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Bi-unitary multiperfect numbers, III

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Dedicated to the memory of Prof. D. Suryanarayana

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Abstract: A divisor d of a positive integer n is called a unitary divisor if $\gcd(d, n/d) = 1$; and d is called a bi-unitary divisor of n if the greatest common unitary divisor of d and n/d is unity. The concept of a bi-unitary divisor is due to D. Surynarayana (1972). Let $\sigma^{**}(n)$ denote the sum of the bi-unitary divisors of n. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis (1987) proved that there are no odd bi-unitary multiperfect numbers. The present paper is part III in a series of papers on even bi-unitary multiperfect numbers. In parts I and II we found all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 5$ and u is odd. There exist exactly six such numbers. In this part we examine the case $a = 6$. We prove that if $n = 2^6 u$ is a bi-unitary triperfect number, then $n = 22848$, $n = 342720$, $n = 51979200$ or $n = 779688000$.

Keywords: Perfect numbers, Triperfect numbers, Multiperfect numbers, Bi-unitary analogues.
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1 Introduction

Throughout this paper, all lower case letters denote positive integers; p and q denote primes. The letters u, v and w are reserved for odd numbers.

A divisor d of n is called a unitary divisor if $\gcd(d, n/d) = 1$. If d is a unitary divisor of n, we write $d \| n$. A divisor d of n is called a *bi-unitary* divisor if $(d, n/d)^{**} = 1$, where the symbol $(a, b)^{**}$ denotes the greatest common unitary divisor of a and b. The concept of a bi-unitary divisor is due to D. Suryanarayana (cf. [5]). Let $\sigma^{**}(n)$ denote the sum of bi-unitary divisors of n. The function $\sigma^{**}(n)$ is multiplicative, that is, $\sigma^{**}(1) = 1$ and $\sigma^{**}(mn) = \sigma^{**}(m)\sigma^{**}(n)$ whenever $(m, n) = 1$.

The concept of a bi-unitary perfect number was introduced by C. R. Wall [6]; a positive integer n is called a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. C. R. Wall [6] proved that there are only three bi-unitary perfect numbers, namely 6, 60 and 90. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis [1] proved that there are no odd bi-unitary multiperfect numbers. Our present paper is part III in a series of papers on even bi-unitary multiperfect numbers. In part I (see [2]) we found all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 3$ and u is odd. We proved that if $1 \leq a \leq 3$ and $n = 2^a u$ is a bi-unitary triperfect number, then $a = 3$ and $n = 120 = 2^3.3.5$. In part II (see [3]) we considered the cases $a = 4$ and $a = 5$. We proved that if $n = 2^4 u$ is a bi-unitary triperfect number, then $n = 2160 = 2^4.3^3.5$, and that if $n = 2^5 u$ is a bi-unitary triperfect number, then $n = 672 = 2^5.3.7$, $n = 10080 = 2^5.3^2.5.7$, $n = 528800 = 2^5.3.5^2.13$ or $n = 22932000 = 2^5.3^2.5^3.7^2.13$.

In the present part we investigate bi-unitary triperfect numbers of the form $n = 2^6 u$. We prove in Theorem 3.1 that if $n = 2^6 u$ is a bi-unitary triperfect number, then $n = 22848 = 2^6.3.7.17$, $n = 342720 = 2^6.3^2.5.7.17$, $n = 51979200 = 2^6.3.5^2.7^2.13.17$ or $n = 779688000 = 2^6.3^2.5^3.7^2.13.17$.

To sum up, the cases $a = 1$ and $a = 2$ give no bi-unitary triperfect numbers, the cases $a = 3$ and $a = 4$ produce both one bi-unitary triperfect number, and the cases $a = 5$ and $a = 6$ yield both four bi-unitary triperfect numbers.

For a general account on various perfect-type numbers, we refer to [4].

2 Preliminaries

We assume that the reader has part I (see [2]) available. We, however, recall Lemmas 2.1 and 2.2 from part I, because they are so important also here.

Lemma 2.1. (I) *If α is odd, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} > \frac{\sigma^{**}(p^{\alpha+1})}{p^{\alpha+1}}$$

for any prime p.
(II) *For any $\alpha \geq 2\ell - 1$ and any prime p,*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} \geq \left(\frac{1}{p-1}\right)\left(p - \frac{1}{p^{2\ell}}\right) - \frac{1}{p^\ell} = \frac{1}{p^{2\ell}}\left(\frac{p^{2\ell+1}-1}{p-1} - p^\ell\right).$$

(III) *If p is any prime and α is a positive integer, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} < \frac{p}{p-1}.$$

Remark 2.1. (I) and (III) of Lemma 2.1 are mentioned in C. R. Wall [6]; (II) of Lemma 2.1 has been used by him [6] without explicitly stating it.

Lemma 2.2. *Let $a > 1$ be an integer not divisible by an odd prime p and let α be a positive integer. Let r denote the least positive integer such that $a^r \equiv 1 \,(mod\ p^{\alpha})$; then r is usually denoted by $ord_{p^{\alpha}}\, a$. We have the following properties.*

(i) If r is even, then $s = r/2$ is the least positive integer such that $a^s \equiv -1 \,(mod\ p^{\alpha})$. Also, $a^t \equiv -1 \,(mod\ p^{\alpha})$ for a positive integer t if and only if $t = su$, where u is odd.

(ii) If r is odd, then $p^{\alpha} \nmid a^t + 1$ for any positive integer t.

Remark 2.2. Let a, p, r and $s = r/2$ be as in Lemma 2.2 ($\alpha = 1$). Then $p|a^t - 1$ if and only if $r|t$. If t is odd and r is even, then $r \nmid t$. Hence $p \nmid a^t - 1$. Also, $p|a^t + 1$ if and only if $t = su$, where u is odd. In particular if t is even and s is odd, then $p \nmid a^t + 1$. In order to check the divisibility of $a^t - 1$ (when t is odd) by an odd prime p, we can confine to those p for which $ord_p\, a$ is odd. Similarly, for examining the divisibility of $a^t + 1$ by p when t is even, we need to consider primes p with $s = ord_p\, a/2$ even.

3 Bi-unitary triperfect numbers of the form $n = 2^6 u$

Theorem 3.1. *Assume that n is a bi-unitary triperfect number with $2^6 \| n$.*

(a) Then $n = 2^6.7^b.17^c.v$, where $b = 1$ or $b = 2$ and v is prime to $2.7.17$.

(b) If $b = 1$, then $n = 2^6.3.7.17 = 22848$ or $n = 2^6.3^2.5.7.17 = 342720$.

(c) If $b = 2$, then $n = 2^6.3.5^2.7^2.13.17 = 51979200$ or $n = 2^6.3^2.5^3.7^2.13.17 = 779688000$.

Proof. Let $n = 2^6 u$, where u is odd, be a bi-unitary triperfect number so that $\sigma^{**}(n) = 3n$. Hence

$$3.2^6.u = 3n = \sigma^{**}(n) = \sigma^{**}(2^6)\sigma^{**}(u) = 7.17.\sigma^{**}(u),$$

so that

$$3.2^6.u = 7.17.\sigma^{**}(u). \tag{3.1}$$

From (3.1), 7 and 17 are factors of u. So we may assume that $u = 7^b.17^c.v$, where v is odd and relatively prime to 7.17. We now have

$$n = 2^6.7^b.17^c.v. \tag{3.1a}$$

Also, from (3.1),

$$3.2^6.7^b.17^c.v = 7.17.\sigma^{**}(7^b)\sigma^{**}(17^c)\sigma^{**}(v),$$

and after simplification we get

$$3.2^6.7^{b-1}.17^{c-1}.v = \sigma^{**}(7^b)\sigma^{**}(17^c)\sigma^{**}(v), \tag{3.1b}$$

where v cannot have more than four odd prime factors.

We prove Theorem 3.1 in this sequence: *(b)*, *(c)*, and *(a)*.

Proof of (b) of Theorem 3.1. Let $b = 1$. Then taking $b = 1$ in $(3.1a)$ we obtain

$$n = 2^6.7.17^c.v. \tag{3.2a}$$

Since $\sigma^{**}(7) = 8$, taking $b = 1$ in $(3.1b)$, we get $3.2^6.17^{c-1}.v = 8.\sigma^{**}(17^c)\sigma^{**}(v)$ and on simplification we obtain

$$3.2^3.17^{c-1}.v = \sigma^{**}(17^c)\sigma^{**}(v), \tag{3.2b}$$

and v has no more than two odd prime factors.

Case $(b = 1, c = 1)$. Let $c = 1$. From $(3.2b)$, we get $3.2^3.v = 18.\sigma^{**}(v)$ or

$$2^2.v = 3.\sigma^{**}(v). \tag{3.2c}$$

This implies $3|v$ so that $v = 3^d.w$, where $(w, 2.3.7.17) = 1$. From $(3.2a)$ and $(3.2c)$ we obtain

$$n = 2^6.7.17.3^d.w, \tag{3.3a}$$

and

$$2^2.3^{d-1}.w = \sigma^{**}(3^d).\sigma^{**}(w), \tag{3.3b}$$

where w has at most one odd prime factor.

We have for $d \geq 3$, $\frac{\sigma^{**}(3^d)}{3^d} \geq \frac{112}{81}$. Hence for $d \geq 3$, from $(3.3a)$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{8}{7}.\frac{18}{17}.\frac{112}{81} = 3.11 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

Let $d = 1$. From $(3.3b)$, $2^2.w = 4.\sigma^{**}(w)$, so that $w = \sigma^{**}(w)$. Hence $w = 1$. Thus $(3.3b)$ is satisfied when $d = 1$. So from $(3.3a)$ $(d = 1)$, $n = 2^6.7.17.3 = 22848$ is a bi-unitary triperfect number.

Let $d = 2$. From $(3.3b)$, $(d = 2)$, we obtain $2^2.3.w = 10.\sigma^{**}(w)$ or

$$2.3.w = 5.\sigma^{**}(w). \tag{3.4}$$

Hence $5|w$. From (3.4), w can have at most one odd prime factor and so $w = 5^e$. Using this in $(3.3a)$ and (3.4), we get

$$n = 2^6.7.17.3^2.5^e, \tag{3.4a}$$

and

$$2.3.5^{e-1} = \sigma^{**}(5^e). \tag{3.4b}$$

If $e \geq 2$, from $(3.4b)$ it follows that $5|\sigma^{**}(5^e)$. This is not possible. Hence $e = 1$ and for this value $(3.4b)$ is satisfied. Thus $n = 2^6.7.17.3^2.5 = 342720$ is a bi-unitary triperfect number.

The case $(b = 1, c = 1)$ is complete.

Case $(b = 1, c \geq 2)$. The relevant equations are $(3.2a)$ and $(3.2b)$ with $c \geq 2$. We now prove that n in $(3.2a)$ cannot be a bi-unitary triperfect number when $c \geq 2$.

We obtain a contradiction to $(3.2b)$, by examining the factors of $\sigma^{**}(17^c)$. We distinguish the following cases:

Case I. Let c be odd so that $c \geq 3$. We have $\sigma^{**}(17^c) = \frac{17^{c+1}-1}{16}$. Since $c+1$ is even, $17^{c+1} \equiv 1 (\text{mod } 9)$. Hence $9|\sigma^{**}(17^c)$. From $(3.2b)$, it follows that $3|v$. Hence $v = 3^d w$, where w is prime to 2.3.7.17; using this in $(3.2a)$ and $(3.2b)$, we obtain

$$n = 2^6.7.17^c.3^d.w, \tag{3.5a}$$

and

$$2^3.17^{c-1}.3^{d+1}.w = \sigma^{**}(17^c).\sigma^{**}(3^d).\sigma^{**}(w), \tag{3.5b}$$

where w has at most one odd prime factor.

Since $c \geq 3$, by Lemma 2.1 ($\ell = 2$), $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}$; also, for $d \geq 3$, $\frac{\sigma^{**}(3^d)}{3^d} \geq \frac{112}{81}$; using these results from $(3.5a)$, we obtain for $d \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{8}{7}.\frac{88452}{83521}.\frac{112}{81} = 3.11 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

If $d = 1$, from $(3.5a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{8}{7}.\frac{88452}{83521}.\frac{4}{3} = 3.000610625 > 3,$$

a contradiction.

Let $d = 2$. From $(3.5a)$ and $(3.5b)$, we have $n = 2^6.7.17^c.3^2.w$, and

$$2^3.17^{c-1}.3^3.w = 10.\sigma^{**}(17^c).\sigma^{**}(w) \text{ or } 2^2.17^{c-1}.3^3.w = 5.\sigma^{**}(17^c).\sigma^{**}(w);$$

the last equation implies that $5|w$ and so $w = 5^e.w'$. Using this, we get

$$n = 2^6.7.17^c.3^2.5^e.w', \tag{3.6a}$$

and

$$2^2.17^{c-1}.3^3.5^{e-1}.w' = \sigma^{**}(17^c).\sigma^{**}(5^e).\sigma^{**}(w'). \tag{3.6b}$$

From $(3.6b)$, we have $w' = 1$. Rewriting $(3.6a)$ and $(3.6b)$, by replacing w' by 1 we get

$$n = 2^6.7.17^c.3^2.5^e, \tag*{$(3.6a)'$}$$

and

$$2^2.17^{c-1}.3^3.5^{e-1}.w' = \sigma^{**}(17^c).\sigma^{**}(5^e). \tag*{$(3.6b)'$}$$

By Lemma 2.1, for $e \geq 3$, $\frac{\sigma^{**}(5^e)}{5^e} \geq \frac{756}{625}$. Hence for $e \geq 3$, from $(3.6a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{8}{7}.\frac{88452}{83521}.\frac{10}{9}.\frac{756}{625} = 3.02461551 > 3,$$

a contradiction.

Hence $e = 1$ or $e = 2$.

If $e = 1$ then from $(3.6a)'$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{8}{7}.\frac{88452}{83521}.\frac{10}{9}.\frac{6}{5} = 3.000610625 > 3,$$

a contradiction.

