.27	-81.64	-84.26	-73.35	-173.46	-354.93	-347.99	-78.00
β	-7.05	-1.43	-1.45	-1.40	-3.33	-6.94	-6.94	-1.42
q	6.9e-156	3.5e-36	2.5e-37	1.4e-32	4.7e-76	7.2e-155	7.4e-152	1.3e-34
$r\ (b=2)$	10.17	2.07	2.09	2.02	4.80	10.01	10.01	2.05
$r\ (b=10)$	3.06	0.62	0.63	0.61	1.45	3.01	3.01	0.62

Table 2. Convergence parameters.

One can distinguish three groups of method that can be classified by their convergence rate, namely Soria (2014), Amdeberhan (1996) and Pilehrood and Pilehrood (2010) with a convergence rate of $r \approx 3$, Pilehrood and Pilehrood (2008) with $r \approx 1.45$ and the others with $r \approx 0.6$.

The rate between two subsequent errors can be calculated from

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} = b^r,$$

as

$$r = \frac{1}{\ln b} \ln\left(\frac{\varepsilon_{n+1}}{\varepsilon_n}\right).$$

For the basis $b = 2$ there is the general tendency that the rates decrease, i.e., move towards the integer values $(2, 10)$, but move away from 5.

The methods of Amdeberhan (1996) and Pilehrood and Pilehrood (2010) have exactly the same error rate, which are only shifted by one index value. The reason is that one is derived from the other such that both use the same generation mechanism.

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