Let $e = 2$. From $(3.6b)'$, we have $2^2.17^{c-1}.3^3.5.w' = 26.\sigma^{**}(17^c)$ or

$$2.17^{c-1}.3^3.5.w' = 13.\sigma^{**}(17^c). \tag{3.6c}$$

From the equation $(3.6c)$, we infer that $w' = 1$. From $(3.6c)$, we find that 13 divides its left-hand side. This is not possible. Hence $d = 2$ is not possible.

Thus $n = 2^6.7.17^c.v$ cannot be a bi-unitary triperfect number when c is odd and $c \geq 2$.

This completes Case I.

Case II. Let c be even, so that $c = 2k$. Then

$$\sigma^{**}(17^c) = \left(\frac{17^k - 1}{16}\right).(17^{k+1} + 1). \tag{3.7}$$

(i) Let k be even. Then $32|17^2 - 1|17^k - 1$. Hence each of the factors on the right of (3.7) is even so that $4|\sigma^{**}(17^c)$. From $(3.2b)$ it follows that v in $(3.2b)$ can have at most one odd prime factor. Since k is even, $9|17^k - 1$ so that $9|\frac{17^k-1}{16}|\sigma^{**}(17^c)$. Hence from $(3.2b)$, $3|v$ and so $v = 3^d$. From $(3.2a)$, we have

$$n = 2^6.7.17^c.3^d. \tag{3.7a}$$

Since $c = 2k$ and k is even, $c \geq 4$. From $(3.7a)$, for $d \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{8}{7}.\frac{88452}{83521}.\frac{112}{81} = 3.111744352 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

Let $d = 1$. From $(3.7a)$, $(d = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{8}{7}.\frac{88452}{83521}.\frac{4}{3} = 3.0006106256 > 3,$$

a contradiction.

Let $d = 2$. From $(3.7a)$, $(d = 2)$,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{8}{7}.\frac{17}{16}.\frac{10}{9} = 2.508680556 < 3,$$

a contradiction.

Hence $c = 2k$ and k is even (or same as $4|c$) is not admissible.

(ii) Let k be odd. We now prove that $k \geq 3$.

On the contrary, let $k = 1$ so that $c = 2$. Since $\sigma^{**}(17^2) = 290$, taking $c = 2$ in $(3.2b)$, we obtain after simplification,

$$2^2.3.17.v = 5.29.\sigma^{**}(v). \tag{3.7b}$$

It follows from $(3.7b)$ that v is divisible by 5 and 29. Since v can have at most two odd prime factors, $v = 5^e.29^f$. From $(3.2a)$, we have $n = 2^6.7.17^2.5^e.29^f$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{8}{7}.\frac{290}{289}.\frac{5}{4}.\frac{29}{28} = 2.760635504 < 3,$$

a contradiction.

Hence we may assume that $k \geq 3$. Hence $\dfrac{17^k - 1}{16} > 1$.

Since k is odd, $16 \| 17^k - 1$. Also, $2 \| 17^{k+1} + 1$. Further, 3 neither divides $17^k - 1$ nor $17^{k+1} + 1$. Hence $\dfrac{17^k - 1}{16}$ and $17^{k+1} + 1$ are relatively prime. Also, $5 | 17^t - 1$ if and only if $4|t$. In particular, t should be even. Since k is odd, $5 \nmid 17^k - 1$. If p and q are odd prime factors of $\dfrac{17^k - 1}{16}$ and $17^{k+1} + 1$, respectively, then $p \neq q,\ p \notin \{3, 5, 17\}$ and $q \notin \{3, 17\}$.

If $\dfrac{k+1}{2}$ is odd, then $290 = 17^2 + 1 | 17^{k+1} + 1$. In this case it follows from (3.7) that $\sigma^{**}(17^c)$ is divisible by three odd prime factors, namely, p, 5 and 29. From $(3.2b)$, it follows that v is divisible by these three odd prime factors; this leads to a contradiction since v cannot have more than two odd prime factors.

If $\dfrac{k+1}{2}$ is even, then $4 | k+1$. And so, $5 | 17^{k+1} - 1$. Hence $5 \nmid 17^{k+1} + 1$. In this case $\sigma^{**}(17^c)$ is divisible by two distinct odd primes p and q; also, $p, q \notin \{3, 5, 17\}$. From $(6b)$ it follows that v is divisible by p and q. Since v has at most two odd prime factors, $v = p^d q^e$. Since $7 \nmid v$, we can assume that $p \geq 11$ and $q \geq 13$. From $(3.2a)$, $n = 2^6.7.17^c.p^d.q^e$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{8}{7}.\frac{17}{16}.\frac{11}{10}.\frac{13}{12} = 2.690559896 < 3,$$

a contradiction.

The proof of Case II is complete.

The case $b = 1$ is finished. This completes the proof of (b) of Theorem 3.1.

Proof of (c) of Theorem 3.1. Let $b = 2$. Since $\sigma^{**}(7^2) = 50$, taking $b = 2$ in $(3.1b)$, we get after simplification, $3.2^5.7.17^{c-1}.v = 5^2.\sigma^{**}(17^c).\sigma^{**}(v)$; this implies that $5^2 | v$. Writing $v = 5^d.w$, where $d \geq 2$, we obtain from $(3.1a)$ and $(3.1b)$,

$$n = 2^6.7^2.17^c.5^d.w, \quad (d \geq 2) \tag{3.8a}$$

and

$$3.2^5.7.17^{c-1}.5^{d-2}.w = \sigma^{**}(17^c).\sigma^{**}(5^d)\sigma^{**}(w), \tag{3.8b}$$

and w has no more than three odd prime factors and prime to 2.5.7.17.

Case $(b=2, d=2)$. Since $\sigma^{**}(5^2)=26$, from $(3.2b)$ $(d=2)$, we get after simplification,

$$3.2^4.7.17^{c-1}.w = 13.\sigma^{**}(17^c).\sigma^{**}(w). \tag{3.8c}$$

From this equation, we infer that $13|w$. Let $w=13^e.w'$. From $(3.8a)$, we get,

$$n = 2^6.7^2.17^c.5^2.13^e.w', \tag{3.9a}$$

and from $(3.8c)$, we have

$$3.2^4.7.17^{c-1}.13^{e-1}.w' = \sigma^{**}(17^c).\sigma^{**}(13^e).\sigma^{**}(w'), \tag{3.9b}$$

where w' has at most two odd prime factors.

Let $c=1$. From $(3.9b)$, $(c=1)$, we get after simplification

$$2^3.7.13^{e-1}.w' = 3.\sigma^{**}(13^e).\sigma^{**}(w'). \tag{3.9c}$$

It follows from $(3.9c)$ that $3|w'$. Let $w'=3^f.w''$. From $(3.9a)$, we have

$$n = 2^6.7^2.17.5^2.13^e.3^f.w'', \tag{3.10a}$$

and from $(3.9c)$,

$$2^3.7.13^{e-1}.3^{f-1}.w'' = \sigma^{**}(13^e).\sigma^{**}(3^f).\sigma^{**}(w''), \tag{3.10b}$$

where w'' has at most one odd prime factor and prime to $2.3.5.7.13.17$.

Let $e=1$ (already $b=2, d=2, c=1$). Taking $e=1$ in $(3.10b)$, we get after simplification,

$$2^2.3^{f-1}.w'' = \sigma^{**}(3^f).\sigma^{**}(w''). \tag{3.10c}$$

If $f=1$, then from $(3.10c)$, we get $w''=\sigma^{**}(w'')$ so that $w''=1$. Thus $(3.10c)$ is satisfied when $f=1$. Taking $e=1$, $f=1$ and $w''=1$ in $(3.10a)$, we see that $n=2^6.7^2.17.5^2.13.3 = 51979200$ is a bi-unitary triperfect number.

If $f=2$, from $(3.10c)$, we find that $5|w''$. But w'' is prime to 5. So we may assume that $f\geq 3$; hence $\frac{\sigma^{**}(3^f)}{3^f}\geq\frac{112}{81}$. From $(3.10c)$, we have

$$\frac{4}{3} = \frac{\sigma^{**}(3^f)}{3^f}.\frac{\sigma^{**}(w'')}{w''} \geq \frac{\sigma^{**}(3^f)}{3^f} \geq \frac{112}{81},$$

which is false.

Hence $e=1$ is complete. Let $e=2$. From $(3.10b)$, $(e=2)$, we get

$$2^3.7.13.3^{f-1}.w'' = 170.\sigma^{**}(3^f).\sigma^{**}(w''). \tag{3.10d}$$

From $(3.10d)$ it is clear that $5|w''$ But this is false.

We may assume that $e\geq 3$; so we can use $\frac{\sigma^{**}(13^e)}{13^e}\geq\frac{30772}{28561}$. From $(3.10a)$, for $f\geq 3$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{50}{49}.\frac{18}{17}.\frac{26}{25}.\frac{30772}{28561}.\frac{112}{81} = 3.112527184 > 3,$$

a contradiction.

Hence when $e\geq 3$, then $f=1$ or $f=2$.

If $f=1$, from $(3.10a)$ $(f=1)$, we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{50}{49}.\frac{18}{17}.\frac{26}{25}.\frac{30772}{28561}.\frac{4}{3}=3.001365498>3,$$

a contradiction.

If $f=2$, from $(3.10b)$, $5|w''$ which is false.

This proves that when $b=2$ and $d=2,\ c=1$ is not possible.

We continue assuming $b=2,\ d=2$ and let $c\geq 2$. The relevant equations are $(3.9a)$ and $(3.9b)$.

If $c=2$, since $\sigma^{**}(17^2)=290$, from $(3.9b)$, we find that $5|w'$ which is false. So, without loss of generality, we may assume that $c\geq 3$. Also, if $e=2$, since $\sigma^{**}(13^2)=170$, from $(3.9b)$, again we see that $5|w'$ which is false. Hence we may assume that $e\neq 2$.

We now assume that $3|n$. From $(3.9a)$, $3|w'$. Let $w'=3^f.w''$. So from $(3.9a)$, we have

$$n=2^6.7^2.17^c.5^2.13^e.3^f.w'', \tag{3.11a}$$

and from $(3.9b)$, we obtain

$$2^4.7.17^{c-1}.13^{e-1}.3^{f+1}.w''=\sigma^{**}(17^c).\sigma^{**}(13^e).\sigma^{**}(3^f).\sigma^{**}(w''); \tag{3.11b}$$

w'' cannot have more than one odd prime factor.

If $f\geq 3$ and $e\geq 3$, from $(3.11a)$, we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{50}{49}.\frac{88452}{83521}.\frac{26}{25}.\frac{30772}{28561}.\frac{112}{81}=3.113160712>3,$$

a contradiction.

Since $e\neq 2$, if $f\geq 3$, then the only possibility is $e=1$. Again from $(3.11a)$, $(e=1)$, we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{50}{49}.\frac{88452}{83521}.\frac{26}{25}.\frac{14}{13}.\frac{112}{81}=3.111744352>3,$$

a contradiction.

Thus $f\geq 3$ does not hold. Hence $f=1$ or $f=2$.

Let $f=1$. If $e=1$, from $(3.11a)$, we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{50}{49}.\frac{88452}{83521}.\frac{26}{25}.\frac{14}{13}.\frac{4}{3}=3.000610625>3,$$

a contradiction.

Since $e\neq 2$, we can assume $e\geq 3$. Again from $(3.11a)$, we obtain

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{50}{49}.\frac{88452}{83521}.\frac{26}{25}.\frac{30772}{28561}.\frac{4}{3}=3.001976401>3,$$

a contradiction.

Hence $f=1$ cannot occur. If $f=2$, from $(3.11b)$, we see that $5|w''$ and this is false.

Thus the case $b=2,\ d=2$ when $3|n$ is complete.

Suppose that $3 \nmid n$ when $b = 2,\ d = 2$.

We return to the equations $(3.9a)$ and $(3.9b)$. In these two equations w' cannot have more than two odd prime factors. Hence we may assume that $w' = p^f q^g$, where $p \geq 11$ and $q \geq 19$. Hence from $(3.9a)$, $n = 2^6.7^2.17^c.5^2.13^e.p^f.q^g$ and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{50}{49}.\frac{17}{16}.\frac{26}{25}.\frac{13}{12}.\frac{11}{10}.\frac{19}{18} = 2.637175771 < 3,$$

a contradiction.

Thus the case $b = 2,\ d = 2$ and $3 \nmid n$ is complete. This also finishes the case $b = 2$ and $d = 2$.

Case $(b = 2, d \geq 3)$. We return to the equations $(3.8a)$ and $(3.8b)$, where we assume that $d \geq 3$.

Case $(b = 2, d = 3)$. Taking $d = 3$ in $(3.8b)$ and since $\sigma^{**}(5^3) = 156 = 2^2.3.13$, we get after simplification,

$$2^3.7.17^{c-1}.5.w = 13.\sigma^{**}(17^c).\sigma^{**}(w). \tag{3.11c}$$

From $(3.11c)$, $13|w$. Hence $w = 13^e.w'$ Substituting this in $(3.8a)$ and $(3.11c)$, we get

$$n = 2^6.7^2.17^c.5^3.13^e.w', \tag{3.12a}$$

and

$$2^3.7.17^{c-1}.5.13^{e-1}.w' = \sigma^{**}(17^c).\sigma^{**}(13^e).\sigma^{**}(w'), \tag{3.12b}$$

where w' has at most one odd prime factor.

Let $c = 1$ (already $b = 2, d = 3$). Since $\sigma^{**}(17) = 18 = 2.3^2$, from $(3.12b)$, $(c = 1)$, we get after simplification

$$2^2.7.5.13^{e-1}.w' = 3^2.\sigma^{**}(13^e).\sigma^{**}(w'). \tag{3.12c}$$

From $(3.12c)$, $3^2|w'$ and so $w' = 3^f$, where $f \geq 2$. Hence from $(3.12a)$ and $(3.12c)$, we have

$$n = 2^6.7^2.17.5^3.13^e.3^f \quad (f \geq 2), \tag{3.13a}$$

and

$$2^2.7.5.13^{e-1}.3^{f-2} = \sigma^{**}(13^e).\sigma^{**}(3^f). \tag{3.13b}$$

Let $e = 1$. From $(3.13b)$ $(e = 1)$, we get

$$2.5.3^{f-2} = \sigma^{**}(3^f). \tag{3.13c}$$

If $f \geq 3$, from $(3.13c)$, $3|\sigma^{**}(3^f)$, a contradiction. Hence $f = 2$. It follows that $(3.13c)$ is satisfied when $f = 2$. Hence from $(3.13a)$, $(e = 1, f = 2)$, $n = 2^6.7^2.17.5^3.13.3^2 = 779688000$, is a bi-unitary triperfect number .

If $e = 2$, since $\sigma^{**}(13^2) = 170$, from $(3.13b)$, 17 is a factor of the left-hand side of $(3.13b)$. But this is not true.

We may assume that $e \geq 3$.

Let $f = 2$. From $(3.13b)$, $(f = 2)$, we get after simplification, $2.7.13^{e-1} = \sigma^{**}(13^e)$; from this equation since $e \geq 3$, we see that $13|\sigma^{**}(13^e)$ which is false. Hence $f \geq 3$.

Thus e and f are both ≥ 3. From $(3.13a)$, we now have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{50}{49}.\frac{18}{17}.\frac{156}{125}.\frac{30772}{28561}.\frac{112}{81} = 3.73503262 > 3,$$

a contradiction.

Thus $c = 1$ is not possible.

Let $c \geq 2$ (with $b = 2, d = 3$). We return to the equations $(3.12a)$ and $(3.12b)$, where now $c \geq 2$. In $(3.12a)$, w' has at most one odd prime factor.

If $3 \nmid n$, then $w' = 1$ or p^f, where $p \geq 11$. In any case $\frac{\sigma^{**}(w')}{w'} < \frac{11}{10}$. Hence if $3 \nmid n$, from $(3.12a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{50}{49}.\frac{17}{16}.\frac{156}{125}.\frac{13}{12}.\frac{11}{10} = 2.998052455 < 3,$$

a contradiction.

Suppose that $3|n$. Then $w' = 3^f$. From $(3.12a)$ and $(3.12b)$,

$$n = 2^6.7^2.17^c.5^3.13^e.3^f, \tag{3.12c}$$

and

$$2^3.7.17^{c-1}.5.13^{e-1}.3^f = \sigma^{**}(17^c).\sigma^{**}(13^e).\sigma^{**}(3^f). \tag{3.12d}$$

If $f \geq 3$, from $(3.12c)$, we get

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{50}{49}.\frac{156}{125}.\frac{112}{81} = 3.274074074 > 3,$$

a contradiction; in the above we used that $\frac{\sigma^{**}(3^f)}{3^f} \geq \frac{112}{81}$ for $f \geq 3$; also, $\frac{\sigma^{**}(17^c)}{17^c} \geq 1$ and $\frac{\sigma^{**}(13^e)}{13^e} \geq 1$.

Hence $f = 1$ or $f = 2$.

If $f = 1$, from $(3.12d)$, it follows that its right-hand side is divisible by 2^4, whereas its left-hand side is divisible unitarily by 2^3.

Let $f = 2$. Taking $f = 2$ in $(3.12c)$ and $(3.12d)$, we obtain

$$n = 2^6.7^2.17^c.5^3.13^e.3^2, \tag{3.13a}$$

and

$$2^2.7.17^{c-1}.13^{e-1}.3^2 = \sigma^{**}(17^c).\sigma^{**}(13^e). \tag{3.13b}$$

Since $\sigma^{**}(17^2) = 290$, taking $c = 2$ in $(3.13b)$, we see that the left-hand side of it should be divisible by 29 and this is not possible. Hence we may assume that $c \geq 3$; hence we can use the result $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}$.

If $e \geq 3$, then $\frac{\sigma^{**}(13^e)}{13^e} \geq \frac{30772}{28561}$. Hence if $e \geq 3$, from $(3.13a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{50}{49}.\frac{88452}{83521}.\frac{156}{125}.\frac{30772}{28561}.\frac{10}{9} = 3.001976401 > 3,$$

a contradiction.

Hence $e = 1$ or $e = 2$.

If $e = 1$, $(3.13b)$ reduces to $2.7.17^{c-1}.3^2 = \sigma^{**}(17^c)$; this implies that $17|\sigma^{**}(17^c)$ which is false.

If $e = 2$, since $\sigma^{**}(13^2) = 170$, taking $e = 2$ in $(3.13b)$, we see that 5 should divide its left-hand side. But this is not possible.

The case $b = 2$ and $d = 3$ is complete.

Case ($b = 2, d \geq 4$). The relevant equations are $(3.8a)$ and $(3.8b)$, where $d \geq 4$.

Case ($b = 2, d \geq 4, 3|n$). Since $3|n$, we have $3|w$. Let $w = 3^e.w'$. Using this in $(3.8a)$ and $(3.8b)$, we get

$$n = 2^6.7^2.17^c.5^d.3^e.w', \quad (d \geq 4) \tag{3.14a}$$

and

$$3.2^5.7.17^{c-1}.5^{d-2}.3^e.w' = \sigma^{**}(17^c).\sigma^{**}(5^d).\sigma^{**}(3^e).\sigma^{**}(w'), \tag{3.14b}$$

and w' has no more than two odd prime factors and is prime to 2.3.5.7.17.

Since $d \geq 3$, we have $\frac{\sigma^{**}(5^d)}{5^d} \geq \frac{756}{625}$ and for $e \geq 3$, $\frac{\sigma^{**}(3^e)}{3^e} \geq \frac{112}{81}$. Hence for $e \geq 3$, from $(3.14a)$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{50}{49}.\frac{756}{625}.\frac{112}{81} = 3.173 > 3,$$

a contradiction.

Hence $e = 1$ or $e = 2$.

If $e = 1$, again from $(3.14a)$, $(e = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{50}{49}.\frac{756}{625}.\frac{4}{3} = 3.06 > 3,$$

a contradiction.

Let $e = 2$ (with $b = 2, d \geq 4$). Taking $e = 2$ in $(3.14a)$ and $(3.14b)$, we get

$$n = 2^6.7^2.17^c.5^d.3^2.w', \quad (d \geq 4) \tag{3.15a}$$

and

$$2^4.7.17^{c-1}.5^{d-3}.3^3.w' = \sigma^{**}(17^c).\sigma^{**}(5^d).\sigma^{**}(w'), \tag{3.15b}$$

and w' has no more than two odd prime factors and is prime to 2.3.5.7.17.

When $e = 2$, we wish to show that n (hence w') is not divisible by 11 or 13 or 19 or 23. If this is proved, then if w' is divisible by two odd primes (in the worst case) say p and q, then $w' = p^f.q^g$, where we can assume that $p \geq 29$ and $q \geq 31$. Also, from $(3.15a)$, $n = 2^6.7^2.17^c.5^d.3^2.p^f.q^g$ so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{50}{49}.\frac{17}{16}.\frac{5}{4}.\frac{10}{9}.\frac{29}{28}.\frac{31}{30} = 2.996523995 < 3,$$

a contradiction. With this the case $b = 2,\ d \geq 4,\ 3|n$ would be complete.

We now prove that n in $(3.15a)$ and $(3.15b)$ is not divisible by s, where $s \in \{11, 13, 19, 23\}$.

We assume that $s|n$ so that $s|w'$. Let $w' = s^f.w''$; substituting this into $(3.15a)$ and $(3.15b)$, we obtain

$$n = 2^6.7^2.17^c.5^d.3^2.s^f.w'', \quad (d \geq 4) \tag{3.16a}$$

and

$$2^4.7.17^{c-1}.5^{d-3}.3^3.s^f.w'' = \sigma^{**}(17^c).\sigma^{**}(5^d).\sigma^{**}(s^f).\sigma^{**}(w''), \tag{3.16b}$$

and w' has no more than one odd prime factor and is prime to $2.3.5.7.17.s$.

We now examine the factors of $\sigma^{**}(5^d)$ in the presence of $(3.16a)$ and $(3.16b)$. We distinguish the following cases:

Case A. Let d be odd. Then

$$\sigma^{**}(5^d) = \frac{5^{d+1}-1}{4} = \frac{(5^t-1)(5^t+1)}{4}, \quad \left(t = \frac{d+1}{2}\right).$$

(i) Let t be even. Then $8|5^t-1$ and trivially $2|5^t+1$. Hence $4|\frac{(5^t-1)(5^t+1)}{4} = \sigma^{**}(5^d)$; it follows from $(3.15b)$ that $w'' = 1$. Rewriting $(3.16a)$ and $(3.16b)$, taking $w'' = 1$, we get

$$n = 2^6.7^2.17^c.5^d.3^2.s^f, \quad (d \geq 4) \qquad (3.16c)$$

and

$$2^4.7.17^{c-1}.5^{d-3}.3^3.s^f = \sigma^{**}(17^c).\sigma^{**}(5^d).\sigma^{**}(s^f). \qquad (3.16d)$$

If $s = 19$ or 23, so that $s \geq 19$, from $(3.16c)$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{50}{49}.\frac{17}{16}.\frac{5}{4}.\frac{10}{9}.\frac{19}{18} = 2.955414841 < 3,$$

a contradiction.

We may assume that $s = 11$ or 13. We have:

(a) $3|5^t-1$, since t is even.

(b) $9|5^t-1 \iff 6|t \iff 7|5^t-1$; $6|t$ implies $5^6-1|5^t-1$ and $5^6-1 = 2^3.3^2.7.31$. Hence $31|5^t-1$ so that $31|\frac{5^t-1}{2}|\sigma^{**}(5^d)$. This is not possible from $(3.16d)$. Hence $9 \nmid 5^t-1$ and $7 \nmid 5^t-1$. As a consequence, $3\|5^t-1$.

(c) Since t is even, $8|5^t-1$; but $16|5^t-1$ implies that $8|\sigma^{**}(5^d)$. This results in an imbalance in the powers of two between two sides of $(3.16d)$. Hence $16 \nmid 5^t-1$ and so $8\|5^t-1$.

(d) $11|5^t-1 \iff 5|t$; and $5|t$ implies that $5^5-1|5^t-1$. Also, $5^5-1 = 2^2.11.71$. Hence $71|\frac{5^t-1}{2}|\sigma^{**}(5^d)$; this is not possible from $(3.16d)$. Thus $11 \nmid 5^t-1$.

(e) $13|5^t-1 \iff 4|t$; this implies $16|5^4-1|5^t-1$. In (c) above, we proved that $16 \nmid 5^t-1$. Hence $13 \nmid 5^t-1$.

(f) $17|5^t-1 \iff 16|t$; this implies $4|t$. As in (e), we get a contradiction. Hence $17 \nmid 5^t-1$.

We have $d \geq 5$, since d is odd and $d \geq 4$. Hence $t = \frac{d+1}{2} \geq 3$. It is clear that $\frac{5^t-1}{24} > 1$, odd and not divisible by 3. Hence $\frac{5^t-1}{24}$ must be divisible by an odd prime say p. Since 5^t-1 is not divisible by any of the primes $5, 7, 11, 13$ and 17, the same is true with respect to $\frac{5^t-1}{24}$. Hence $p|\frac{5^t-1}{24}|\sigma^{**}(5^d)$ and $p \notin \{2, 3, 5, 7, 11, 13, 17\}$. This contradicts $(3.16d)$ since $s = 11$ or 13.

The case that t is even is complete.

(ii) Let t be odd.

(a) $4\|5^t-1$ since t is odd. Hence $\frac{5^t-1}{4}$ is odd and > 1 since $t \geq 3$.

(b) 5^t-1 is not divisible by $3, 7, 13, 17$ or 23, since t is odd; trivially not divisible by 5.

(c) $19|5^t-1 \iff 9|t$; this implies that $5^9-1|5^t-1$. Also, $5^9-1 = 2^2.19.31.829$. Hence $\frac{5^t-1}{4} \mid \sigma^{**}(5^d)$, is divisible by 31 and 829. It follows from $(3.16b)$ that w'' is divisible by 31 and 829. This is not possible since w'' cannot have more than one odd prime factor. Hence $19 \nmid 5^t-1$.

(d) Let $s \neq 11$. We claim that $11 \nmid 5^t-1$. Suppose that $11|5^t-1$. This is if and only if $5|t$. Hence $11|5^t-1$ implies $5^5-1|5^t-1$. Also, $5^5-1 = 2^2.11.71$. It follows that $\frac{5^t-1}{4}|\sigma^{**}(5^d)$, is divisible by 11 and 71. Since $s \neq 11$, from $(3.16b)$ we infer that w'' is divisible by 11 and 71. This is not possible since w'' cannot have more than one odd prime factor. Hence when $s \neq 11$, $11 \nmid 5^t-1$.

(e) Let $s = 11$. We prove that $\frac{5^t-1}{4}$ has a prime factor $\neq 11$; if $11 \nmid 5^t-1$, then this is trivially true. We assume that $11|5^t-1$. If $\frac{5^t-1}{4}$ is divisible by 11 alone, then we must have $\frac{5^t-1}{4} = 11^\alpha$ for some positive integer α. If $\alpha \geq 2$, then $11^2|5^t-1$; this is if and only if $55|t$. In particular $11|t$. Hence $5^{11}-1|5^t-1$ and $5^{11}-1 = 2^2.12207031$. It follows that $12207031|\frac{5^t-1}{4} = 11^\alpha$, which is impossible. Hence $\frac{5^t-1}{4} = 11$ or $5^t = 45$, which is not possible. Thus $\frac{5^t-1}{4}$ must be divisible by an odd prime say $p \neq 11$. Clearly, $p \notin \{3,5,7,11,13,17,19,23\}$. Hence $p|\frac{5^t-1}{4}|\sigma^{**}(5^d)$. From $(3.16b)$, we find that $p|w''$.

Thus if $s \neq 11$, from (a)–(d), it follows that $\frac{5^t-1}{4}$ is not divisible by any prime in the set $\{3,5,7,11,13,17,19,23\}$. In particular, if $p|\frac{5^t-1}{4}|\sigma^{**}(5^d)$ and $p \notin \{3,5,7,11,13,17,19,23\}$, from $(3.16b)$, we infer that $p|w''$.

Hence when t is odd, we can conclude that there is an odd prime $p|\frac{5^t-1}{4}$ and $p|w''$.

Let $s \in \{11,13,19,23\}$. We now prove that we can find an odd prime $q|5^t+1$ and $q|w''$ when t is odd. We have

(f) $2\|5^t+1$ and $3|5^t+1$.

(g) 5^t+1 is not divisible by 13 and 17 since t is odd.

(h) 5^t+1 is not divisible by 11 and 19 for any t.

(i) $23|5^t+1 \iff t = 11u$, where u is odd. Hence $23|5^t+1$ implies $5^{11}+1|5^t+1$. Also, $5^{11}+1 = 2.3.23.67.5281$. So, 5^t+1, a factor of $\sigma^{**}(5^d)$, is divisible by 67 and 5281. From $(3.16b)$, it follows that w'' is divisible by 67 and 5281. This cannot happen. Hence $23 \nmid 5^t+1$.
Thus 5^t+1 is not divisible by any of $11, 13, 17, 19$ and 23.

(j) We may note that $7|5^t+1 \iff 9|5^t+1 \iff t = 3u$, where u is odd. Assume that $7 \nmid 5^t+1$. Then $9 \nmid 5^t+1$. Hence $3\|5^t+1$. Also, $\frac{5^t+1}{6} > 1$, odd and not divisible by any of the primes $3,5,7,11,13,17,19$ and 23. Let $q|\frac{5^t+1}{6}$ so that $q|\sigma^{**}(5^d)$. Then $q \notin \{3,5,7,11,13,17,19,23\}$. From $(3.16b)$, $q|w''$.
Suppose that $7|5^t+1$ so that $9|5^t+1$. We note that $27|5^t+1 \iff t = 9u$, where u is odd. Hence $27|5^t+1$ implies $5^9+1|5^t+1$. Also, $5^9+1 = 2.3^3.7.5167$. Hence

$5167|5^t+1|\sigma^{**}(5^d)$. From $(3.16b)$, it follows that $5167|w''$. We already proved that there is an odd prime $p|\dfrac{5^t-1}{4}$ and $p|w''$. Now, $\dfrac{5^t-1}{4}$ and 5^t+1 are relatively prime. Since p and 5167 respectively divide these factors, it follows that w'' is divisible by these two odd primes. This cannot happen. Hence $27 \nmid 5^t+1$. Thus $7|5^t+1$ implies $9\|5^t+1$.

We have $\dfrac{5^t+1}{18}$ is >1, odd and not divisible by 3. By our assumption, $7|\dfrac{5^t+1}{18}$ and from $(3.16b)$, $7^2 \nmid \dfrac{5^t+1}{18}$, since $\dfrac{5^t+1}{18}|\sigma^{**}(5^d)$. Hence $7\|\dfrac{5^t+1}{18}$. If $\dfrac{5^t+1}{18}$ is divisible by 7 alone, then we must have $\dfrac{5^t+1}{18}=7$ or $5^t=125$ or $t=3$.

We now prove that $t=3$ is not possible. Suppose that $3=t=\dfrac{d+1}{2}$ so that $d=5$. We have $\sigma^{**}(5^5)=\dfrac{5^6-1}{4}=2.3^2.7.31$. Taking $d=5$ in $(3.16b)$, we get after simplification

$$2^3.3.17^{c-1}.5^2.s^f.w''=31.\sigma^{**}(17^c).\sigma^{**}(s^f).\sigma^{**}(w''); \qquad (3.16e)$$

this implies that $31|w''$ so that $w''=31^g$. Substituting $w''=31^g$ in $(3.16a)$ and $(3.16e)$, we get

$$n=2^6.7^2.17^c.5^d.3^2.s^f.31^g, \qquad (3.17a)$$

and

$$2^3.3.17^{c-1}.5^2.s^f.31^{g-1}=\sigma^{**}(17^c).\sigma^{**}(s^f).\sigma^{**}(31^g). \qquad (3.17b)$$

We obtain a contradiction by examining the factors of $\sigma^{**}(17^c)$.

Let c be odd. Then $9|\sigma^{**}(17^c)$. This is not possible from $(3.17b)$.

We may assume that c is even so that $c=2k$. Then

$$\sigma^{**}(17^c)=\left(\frac{17^k-1}{16}\right).(17^{k+1}+1).$$

(i) If k is even, $9|17^k-1$ and so $9|\dfrac{17^k-1}{16}|\sigma^{**}(17^c)$ and this leads to a contradiction from $(3.17b)$.

(ii) Let k be odd. First we note that $k>1$. If $k=1$, then $c=2$. We have $\sigma^{**}(17^2)=290$. Taking $c=2$ in $(3.17b)$, we see that 29 divides its right-hand side but 29 does not divide its left-hand side. Hence $k=1$ cannot occur.

We may assume that $k\geq 3$. Since k is odd, $16\|17^k-1$; also, 17^k-1 is not divisible by $3,5,7,11,13,23$ and 31 since k is odd. $19|17^k-1 \iff 9|k$. In such a case $17^9-1|17^k-1$. Also, $17^9-1=2^4.19.307.1270657$. In particular, $307|\dfrac{17^k-1}{16}|\sigma^{**}(17^c)$. But this is not possible can be seen from $(3.17b)$. Hence $19\nmid 17^k-1$.

Thus $\dfrac{17^k-1}{16}>1$ and is odd; also it is not divisible by $3,5,7,11,13,17,19,23$ and 31. If p is an odd prime factor of $\dfrac{17^k-1}{16}|\sigma^{**}(17^c)$, then $p\notin\{3,5,7,11,13,17,19,23,31\}$. But this is not possible from $(3.17b)$.

Thus $d=5$ (or $t=3$) is not admissible.

Hence $\dfrac{5^t+1}{18}$ is not divisible by 7 alone. As a consequence, we can find an odd prime $q|\dfrac{5^t+1}{18}$ and $q \neq 7$. Since $\dfrac{5^t+1}{18}$ is not divisible by $3, 5, 7, 11, 13, 17, 19$ and 23, $q \notin \{3, 5, 7, 11, 13, 17, 19, 23\}$. It follows from $(3.16b)$ that $q|w''$. Since $p|w''$, $q|w''$ and $p \neq q$ it follows that w'' is divisible by two odd primes which is not possible.

This completes the case when $t = \dfrac{d+1}{2}$ is odd.

Case B. It remains to examine the case when d is even. Let $d = 2k$. Then

$$\sigma^{**}(5^d) = \left(\frac{5^k - 1}{4}\right).(5^{k+1} + 1).$$

(iii) If k is even, we get a contradiction just as in (i) of Case A where t was even.

(iv) Assume that k is odd. Since $d = 2k \geq 4$, we have $k \geq 3$. Again as in (ii) of Case A, on similar lines, we can show that $\dfrac{5^k-1}{4}$ is divisible by an odd prime $p|w''$.

It remains to examine $5^{k+1} + 1$ when k is odd.

(v) Since $k + 1$ is even, $2||5^{k+1} + 1$ and not divisible by $3, 7$ and 23.

(vi) Since $5^t + 1$ is not divisible by 11 and 19 for any positive integer t; the same is true with respect to $5^{k+1} + 1$.

(vii) $17|5^{k+1} + 1 \iff k + 1 = 8u$, where u is odd. Hence $17|5^{k+1} + 1$ implies $5^8 + 1|5^{k+1} + 1$. Also, $5^8 + 1 = 2.17.11489$. Hence $11489|5^{k+1} + 1|\sigma^{**}(5^d)$. From $(3.16b)$, $11489|w''$. Since $p|\dfrac{5^k-1}{4}$ divides w'', it follows that w'' is divisible by two odd primes. This cannot happen. Hence $17 \nmid 5^{k+1} + 1$.

(viii) If $13 \nmid 5^{k+1}+1$, then it follows from (v)–(vi) that $\dfrac{5^{k+1}+1}{2} > 1$, is odd and not divisible by any of the primes $3, 5, 7, 11, 13, 19$ and 23. Thus if $q|\dfrac{5^{k+1}+1}{2}$, then q is odd and $q \notin \{3, 5, 7, 11, 13, 19, 23.\}$. From $(3.16b)$, $q|w''$.
Suppose that $13|5^{k+1} + 1$. Suppose that $13^2|5^{k+1} + 1$; this is if and only if $k + 1 = 26u$, where u is odd. This implies that $5^{26} + 1|5^{k+1} + 1$ and $5^{26} + 1 = 2.13^2.53.83181652304609$. Thus $\dfrac{5^{k+1}+1}{2}$ is divisible by two odd primes and these primes divide w'' by $(3.16b)$. But this is not possible. Hence $13||\dfrac{5^{k+1}+1}{2}$. It follows that $\dfrac{5^{k+1}+1}{26} > 1$, odd and not divisible by any of $3, 5, 7, 11, 13, 17, 19$ and 23. Hence if $q|\dfrac{5^{k+1}+1}{26}$, then q is odd and $q \notin \{3, 5, 7, 11, 13, 19, 23\}$. From $(3.16b)$, $q|w''$.

Thus p and q divide w''. This is not possible.

This completes the Case B.

Hence $s \nmid n$, where $s \in \{11, 13, 19, 23\}$.

Thus, n in $(3.16a)$ satisfying $(3.16b)$ cannot be a bi-unitary triperfect number when $b = 2$, $d \geq 4$ and $3|n$.

Case $(b = 2,\ d \geq 4,\ 3 \nmid n)$. The relevant equations are $(3.8a)$ and $(3.8b)$. We obtain a contradiction by examining the factors of $\sigma^{**}(17^c)$, and hence n in $(3.8a)$ cannot be a bi-unitary triperfect number.

For the validity of $(3.8b)$, we show that the only choice for c is that $c = 2k$, where k is odd. In such a case, we prove that $\dfrac{17^k - 1}{16}$ and $17^{k+1} + 1$ should be divisible by two odd primes p and q, and each of them exceeds 41. We can assume that $p \geq 43$ and $q \geq 47$. If at all w has a third prime factor say r, then obviously $r \geq 11$, from $(3.8b)$. Hence $n = 2^6.7^2.17^c.5^d.p^e.q^f.r^g$. We have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{50}{49}.\frac{17}{16}.\frac{5}{4}.\frac{43}{42}\frac{47}{46}.\frac{11}{10} = 2.899557597 < 3,$$

a contradiction.

If c is odd or $4|c$, then $9|\sigma^{**}(17^c)$. This implies that $3|w$, from $(3.8b)$. This is not true since by our assumption $3 \nmid n$.

Let $c = 2k$, where k is odd. We have

(a) $16||17^k - 1$ since k is odd. Also, $17^k - 1$ is not divisible by $3, 5, 7, 11, 13, 23, 29, 31, 37$ and 41, since k is odd; not divisible by 17 trivially.

(b) $19|17^k - 1$ implies $9|k$. This implies that $17^9 - 1|17^k - 1$. Also, $17^9 - 1 = 2^4.19.307.1270657$. Hence $19, 307$ and 1270657 divide $\dfrac{17^k - 1}{16}|\sigma^{**}(17^c)$; from $(3.8b)$, it follows that these three prime factors divide w. Since w has at most three prime factors, from $(3.8a)$, we have $n = 2^6.7^2.17^c.5^d.19^e.(307)^f.(1270657)^g$ so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{50}{49}.\frac{17}{16}.\frac{5}{4}.\frac{19}{18}.\frac{307}{306}.\frac{1270657}{1270656} = 2.668567854 < 3,$$

a contradiction. Hence $19 \nmid 17^k - 1$.

Thus $\dfrac{17^k - 1}{16}$ is odd and not divisible by any of the primes from 3 to 41. We now prove that $\dfrac{17^k - 1}{16} > 1$ or $k > 1$.

Assume that $k = 1$ so that $c = 2$. We have $\sigma^{**}(17^2) = 290$. Taking $c = 2$ in $(3.8b)$, we get after simplification

$$3.2^4.7.17.5^{d-2}.w = 29.\sigma^{**}(5^d).\sigma^{**}(w), \tag{3.18}$$

so that $29|w$. Let $w = 29^e.w'$. From $(3.8a)$ and (3.18), we obtain

$$n = 2^6.7^2.17^2.5^d.29^e.w', \qquad (3.18a),$$

and

$$3.2^4.7.17.5^{d-2}.29^{e-1}.w' = \sigma^{**}(5^d).\sigma^{**}(29^e).\sigma^{**}(w'), \qquad (3.18b)$$

where w' has at most two odd prime factors.

If p_1 and p_2 denote these two prime factors of w', then it follows from $(3.18b)$ that $p_1 \geq 11$ and $p_2 \geq 13$. Also, $n = 2^6.7^2.17^2.5^d.29^e.p_1^f.p_2^g$. We have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{50}{49}.\frac{290}{289}.\frac{5}{4}.\frac{29}{28}.\frac{11}{10}.\frac{13}{12} = 2.937283312 < 3,$$

a contradiction.

Hence $k = 1$ is not admissible. We may assume that $k \geq 3$, since k is odd. Thus $\frac{17^k - 1}{16}$, odd and not divisible by any prime from 3 to 41. Let $p|\frac{17^k - 1}{16}$. Then $p \geq 43$.

We now consider the factor $17^{k+1} + 1$, where k is odd and ≥ 3.

(c) $17^{k+1} + 1$ is not divisible by $3, 7, 11, 13, 23$ and 31 since $k + 1$ is even; also, $2\|17^{k+1} + 1$.

(d) $19 \nmid 17^t + 1$ for any positive integer t. In particular, $19 \nmid 17^{k+1} + 1$.

(e) $37|17^{k+1} + 1 \iff k + 1 = 18u$, where u is odd; this implies that $17^8 + 1|17^{k+1} + 1$. Also, $17^8+1 = 2.5.29.37.109.181.2089.83233.382069$. Hence $17^{k+1}+1$ is divisible by seven odd prime factors ≥ 29 and from $(3.8b)$, these seven prime factors divide w. This contradicts the fact that w has no more than three odd prime factors. This proves that $37 \nmid 17^{k+1} + 1$.

(f) $41|17^{k+1} + 1 \implies k + 1 = 20u$; This implies that $17^{20} + 1|17^{k+1} + 1$. Also, $17^{20} + 1 = 2.p_1.p_2.p_3$, where $p_1 = 41,\ p_2 = 41761$ and $p_3 = 118\,684\,412\,830\,256\,8601$. Hence $17^{k+1} + 1|\sigma^{**}(17^c)$ is divisible by p_1, p_2 and p_3. From $(3.8b)$ it follows that these three primes divide w. We have already shown that $p|w$, where $p|\ \frac{17^k - 1}{16}$ and $p \geq 43$. Thus w is divisible by four odd primes p, p_1, p_2 and p_3. This is not possible. Hence $41 \nmid 17^{k+1}+1$.

(g) We may note that $5|17^{k+1} + 1 \iff 29|17^{k+1} + 1 \iff k + 1 = 2u$.

Suppose that $5 \nmid 17^{k+1} + 1$. Then $29 \nmid 17^{k+1} + 1$. From (c)–(f) above, it follows that $\frac{17^{k+1} + 1}{2}$ is odd, > 1 and not divisible by any prime from 3 to 41. If $q|\frac{17^{k+1} + 1}{2}$, then from $(3.8b)$ it follows that $q|w$ and $q \geq 43$.

Suppose that $5|17^{k+1} + 1$ so that $29|17^{k+1} + 1$. Let us assume that $\frac{17^{k+1} + 1}{2} = 5^\alpha.29^\beta$, where α and β are positive integers. If $\alpha \geq 2$, then $5^2|17^{k+1} + 1$. But this is if and only if $k + 1 = 10u$; in such a case $17^{10} + 1|17^{k+1} + 1$. Also, $17^{10} + 1 = 2.5^2.29.21881.63541$. Hence $21881|\frac{17^{k+1} + 1}{2} = 5^\alpha.29^\beta$. This is obviously false. Hence $\alpha = 1$.

Similarly, if $\beta \geq 2$, $29^2|17^{k+1}+1$; this is if and only if $k+1 = 58u$ so that $17^{58}+1|17^{k+1}+1$. Also, $17^{58} + 1 = 2.5.4908077.P$, where

$$P = 5627\,688\,836\,691\,687\,811\,685\,586\,936\,872\,121\,257\,317\,104\,508\,544\,673\,081\,805\,033.$$

In particular, $4908077|\frac{17^{k+1} + 1}{2} = 5^\alpha.29^\beta$. But this cannot happen. Hence $\beta = 1$.

Thus $\frac{17^{k+1} + 1}{2} = 5.29$ or $17^{k+1} = 289$ so that $k + 1 = 2$ or $k = 1$. But $k \geq 3$. This contradiction proves that $\frac{17^{k+1} + 1}{2}$ must be divisible by a prime $q \neq 5$ and 29. It now follows that $\frac{17^{k+1} + 1}{2}$ is divisible by an odd prime q not in $[3, 41]$; also, since $q|\sigma^{**}(17^c)$, from $(3.18b)$, we have $q|w$. The primes p and q are different since they divide $\frac{17^k - 1}{16}$ and $17^{k+1} + 1$ respectively which are relatively prime. As mentioned in the beginning of the case $b = 2,\ d \geq 4,\ 3 \nmid n$ we obtain a contradiction.

The case $b = 2$ is complete. The proof of (c) of Theorem 3.1 is complete.

Proof of (a) of Theorem 3.1.

Case $b \geq 3$. We return to the equations $(3.1a)$ and $(3.1b)$, where $b \geq 3$. We claim that n in $(3.1a)$ cannot be a bi-unitary triperfect number. On the contrary we assume that n in $(3.1a)$ is a bi-unitary triperfect number and obtain a contradiction.

Case $b \geq 3$ *with* $3|n$. From $(3.1a)$, $3|v$. Let $v = 3^d u$, where $(u, 2.3.7.17) = 1$. Substituting $v = 3^d u$ in $(3.1a)$ and $(3.1b)$, we obtain

$$n = 2^6.7^b.17^c.3^d.u, \quad (b \geq 3) \tag{3.19a}$$

and

$$2^6.3^{d+1}.7^{b-1}.17^{c-1}.u = \sigma^{**}(7^b).\sigma^{**}(17^c).\sigma^{**}(3^d).\sigma^{**}(u), \tag{3.19b}$$

where u has at most three odd prime factors.

By Lemma 2.1, since $b \geq 3$, $\frac{\sigma^{**}(7^b)}{7^b} > \frac{2752}{2401}$. Also, $\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521}$ when $c \geq 3$ and $\frac{\sigma^{**}(3^d)}{3^d} \geq \frac{112}{81}$ when $d \geq 3$. Hence if $c \geq 3$ and $d \geq 3$, from $(3.19a)$, we get

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{2752}{2401}.\frac{88452}{83521}.\frac{112}{81} = 3.120816493 > 3,$$

a contradiction.

Hence when $c \geq 3$, then $d = 1$ or $d = 2$. Let $c \geq 3$.

If $d = 1$, from $(3.19a)$, $(d = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{2752}{2401}.\frac{88452}{83521}.\frac{4}{3} = 3.009358761 > 3,$$

a contradiction.

Let $d = 2$. Taking $d = 2$ in $(3.19b)$, since $\sigma^{**}(9) = 10$, it follows that $5|u$. Let $u = 5^e.w$. Using this in $(3.19a)$ and $(3.19b)$, we get,

$$n = 2^6.7^b.17^c.3^2.5^e.w, \quad (b \geq 3,\ c \geq 3) \tag{3.20a}$$

and

$$2^5.3^3.7^{b-1}.17^{c-1}.5^{e-1}.w = \sigma^{**}(7^b).\sigma^{**}(17^c).\sigma^{**}(5^e).\sigma^{**}(w), \tag{3.20b}$$

where w has at most two odd prime factors.

We have $\frac{\sigma^{**}(5^e)}{5^e} \geq \frac{756}{625}$ for $e \geq 3$. Hence from $(3.20a)$, for $e \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{2752}{2401}.\frac{88452}{83521}.\frac{10}{9}.\frac{756}{625} = 3.033433631 > 3,$$

a contradiction.

Hence $e = 1$ or $e = 2$.

If $e = 1$, we have from $(3.20a)$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{2752}{2401}.\frac{88452}{83521}.\frac{10}{9}.\frac{6}{5} = 3.009358761 > 3,$$

a contradiction.

Let $e = 2$ $(c \geq 3, d = 2)$. Since $\sigma^{**}(5^2) = 26$, taking $e = 2$ in $(3.20b)$, we obtain

$$2^5.3^3.7^{b-1}.17^{c-1}.5^{e-1}.w = 26.\sigma^{**}(7^b).\sigma^{**}(17^c).\sigma^{**}(w)$$

or

$$2^4.3^3.7^{b-1}.17^{c-1}.5^{e-1}.w = 13.\sigma^{**}(7^b).\sigma^{**}(17^c).\sigma^{**}(w); \tag{3.20c}$$

from this equation it follows that $13|w$. Let $w = 13^f.w'$. Now from $(3.20a)$ and $(3.20c)$, we obtain

$$n = 2^6.7^b.17^c.3^2.5^2.13^f.w', \quad (b \geq 3,\ c \geq 3) \tag{3.21a}$$

and

$$2^4.3^3.7^{b-1}.17^{c-1}.5.13^{f-1}.w' = \sigma^{**}(7^b).\sigma^{**}(17^c).\sigma^{**}(13^f).\sigma^{**}(w'), \tag{3.21b}$$

where w' has no more than one odd prime factor.

By examining the factors of $\sigma^{**}(7^b)$ we show that if b is odd or $4|b$, then we obtain a contradiction. If $b = 2k$, where k is odd, we prove that $\dfrac{7^k-1}{6}$ is divisible by a prime $p \geq 29$. From $(3.21b)$, $p|w'$ and so $w = p^g$. So from $(3.21a)$, we have $n = 2^6.7^b.17^c.3^2.5^2.13^f.p^g$; hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{10}{9}.\frac{26}{25}.\frac{13}{12}.\frac{29}{28} = 2.988378605 < 3,$$

a contradiction.

We now justify the above.

If b is odd or $4|b$, we have $8|\sigma^{**}(7^b)$. From $(3.21b)$, we find that in this case, 2^5 divides its right-hand side but its left-hand side is unitarily divisible by 2^4. This is a contradiction.

In what follows we will be using several results on the divisibility of $7^k - 1$ by various primes. We refer to Appendix C of [2] for these results.

Let $b = 2k$, where k is odd. Since $b \geq 3$, we have $k \geq 3$.

(a) $2\|7^k - 1$ since k is odd; and $3|7^k - 1$.

(b) $7^k - 1$ is not divisible by $5, 11, 13, 17$ and 23, since k is odd; trivially not divisible by 7.

(c) Assume $27|7^k - 1$. This implies $9|k$ and so $7^9 - 1|7^k - 1$. Also, $7^9 - 1 = 2.3^3.19.37.1063$. It follows that $\dfrac{7^k-1}{6}$, a factor of $\sigma^{**}(7^b)$, is divisible by $19, 37$ and 1063. From $(3.21b)$, these three primes divide w'. But w' is divisible at most by one odd prime factor. Hence $27 \nmid 7^k - 1$.

(d) We note that $9|7^k - 1 \iff 19|7^k - 1 \iff 3|k$. Hence if $9 \nmid 7^k - 1$ then $19 \nmid 7^k - 1$; in this case $\dfrac{7^k-1}{6}$ is not divisible by 3 and 19. Thus from (a) and (b), $\dfrac{7^k-1}{6} > 1$, odd and not divisible by $3, 5, 7, 11, 17, 19$ and 23. Hence if $p|\dfrac{7^k-1}{6}$, then from $(3.21b)$, $p|w'$ and $p \geq 29$.

Suppose that $9|7^k - 1$ so that $19|7^k - 1$. By (c), $9\|7^k - 1$. Then $\dfrac{7^k-1}{18}$ is odd and > 1; also not divisible by 3. Suppose $\dfrac{7^k-1}{18}$ is divisible by 19 alone so that $\dfrac{7^k-1}{18} = 19^\alpha$. If $\alpha \geq 2$, then $19^2|7^k - 1$; this is if and only if $57|k$ and so $19|k$. But $419|7^{19} - 1|7^k - 1$. Hence

$419|\frac{7^k-1}{18}=19^\alpha$ which is impossible. Hence $\alpha=1$ and so $\frac{7^k-1}{18}=19$ or $k=3$. Hence $b=6$.

We now prove that $b=6$ is not admissible. We have $\sigma^{**}(7^6)=2.3.19.1201$. Taking $b=6$ in $(3.21b)$, we see that 19 and 1201 divide w'. But w' has at most one odd prime factor. This proves that $b=6$ is not possible. Hence $\frac{7^k-1}{18}$ must be divisible by an odd prime say $p\neq 19$. It follows that $p\notin\{3,5,7,11,17,19,23\}$. From $(3.21b)$, $p|w'$ and $p\geq 29$.

The case $b\geq 3, c\geq 3,\ 3|n$ is complete.

We may assume that $b\geq 3, 3|n$ and $c=1$ or $c=2$. We return to $(3.19a)$ and $(3.19b)$.

Let $c=1$. Since $\sigma^{**}(17)=18$, taking $c=1$ in $(3.19a)$ and $(3.19b)$, we get

$$n=2^6.7^b.17.3^d.u, \quad (b\geq 3) \tag{3.22a}$$

and

$$2^5.3^{d-1}.7^{b-1}.u=\sigma^{**}(7^b).\sigma^{**}(3^d).\sigma^{**}(u), \tag{3.22b}$$

where u has at most three odd prime factors.

From $(3.22a)$, we have for $d\geq 3$,

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{2752}{2401}.\frac{18}{17}.\frac{112}{81}=3.120181406>3,$$

a contradiction.

Hence $d=1$ or $d=2$.

If $d=1$, then $n=2^6.7^b.17.3.u$ and so we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{2752}{2401}.\frac{18}{17}.\frac{4}{3}=3.008746>3,$$

a contradiction.

Let $d=2$. Since $\sigma^{**}(3^2)=10$, taking $d=2$ in $(3.22b)$ we see that $5|u$. Let $u=5^e.w$. With this u, from $(3.22a)$, $(d=2)$, and $(3.22b)$, $(d=2)$, we get

$$n=2^6.7^b.17.3^2.5^e.w, \quad (b\geq 3) \tag{3.22c}$$

and

$$2^4.3.7^{b-1}.5^{e-1}.w=\sigma^{**}(7^b).\sigma^{**}(5^e).\sigma^{**}(w), \tag{3.22d}$$

where w can have at most two prime factors and $(w,2.3.5.7.17)=1$.

If $e\geq 3$, from $(3.22c)$, we have

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{2752}{2401}.\frac{18}{17}.\frac{10}{9}.\frac{756}{625}=3.032816327>3,$$

a contradiction. Hence $e=1$ or $e=2$. If $e=1$, we have $n=2^6.7^b.17.3^2.5.w$ and so

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{119}{64}.\frac{2752}{2401}.\frac{18}{17}.\frac{10}{9}.\frac{6}{5}=3.008746356>3,$$

a contradiction.

Let $e = 2$. Since $\sigma^{**}(5^2) = 26$, taking $e = 2$ in $(3.22d)$, we find that $13|w$. Let $w = 13^f.w'$. From $(3.22c)$ and $(3.22d)$, we get

$$n = 2^6.7^b.17.3^2.5^2.13^f.w', \quad (b \geq 3) \tag{3.23a}$$

and

$$2^3.3.7^{b-1}.5.13^{f-1}.w' = \sigma^{**}(7^b).\sigma^{**}(13^f).\sigma^{**}(w'); \tag{3.23b}$$

w' has no more than one odd prime factor and w' is prime to 2.3.5.7.13.17.

We obtain a contradiction from $(3.23b)$ by examining the factors of $\sigma^{**}(7^b)$.

If b is odd or $4|b$, then $8|\sigma^{**}(7^b)$. Hence the right-hand side of $(3.23b)$ is divisible by 2^4 while its left-hand side unitarily by 2^3.

We may assume that $b = 2k$, and k is odd; $b \geq 3$ implies $k \geq 3$. We have

$$\sigma^{**}(7^b) = \left(\frac{7^k - 1}{6}\right).(7^{k+1} + 1).$$

(a) $2\|7^k - 1$.

(b) $3\|7^k - 1$, since 3 is a unitary divisor of the left-hand side of $(3.23b)$.

(c) $7^k - 1$ is not divisible by $5, 11, 13, 17$ and 23 since k is odd; not divisible by 7 trivially.

(d) Since $7^k - 1$ is not divisible by 9 and hence not divisible by 19.

From (a)–(d), we conclude that $\frac{7^k - 1}{6}$ is odd, > 1 and not divisible by any prime from 3 to 23. Hence if $p|\frac{7^k-1}{6}$, then from $(3.23b)$, $p|w'$ and $p \geq 29$. Hence $w' = p^g$ and $n = 2^6.7^b.17.3^2.5^2.13^f.p^g$. We have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{18}{17}.\frac{10}{9}.\frac{26}{25}.\frac{13}{12}.\frac{29}{28} = 2.978038194 < 3,$$

a contradiction.

The case $c = 1$ is complete.

Let $c = 2$. The relevant equations are $(3.19a)$ and $(3.19b)$. Since $\sigma^{**}(17^2) = 290 = 2.5.29$, taking $c = 2$ in $(3.19b)$, we find that u is divisible by 5 and 29. Hence, $u = 5^e.29^f.w$. From $(3.19a)$, $(c = 2)$, and $(3.19b)$, $(c = 2)$, we obtain the following:

$$n = 2^6.7^b.17^2.3^d.5^e.29^f.w, \quad (b \geq 3) \tag{3.24a}$$

and

$$2^5.3^{d+1}.7^{b-1}.17.5^{e-1}.29^{f-1}.w = \sigma^{**}(7^b).\sigma^{**}(3^d).\sigma^{**}(5^e).\sigma^{**}(29^f).\sigma^{**}(w), \tag{3.24b}$$

w is prime to 2.3.5.7.17.29 and has no more than one prime factor.

From Lemma 2.1, we have $\frac{\sigma^{**}(5^e)}{5^e} \geq \frac{26}{25}$ for all $e \geq 1$. Using $\frac{\sigma^{**}(3^d)}{3^d} \geq \frac{112}{81}$, for $d \geq 3$, from $(3.24a)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{2752}{2401}.\frac{290}{289}.\frac{112}{81}.\frac{26}{25} = 3.075316052 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

Let $d = 1$. Since $\sigma^{**}(3) = 4$, taking $d = 1$ in $(3.24b)$, we find that $w = 1$. Taking $w = 1$ in $(3.24a)$ and $(3.24b)$, we get

$$n = 2^6.7^b.17^2.3.5^e.29^f, \quad (b \geq 3) \tag{3.24c}$$

and

$$2^3.7^{b-1}.17.3^2.5^{e-1}.29^{f-1} = \sigma^{**}(7^b).\sigma^{**}(5^e).\sigma^{**}(29^f). \tag{3.24d}$$

If $e = 1$, from $(3.24c)$, $(e = 1)$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{2752}{2401}.\frac{290}{289}.\frac{4}{3}.\frac{6}{5} = 3.421711542 > 3,$$

a contradiction.

Let $e = 2$. Since $\sigma^{**}(5^2) = 26$, taking $e = 2$ in $(3.24d)$, we find that 13 divides its left-hand side which is false.

For $e \geq 3$, using $\frac{\sigma^{**}(5^e)}{5^e} \geq \frac{756}{625}$, from $(3.24c)$ we get

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{2752}{2401}.\frac{290}{289}.\frac{4}{3}.\frac{756}{625} = 3.449085234 > 3,$$

a contradiction.

The case $d = 1$ is complete.

Let $d = 2$. Taking $d = 2$ in $(3.24a)$ and $(3.24b)$, we get

$$n = 2^6.7^b.17^2.3^2.5^e.29^f.w, \quad (b \geq 3) \tag{3.24e}$$

and

$$2^4.3^3.7^{b-1}.17.5^{e-2}.29^{f-1}.w = \sigma^{**}(7^b).\sigma^{**}(5^e).\sigma^{**}(29^f).\sigma^{**}(w), \tag{3.24f}$$

where w is prime to 2.3.5.7.17.29 and has no more than one prime factor.

We shall obtain a contradiction by examining the factors of $\sigma^{**}(7^b)$.

If b is odd or $4|b$, then $8|\sigma^{**}(7^b)$. This results in imbalance in powers of two between both sides of $(3.24f)$.

Let $b = 2k$, where k is odd. Since $b \geq 3$, we have $k \geq 3$. Also,

$$\sigma^{**}(7^b) = \left(\frac{7^k - 1}{6}\right).(7^{k+1} + 1).$$

We consider $7^{k+1} + 1$, where k is odd.

(a) $2 \| 7^{k+1} + 1$ and $3 \nmid 7^{k+1} + 1$; trivially not divisible by 7.

(b) $29 \nmid 7^t + 1$ for any t; in particular $29 \nmid 7^{k+1} + 1$.

(c) Suppose that $5 \nmid 7^{k+1} + 1$ and $17 \nmid 7^{k+1} + 1$. Then from (a) and (b) it is clear that $\frac{7^{k+1} + 1}{2}$ is > 1, odd and every prime factor of it is not in $\{3, 5, 7, 17, 29\}$. Hence each prime factor of $\frac{7^{k+1} + 1}{2}$ divides w from $(3.24f)$.

(d) Suppose that $5|7^{k+1}+1$ and $17 \nmid 7^{k+1}+1$; $5|7^{k+1}+1 \implies k+1=2u$. Hence $7^2+1|7^{k+1}+1$. Thus $5^2|7^{k+1}+1$. Assume that $\frac{7^{k+1}+1}{2}=5^\alpha$, where $\alpha \geq 2$. If $\alpha \geq 3$, then $5^3|7^{k+1}+1$. This is if and only if $k+1=10u$. Also, $7^{10}+1=2.5^3.281.4021$. It follows that $281|\frac{7^{10}+1}{2}|\frac{7^{k+1}+1}{2}=5^\alpha$ and this is impossible. Hence $\alpha=2$ so that $\frac{7^{k+1}+1}{2}=5^2$. Hence $k=1$. But $k \geq 3$. Thus $\frac{7^{k+1}+1}{2}$ is divisible by an odd prime $q \neq 5$. Also, by our assumption $q \neq 17$. Hence from (a) and (b), $q \notin \{3,5,7,17,29\}$. Since $\frac{7^{k+1}+1}{2}|\sigma^{**}(7^b)$, from $(3.24f)$, $q|w$.

(e) Suppose $17|7^{k+1}+1$ and $5 \nmid 7^{k+1}+1$. From $(3.24b)$, 17 is a unitary divisor of its left-hand side. Since $17|7^{k+1}+1|\sigma^{**}(7^b)$ it follows that $17\|7^{k+1}+1$. If $7^{k+1}+1$ is divisible by 17 alone, then we must have $\frac{7^{k+1}+1}{2}=17$ or $7^{k+1}=33$ which is not possible. Hence $\frac{7^{k+1}+1}{2}$ which is >1 and odd should be divisible by an odd prime $q \neq 17$. By our assumption $q \neq 5$. Hence from (a) and (b), $q \notin \{3,5,7,17,29\}$. From $(3.24f)$, $q|w$.

(f) Suppose that $7^{k+1}+1$ is divisible by both 5 and 17. Then $5^2|7^{k+1}+1$ and $17\|7^{k+1}+1$. Assume that $5^3|7^{k+1}+1$. This is if and only if $k+1=10u$. Also, $7^{10}+1=2.5^3.281.4021$. Thus 281 and 4021 divide $7^{k+1}+1$ which is a divisor of $\sigma^{**}(7^b)$. From $(3.24f)$, it follows that w is divisible by 281 and 4021. This is not possible. Hence $5^2\|7^{k+1}+1$. Thus $\frac{7^{k+1}+1}{2.5^2.17}$ is odd and >1. It must be divisible by an odd prime q and $q \notin \{3,5,7,17,29\}$. From $(3.24f)$, $q|w$.

(g) From (a)–(f), it follows that $\frac{7^{k+1}+1}{2}$ is divisible by an odd prime $q|w$. Since w has no more than one prime factor, $w=q^f$.

We shall now consider 7^k-1 when k is odd. We have

(h) $2\|7^k-1$ and $3|7^k-1$.

(i) $9|7^k-1$ if and only if $19|7^k-1$ if and only if $3|k$. Suppose $9|7^k-1$. Then $19|7^k-1$. Hence $19|\sigma^{**}(7^b)$. From $(3.24f)$, since $w=q^f$, $q=19$. Since $q|\frac{7^{k+1}+1}{2}$, $19|7^k-1$, 7^k-1 and $\frac{7^{k+1}+1}{2}$ are relatively prime, $q \neq 19$. This proves that $9 \nmid 7^k-1$ (as a consequence $3\|7^k-1$) and so $19 \nmid 7^k-1$.

(j) Since k is odd, 7^k-1 is not divisible by 5 and 17. Also, $29|7^k-1$ if and only if $7|k$. We have $7^7-1=2.3.29.4733$. It follows from $(3.24f)$ that $4733|w=q^f$. But $q \neq 4733$ since q and 4733 are prime factors of relatively prime factors. Hence $29 \nmid 7^k-1$.

(k) Thus $\frac{7^k-1}{6}$ is >1, odd and not divisible by any prime in $\{3,5,7,17,29\}$. If $p|\frac{7^k-1}{6}$, then p is an odd prime $\notin \{3,5,7,17,29\}$. From $(3.24f)$, $p|w=q^f$. This is not possible since $p \neq q$.

With this contradiction, the case $d=2$ is complete. Also, the case $c=2, 3|n$, is complete.

The case $b \geq 3$ with $3|n$ is complete.

Case $b \geq 3$ *with* $3 \nmid n$. We return to the equations $(3.1a)$ and $(3.1b)$. We assume that $b \geq 3$ and $3 \nmid n$. We show that n cannot be a bi-unitary triperfect number. We first settle this when $5 \nmid n$. We examine $\sigma^{**}(17^c)$ to obtain a contradiction. We distinguish the following cases:

(i) If c is odd or $4|c$, then $9|\sigma^{**}(17^c)$. From $(3.1b)$, it follows that $3|v$. But this is not true since $3 \nmid n$ has been assumed.

(ii) Let $c = 2k$, where k is odd.

(a) Then $17^k - 1$ is not divisible by $3, 5, 7, 11, 13, 23, 29$ and 37; trivially not divisible by 17.

(b) Suppose $19|17^k - 1$. This implies $9|k$ and as a consequence $17^9 - 1|17^k - 1$. We have $17^9 - 1 = 2.19.307.1270657$. Hence all the three odd prime factors of $17^9 - 1$ divide $\dfrac{17^k - 1}{6}|\sigma^{**}(17^c)$. From $(3.1b)$, these three prime factors divide v. Since v is divisible by not more than four prime factors, let p denote the possible fourth prime factor. We can assume that $p \geq 11$. Hence $n = 2^6.7^b.17^c.19^d.307^e.(1270657)^f.p^g$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{19}{18}.\frac{307}{306}.\frac{1270657}{1270656}.\frac{11}{10} = 2.953428577 < 3,$$

a contradiction. Hence $19 \nmid 17^k - 1$.

(c) $32 \nmid 17^k - 1$, since k is odd. Hence $16\|17^k - 1$.

(d) We now prove that $k > 1$. Let $k = 1$. Then $c = 2$. Since $\sigma^{**}(17^2) = 290,\ 5|\sigma^{**}(17^2)$. Taking $c = 2$ in $(3.1b)$, we find that $5|v$. This is false since $5 \nmid n$ by our assumption. Hence $k \geq 3$.

From (a)–(d), it follows that $\dfrac{17^k - 1}{16} > 1$, odd and not divisible by any of the primes $3, 5, 7, 11, 13, 17, 19, 23, 29$ and 37. Hence $\dfrac{17^k - 1}{16}$ must be divisible by a prime $p \geq 41$. Let the other three prime factors of v be p_1, p_2 and p_3, where $p_1 \geq 11$, $p_2 \geq 13$ and $p_3 \geq 19$. Hence $n = 2^6.7^b.17^c.p_1^d.p_2^e.p_3^f.p^g$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{11}{10}.\frac{13}{12}.\frac{19}{18}.\frac{41}{40} = 2.971682922 < 3,$$

a contradiction.

Thus $n = 2^6.7^b.17^c.v$ $(b \geq 3)$ is not a bi-unitary triperfect number if $3 \nmid n$ and $5 \nmid n$.

We prove that $n = 2^6 7^b 17^c v$, where $b \geq 3,\ 5|n,\ 3 \nmid n$ and $(v, 2.3.7.17) = 1$ cannot be a bi-unitary triperfect number.

We assume the contrary and obtain a contradiction.

Since $5|n$, we can write $v = 5^d w$, where $(w, 2.3.5.7.17) = 1$. Hence

$$n = 2^6 7^b 17^c 5^d w, \quad (b \geq 3). \tag{3.25a}$$

If n is a bi-unitary triperfect number, then $\sigma^{**}(n) = 3n$. Hence from $(3.25a)$ and since $\sigma^{**}(2^6) = 119 = 7.17$, the equation $\sigma^{**}(n) = 3n$ on simplification transforms into

$$3.2^6.7^{b-1}.17^{c-1}.5^d.w = \sigma^{**}(7^b)\sigma^{**}(17^c)\sigma^{**}(5^d)\sigma^{**}(w), \tag{3.25b}$$

where

$$w \text{ cannot have more than three odd prime factors.} \tag{3.25c}$$

It may be noted that $c \geq 2$ can be assumed; $c = 1$ implies that $\sigma^{**}(17^c) = 18$ and so 9 divides the left-hand side of $(3.25b)$. This is not possible since w is prime to 3.

Trivially an odd prime factor of the left-hand side of $(3.25b)$ divides w if and only if it does not belong to $\{3, 5, 7, 17\}$.

We essentially use the following lemmas (Lemmas 3.1, 3.2 and 3.3) to prove that n given in $(3.25a)$ cannot be a bi-unitary triperfect number:

Lemma 3.1. *Let n be as in $(3.25a)$ with $w = p_1^e p_2^f p_3^g$, where p_1, p_2 and p_3 are distinct odd primes with $p_1 \geq 29$, $p_2 \geq 1009$ and $p_3 \geq 1013$ and e, f, and g are positive integers. Then $\sigma^{**}(n) < 3n$. Hence n cannot be a bi-unitary perfect number.*

Proof. We have $n = 2^6 7^b 17^c 5^d p_1^e p_2^f p_3^g$ so that by Lemma 2.1,

$$\frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{5}{4}.\frac{29}{28}.\frac{1009}{1008}.\frac{1013}{1012} = 2.989869702 < 3. \qquad \square$$

Lemma 3.2. *Let $n = 2^6 7^b 17^c 5^d w$ $(b \geq 3)$ be as given in $(3.25a)$.*

(I) If b is odd or $4|b$, then n cannot be a bi-unitary triperfect number.

(II) If $b = 6$, then n cannot be a bi-unitary triperfect number.

(III) Let $b = 2k$, where $k \geq 5$ is odd. We have

$$\sigma^{**}(7^b) = \left(\frac{7^k - 1}{6}\right).(7^{k+1} + 1).$$

If n is a bi-unitary triperfect number, then:

(A) $\frac{7^k - 1}{6}$ is divisible by an odd prime $p' > 2520$ dividing w.

(B) $7^{k+1} + 1$ is divisible by an odd prime $q' \geq 1201$ dividing w.

(C) n is not divisible by 11 or 13 or 19 or 23.

Proof. We assume that n is a bi-unitary triperfect number. Then $(3.25b)$ holds.

Proof of (I). Let b be odd. We have

$$\sigma^{**}(7^b) = \frac{7^{b+1} - 1}{6} = \frac{(7^t - 1)(7^t + 1)}{6},$$

where $t = \frac{b+1}{2}$.

(i) Let t be even. Then $48 = 7^2 - 1 | 7^t - 1$ and $2 \| 7^t + 1$. Hence $16 | \frac{(7^t - 1)(7^t + 1)}{6} = \sigma^{**}(7^b)$. It follows from $(3.25b)$ that 2^6 divides its right-hand side, whereas 2^6 unitarily divides its left-hand side. Hence $w = 1$ so that from $(3.25a)$, $n = 2^6 7^b 17^c 5^d$. We have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{5}{4} = 2.881062826 < 3,$$

a contradiction.

(ii) Let t be odd. Then $8|7^t+1$ and $2\|7^t-1$. Hence $8|\frac{(7^t-1)(7^t+1)}{6} = \sigma^{**}(7^b)$. From $(3.25b)$, it follows that, 2^5 divides its right-hand side and 2^6 unitarily divides its left-hand side. Hence w cannot have more than one odd prime factor. We obtain a contradiction by showing that w is divisible by two odd prime factors.

We have $8|7^t+1$. If $16|7^t+1$ then since $2\|7^t-1$, it follows that $16|\sigma^{**}(7^b)$ and we obtain a contradiction as in (i). So we may assume that $16 \nmid 7^t+1$ and hence $8\|7^t+1$.

Since t is odd, 7^t+1 is not divisible by $3, 5$ and 17; also not divisible by 7 trivially. We have that $\frac{7^t+1}{8}$ is odd and > 1 since $t \geq 2$ as $b \geq 3$. Hence we can find an odd prime $q|\frac{7^t+1}{8}|\sigma^{**}(7^b)$; also, $q \notin \{3, 5, 7, 17\}$. From $(3.25b)$, it follows that $q|w$.

We now consider the factor 7^t-1 when t is odd.

(a) We have $2\|7^t-1$ and $3|7^t-1$.

(b) We may note that $9|7^t-1 \iff 3|t \iff 19|7^t-1$. Hence $9|7^t-1 \implies 19|7^t-1$ so that $19|\frac{7^t-1}{6}|\sigma^{**}(7^b)$. From $(3.25b)$, we see that $19|w$. Already w is divisible by q. Since $q|7^t+1$, $19|7^t-1$ and q is odd, $q \neq 19$. Thus w is divisible by two odd primes, whereas it should be divisible by not more than one odd prime. Hence $9 \nmid 7^t-1$; also, $19 \nmid 7^t-1$ and $3\|7^t-1$.

(c) Since t is odd, 7^t-1 is not divisible by 5 or 17; not divisible by 7 trivially. Thus $\frac{7^t-1}{6}$ is odd, > 1 and not divisible by $3, 5, 7$ or 17. Hence we can find an odd prime $p|\frac{7^t-1}{6}|\sigma^{**}(7^b)$ and $p \notin \{3, 5, 7, 17\}$. From $(3.25b)$, $p|w$. Since $\frac{7^t-1}{6}$ and 7^t+1 are relatively prime, we must have $p \neq q$. Hence w is divisible by two odd primes. But this cannot happen.

The proof of (I) when b is odd is complete.
Now let $b = 2k$, where k is even. This is same as $4|b$.

(iii) Since k is even, $8|7^k-1$ and since $k+1$ is odd, $8|7^{k+1}+1$, so that $32|\sigma^{**}(7^b)$. It follows that 2^8 divides the right-hand side of $(3.25b)$, but 2^6 divides its left-hand side unitarily. This is a contradiction.

The proof of (I) is complete.
<u>*Proof of (II).*</u> Let $b = 6$. Then $\sigma^{**}(7^6) = \left(\frac{7^3-1}{6}\right).(7^4+1) = 2.3.19.1201.$

If we assume that n is a bi-unitary triperfect number, taking $b = 6$ in $(3.25b)$ we get,

$$2^5.7^5.17^{c-1}.5^d.w = 19.1201.\sigma^{**}(17^c)\sigma^{**}(5^d)\sigma^{**}(w). \tag{3.25d}$$

It follows from $(3.25d)$ that w is divisible by 19 and 1201. So we can write, $w = 19^e.(1201)^f.w'$, where w' is prime to 2.3.5.7.17.19.1201. Hence from $(3.25a)$,

$$n = 2^6 7^6 17^c 5^d 19^e (1201)^f w', \tag{3.26a}$$

and from $(3.25d)$,

$$2^5.7^5.17^{c-1}.5^d.19^{e-1}.(1201)^{f-1}.w' = \sigma^{**}(17^c)\sigma^{**}(5^d)\sigma^{**}(19^e)\sigma^{**}((1201)^f)\sigma^{**}(w'), \tag{3.26b}$$

where

$$w' \text{ has at most one odd prime factor.} \tag{3.26c}$$

By examining the factors of $\sigma^{**}(5^d)$ we arrive at a contradiction to $(3.26c)$.

If d is odd, then $3|5^{d+1}-1$. Hence $3|\frac{5^{d+1}-1}{4}|\sigma^{**}(5^d)$. It follows from $(3.26b)$ that its right-hand side is divisible by 3 but its left-hand side is not.

Let $d = 2\ell$, so that $\sigma^{**}(5^d)) = \left(\frac{5^\ell - 1}{4}\right).(5^{\ell+1}+1)$.

If ℓ is even, then $3|5^\ell - 1$ and so $3|\sigma^{**}(5^d)$. This leads to a contradiction as above.

We may assume that $d = 2\ell$ and ℓ is odd.

If $\ell = 1$, then $d = 2$ and so $\sigma^{**}(5^d) = 26 = 2.13$. Taking $d = 2$ in $(3.26b)$, we infer that $13|w'$ and in view of $(3.26c)$, $w' = 13^g$, say. From $(3.26a)$, we obtain, $n = 2^6 7^6 17^c 5^2 19^e (1201)^f 13^g$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{26}{25}.\frac{19}{18}.\frac{1201}{1200}.\frac{13}{12} = 2.743348734 < 3,$$

a contradiction.

We may assume that $\ell > 1$ so that $\ell \geq 3$ since ℓ is odd.

Since ℓ is odd, $4\|5^\ell - 1$ and $5^\ell - 1$ is not divisible by 7 or 17; trivially not divisible by 5.

$19|5^\ell - 1 \iff 9|\ell$; this implies that $5^9 - 1|5^\ell - 1$. We have $5^9 - 1 = 2^2.19.31.829$. Hence 31 and 829 divide $\frac{5^\ell - 1}{4}|\sigma^{**}(5^d)$. From $(3.26b)$ it follows that w' is divisible by 31 and 829; this is not possible because of $(3.26c)$. Thus $19 \nmid 5^\ell - 1$. Also, $1201|5^\ell - 1 \iff 600|\ell$. Since ℓ is odd, $1201 \nmid 5^\ell - 1$.

Thus $\frac{5^\ell - 1}{4}$ is odd, > 1 and not divisible by $5, 7, 17, 19$ or 1201. Let $p|\frac{5^\ell - 1}{4}|\sigma^{**}(5^d)$ so that p is odd and $p \notin \{5, 7, 17, 19, 1201\}$. From $(3.26b)$, $p|w'$.

We now consider the factor $5^{\ell+1}+1$, where ℓ is odd. We have $2\|5^{\ell+1}+1$ and it is not divisible by $5, 7$ or 19; $17|5^{\ell+1}+1 \iff \ell + 1 = 8u$ (u odd); this implies that $5^8 + 1|5^{\ell+1}+1$. Since $11489|5^8 + 1$, it follows that $\sigma^{**}(5^d)$ is divisible by 11489 and from $(3.26b)$, $11489|w'$. Since $\frac{5^\ell - 1}{4}$ and $5^{\ell+1}+1$ are relatively prime, p and 11489 divide these factors respectively, we must have $p \neq 11489$. Thus w' is divisible by two odd primes contradicting $(3.26c)$. Hence $17 \nmid 5^{\ell+1}+1$.

Also, $1201|5^{\ell+1}+1 \iff \ell+1 = 300u$, where u is odd; in particular, $\ell+1 = 12u'$, where u' is odd. Hence $5^{12}+1|5^{\ell+1}$, and $5^{12}+1 = 2.313.39001$. Hence 313 and 39001 divide $5^{\ell+1}+1|\sigma^{**}(5^d)$. From $(3.26b)$ we see that w' is divisible by these two odd primes contradicting $(3.26c)$. Hence $1201 \nmid 5^{\ell+1}+1$.

Thus if $q|\frac{5^{\ell+1}+1}{2}$, then q is odd and $q \notin \{5, 7, 17, 19, 1201\}$; hence from $(3.26b)$, $q|w'$. Hence w' is divisible by two odd primes p and q, $p \neq q$ contradicting $(3.26c)$.

We have proved that when $b = 6$, n in $(3.25a)$ cannot be a bi-unitary triperfect number.

The proof of (II) is complete.

Proof of (III)(A). Let $b = 2k$, where $k \geq 5$ and odd.

We assume that n given in $(3.25a)$ is a bi-unitary triperfect number and hence $(3.25b)$ holds. Let

$$S'_7 = \{p|7^k - 1 \,:\, p \in [3, 2520] - \{3, 19, 37, 1063\} \text{ and } ord_p 7 \text{ is odd}\}.$$

From Lemma 2.4 (a) of [2], it follows that if S'_7 is non-empty, then we can find a prime $p'|\frac{7^k-1}{6}|\sigma^{**}(7^b)$ and $p' \geq 2521$; that is, III (A) of Lemma 3.2 holds. Also, from $(3.25b)$, $p'|w$.

We may assume that S'_7 is empty. Since $p \nmid 7^k - 1$ if $ord_p 7$ is even, it follows that:

(A_1) $\frac{7^k-1}{6}$ is not divisible by any prime in $[3, 2520]$ except possibly $3, 9, 37$ and 1067.

(A_2) We have $3|7^k - 1$ and since k is odd, $2\|7^k - 1$. Also, $27 \nmid 7^k - 1$. If this is not so, then $9|\frac{7^k-1}{6}|\sigma^{**}(7^b)$ and from $(3.25b)$ it follows that $3|w$ and this is not true. We settle the divisibility of $7^k - 1$ by 9 later.

(A_3) We note that $37|7^k - 1 \iff 9|k \iff 1063|7^k - 1$. We assume that $37|7^k - 1$. Hence $7^9 - 1|7^k - 1$. Also, $7^9 - 1 = 2.3^3.19.37.1063$. Hence $3^2|\frac{7^9-1}{6}|\frac{7^k-1}{6}|\sigma^{**}(7^b)$. From $(3.25b)$, $3|w$. This is false. Hence $37 \nmid 7^k - 1$.

(A_4) We have $9|7^k - 1 \iff 3|k \iff 19|7^k - 1$.

(A_5) Suppose $19 \nmid 7^k - 1$. Then $9 \nmid 7^k - 1$ and so $3\|7^k - 1$. Hence from A_1, A_2 and A_3, $\frac{7^k-1}{6}$ is not divisible by any prime in $[3, 2520]$. Since $\frac{7^k-1}{6} > 1$ and odd, if $p'|\frac{7^k-1}{6}|\sigma^{**}(7^b)$, then $p' > 2560$ and $p'|w$ by $(3.25b)$.

Hence (III)(A) of Lemma 3.2 follows.

(A_6) Suppose that $19|7^k - 1$ so that $9|7^k - 1$. Hence $9\|7^k - 1$. It follows that $\frac{7^k-1}{18}$ is odd, > 1 and not divisible by 3. We can show that $\frac{7^k-1}{18}$ is not divisible by 19 alone. Hence we can find an odd prime $p'|\frac{7^k-1}{18}$ and $p' \neq 19$. We have $p'|\frac{7^k-1}{18}|\frac{7^k-1}{6}|\sigma^{**}(7^b)$ and it follows A_1 to A_4 that $p' > 2503$. From $(3.25b)$, it is clear that $p'|w$.

This completes the proof of (III)(A).

Proof of (III)(B):

(B_1) Let

$$T'_7 = \{q|7^{k+1} + 1 \ : \ q \in [3, 1193] - \{5, 13, 181, 193, 409\}\} \text{ and } s = \frac{1}{2} ord_q 7 \text{ is even.}$$

By Lemma 2.4 (b) of [2], if T'_7 is non-empty, then we can find a prime $q'|\frac{7^{k+1}+1}{2}|\sigma^{**}(7^b)$ and $q' > 1193$. By $(3.25b)$, it follows that $q'|w$. This upholds III(B) of Lemma 3.2.

(B_2) We may assume that T'_7 is empty. Since $q \nmid 7^{k+1} + 1$ if $s = \frac{1}{2} ord_q 7$ is not even, from $T'_7 = \emptyset$, we can conclude that $\frac{7^{k+1}+1}{2}$ is not divisible by any prime in $[3, 1193]$ except possibly $5, 13, 181, 193$ and 409.

(B_3) We may note that $193|7^{k+1}+1 \iff 12|k \iff 409|7^{k+1}+1$. Suppose that $193|7^{k+1}+1$. This implies that $7^{12} + 1|7^{k+1} + 1$. Also, $7^{12} + 1 = 2.73.193.409.1201$. Hence $7^{k+1} + 1|\sigma^{**}(7^b)$ is divisible by four odd primes $73, 193, 409$ and 1201. From $(3.25b)$, these four odd primes divide w. This contradicts $(3.25c)$. Thus $\frac{7^{k+1}+1}{2}$ is not divisible by 193 and 409.

(B_4) We note that $13|7^{k+1}+1$ if and only if $k+1=6u$ if and only if $181|7^{k+1}+1$. Assume that $13|7^{k+1}+1$ so that $181|7^{k+1}+1$ and $k+1=6u$. Hence $7^6+1|7^{k+1}+1$. Also, $7^6+1=2.5^2.13.181$. So, $5^2|7^{k+1}+1$. We now show that $5^3 \nmid 7^{k+1}+1$. We have $5^3|7^{k+1}+1$ if and only if $k+1=10u$; also, $7^{10}+1=2.5^3.281.4021$. Thus $5^3|7^{k+1}+1$ implies that 281 and 4021 divide $7^{k+1}+1|\sigma^{**}(7^b)$. From $(3.25b)$, it follows that 281 and 4021 are factors of w. Already, 13 and 181 are factors of $7^{k+1}+1|\sigma^{**}(7^b)$ and from $(3.25b)$, 13 and 181 divide w also. Thus four prime factors divide w contradicting $(3.25c)$. Hence $5^3 \nmid 7^{k+1}+1$ and so $5^2\|7^{k+1}+1$.

Clearly, $\dfrac{7^{k+1}+1}{50}$ is odd, >1 and not divisible by 5. We note that $13^2|7^{k+1}+1$ if and only if $k+1=78u$. Hence $13^2|7^{k+1}+1$ implies that $7^{78}+1|7^{k+1}+1$. From Appendix G of [2], we can see that $7^{78}+1$ has more than three prime factors dividing w. This cannot happen. Hence $13^2 \nmid 7^{k+1}+1$ and so $13\|7^{k+1}+1$. Further, $181^2|7^{k+1}+1$ if and only if $k+1=1068u$; also, from Appendix G of [2], $7^{1068}+1$ has more than three prime factors dividing w. This contradicts $(3.25c)$. Hence $181^2 \nmid 7^{k+1}+1$ and so $181\|7^{k+1}+1$.

Thus $13|7^{k+1}+1$ implies that 13 and 181 are unitary divisors of $\dfrac{7^{k+1}+1}{50}$; if it is divisible by 13 and 181 alone, then we should have $\dfrac{7^{k+1}+1}{50}=13.181$ and so $k=5$ or $b=10$. We now prove that $b=10$ is not possible.

We have $\sigma^{**}(7^{10})=\left(\dfrac{7^5-1}{6}\right).(7^6+1)=2.5^2.13.181.2801$. Thus $\sigma^{**}(7^{10})$ is divisible by three prime factors dividing w in $(3.25b)$. From $(3.25c)$, we have $w=13^e.181^f.(2801)^g$. Taking $b=10$ in $(3.25a)$ and $(3.25b)$, we get

$$n=2^6.7^{10}.17^c.5^d.13^e.181^f.(2801)^g, \tag{3.27a}$$

and

$$\begin{aligned}&3.2^5.7^9.17^{c-1}.5^{d-2}.13^{e-1}.181^{f-1}.(2801)^{g-1}\\&\quad=\sigma^{**}(17^c).\sigma^{**}(5^d).\sigma^{**}(13^e).\sigma^{**}(181^f).\sigma^{**}((2801)^g),\end{aligned} \tag{3.27b}$$

where $c\ge 2$ and $d\ge 2$.

We obtain a contradiction by examining the factors of $\sigma^{**}(17^c)$ in different cases.

If c is odd or $4|c$, then $9|\sigma^{**}(17^c)$. It follows from $(3.27b)$ that this cannot happen.

Hence we may assume that $c=2\ell$ and ℓ is odd. Since ℓ is odd, $17^\ell-1$ is not divisible by $3,5,7$ and 13; trivially not divisible by 17. Also, 17^t-1 is divisible by 32 if and only if t is even; divisible by 181 if and only if $36|t$ and by 2801 if and only if $56|t$. In these cases, all the values of t must be even. Since ℓ is odd, $17^\ell-1$ is not divisible by 32 or 181 or 2801. Since $16|17^\ell-1$ and $32\nmid 17^\ell-1$, we have $16\|17^\ell-1$. Hence $\dfrac{17^\ell-1}{16}$ is odd.

If $\ell=1$, then $c=2$ and $\sigma^{**}(17^2)=290$. Hence $29|\sigma^{**}(17^2)$. Taking $c=2$ in $(3.27b)$, we find that 29 should divide the right-hand side of it. This is is not possible.

Hence $\ell\ge 3$ and so $\dfrac{17^\ell-1}{16}>1$. Thus $\dfrac{17^\ell-1}{16}>1$, odd and not divisible by $3,5,7,13,17$, 181 and 2801. Since $\dfrac{17^\ell-1}{16}$ is a factor of $\sigma^{**}(17^c)$, this cannot happen by virtue of $(3.27b)$. Therefore, $b=10$ is not possible.

This proves that $\dfrac{7^{k+1}+1}{50}$ must be divisible by an odd prime $q' \notin \{5, 13, 181\}$. Now $q'|\dfrac{7^{k+1}+1}{50}|\dfrac{7^{k+1}+1}{2}$ and we already proved that $\dfrac{7^{k+1}+1}{2}$ is not divisible by any prime in $[3, 1193] - \{5, 13, 181\}$, it follows that $q' > 1193$ (or $q' \geq 1201$).

Thus we proved (III)(B) when $13|7^{k+1}+1$.

(B_5) Assume that $13 \nmid 7^{k+1}+1$ and hence $181 \nmid 7^{k+1}+1$. If $5 \nmid 7^{k+1}+1$, then none of the primes in $[3, 1193]$ is a factor of $\dfrac{7^{k+1}+1}{2}$ and so every prime factor of it exceeds 1193. This upholds the statement in (III)(B). Hence we may assume that $5|\dfrac{7^{k+1}+1}{2}$. This is if and only if $k+1 = 2u$; hence $7^2+1 = 50|7^{k+1}+1$. Thus $5^2|7^{k+1}+1$. If $5^3|7^{k+1}+1$ then $k+1 = 10u$. Hence $7^{10}+1 = 2.5^3.281.4021$ is a factor of $7^{k+1}+1$. From $(3.25b)$, it follows that w is divisible by 281 and 4021; also, w is divisible by $p' > 2521$ dividing $\dfrac{7^k-1}{6}$ from (III)(A). Hence from $(3.25a)$, $n = 2^6.7^b.17^c.5^d.281^e.(4021)^f.p'^g$, and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{5}{4}.\frac{281}{280}.\frac{4021}{4020}.\frac{2521}{2520} = 2.893219225 < 3,$$

a contradiction.

Hence $5^3 \nmid 7^{k+1}+1$ and so $5^2 \| 7^{k+1}+1$. If $\dfrac{7^{k+1}+1}{2}$ is divisible by 5 alone, then we must have $\dfrac{7^{k+1}+1}{2} = 5^2$ or $k = 1$. But $k \geq 5$. This contradiction proves that $\dfrac{7^{k+1}+1}{2}$ must be divisible by an odd prime $q' \neq 5$. By our assumption, $7^{k+1}+1$ is not divisible by 13 and from B_1, B_2 and B_3, it follows that $q' > 1193$ and from $(3.25b)$, $q'|w$, since q' is a factor of $\sigma^{**}(7^b)$.

Thus the proof of (III)(B) is complete.

Proof of (III)(C). Let n be as given in $(3.25a)$ and $(3.25b)$. First we observe that $c \geq 3$. When $c = 2$, $\sigma^{**}(17^2) = 290$. Taking $c = 2$ in $(3.25b)$, we see that $29|w$. Let p' and q' be the primes dividing w obtained in (III)(A) and (III)(B), where $p' \geq 2521$ and $q' > 1193$; so, $q' \geq 1201$. Now the primes $29, p'$ and q' satisfy the hypothesis of Lemma 3.1. Hence n cannot be a bi-unitary triperfect number contrary to our assumption. Hence $c \geq 3$.

(i) Suppose that $11|n$. Hence from $(3.25a)$, $11|w$. By $(3.25c)$, $w = 11^e p'^f q'^g$. From $(3.25a)$ and $(3.25b)$, we have

$$n = 2^6.7^b.17^c.5^d.11^e.p'^f.q'^g, \tag{3.28a}$$

and

$$3.2^6.7^{b-1}.17^{c-1}.5^d.11^e.p'^f.q'^g = \sigma^{**}(7^b)\sigma^{**}(17^c)\sigma^{**}(5^d)\sigma^{**}(11^e)\sigma^{**}(p'^f)\sigma^{**}(q'^g). \tag{3.28b}$$

When $e = 1$, $\sigma^{**}(11^e) = 12$. Hence $4|\sigma^{**}(11^e)$. From $(3.28b)$, it follows that 2^7 divides its right-hand side, whereas 2^6 is a unitary divisor of its left-hand side. This is a contradiction.

If $e = 2$, $\sigma^{**}(11^e) = 122 = 2.61$. Taking $e = 2$ in $(3.28b)$, we find that 61 divides its left-hand side but it cannot divide its right-hand side. Hence we may assume that $e \geq 3$.

Hence, without loss of generality, we can assume that $b \geq 9,\ c \geq 3$, and $e \geq 3$. By Lemma 2.1, we have

$$\frac{\sigma^{**}(7^b)}{7^b} \geq \frac{6723200}{5764801} \quad (b \geq 9), \tag{3.28c}$$

$$\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{88452}{83521} \quad (c \geq 3) \tag{3.28d}$$

and $\frac{\sigma^{**}(11^e)}{11^e} \geq \frac{15984}{14641}$, $(e \geq 3)$. Also, if $d \geq 3$, then $\frac{\sigma^{**}(5^d)}{5^d} \geq \frac{756}{625}$. From these results and $(3.28a)$, when $d \geq 3$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{6723200}{5764801}.\frac{88452}{83521}.\frac{756}{625}.\frac{15984}{14641} = 3.032684127 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

If $d = 1$, from $(3.28a)$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{6723200}{5764801}.\frac{88452}{83521}.\frac{6}{5}.\frac{15984}{14641} = 3.008615205 > 3,$$

a contradiction.

If $d = 2,\ \sigma^{**}(5^d) = 26$. Hence from $(3.28b)$, it follows that 13 divides its right-hand side but it cannot divide its left-hand side.

Hence $11 \nmid n$.

(ii) Suppose $13|n$. Hence $w = 13^e p'^f q'^g$. From $(3.25a)$ and $(3.25b)$, we get

$$n = 2^6.7^b.17^c.5^d.13^e.p'^f.q'^g, \tag{3.29a}$$

and

$$3.2^6.7^{b-1}.17^{c-1}.5^d.13^e.p'^f.q'^g = \sigma^{**}(7^b)\sigma^{**}(17^c)\sigma^{**}(5^d)\sigma^{**}(13^e)\sigma^{**}(p'^f)\sigma^{**}(q'^g). \tag{3.29b}$$

By Lemma 2.1, for $d \geq 5$, $\frac{\sigma^{**}(5^d)}{5^d} \geq \frac{19406}{15625}$ and for $e \geq 3$, $\frac{\sigma^{**}(13^e)}{13^e} \geq \frac{30772}{28561}$. Since we have $b \geq 9$ and $c \geq 3$, from $(3.29a)$, we obtain for $d \geq 5$ and $e \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{6723200}{5764801}.\frac{88452}{83521}.\frac{19406}{15625}.\frac{30772}{28561} = 3.073045463 > 3,$$

a contradiction. Thus $d \geq 5$ implies that $e = 1$ or $e = 2$.

If $d \geq 5$ and $e = 1$, from $(3.29a)$, we get,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{6723200}{5764801}.\frac{88452}{83521}.\frac{19406}{15625}.\frac{14}{13} = 3.071647353 > 3,$$

a contradiction.

Let $d \geq 5$ and $e = 2$. Taking $e = 2$ in $(3.29a)$, we get

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{5}{4}.\frac{170}{169}.\frac{2521}{2520}.\frac{1201}{1200} = 2.901676629 < 3,$$

a contradiction.

Thus $d \geq 5$ cannot occur. Hence d takes the choices $1, 2, 3$ and 4.

Let $d = 1$. Taking $d = 1$ in $(3.29a)$, we get

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{6}{5}.\frac{13}{12}.\frac{2521}{2520}.\frac{1201}{1200} = 2.999992261 < 3,$$

a contradiction.

If $d = 2$, from $(3.29a)$, we obtain

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{26}{25}.\frac{13}{12}.\frac{2521}{2520}.\frac{1201}{1200} = 2.599993292 < 3,$$

a contradiction. Since $\sigma^{**}(5^3) = 2^2.3.13$ and $\sigma^{**}(5^4) = 2^2.3^2.7$, $\sigma^{**}(5^d)$ is divisible by 4 when $d = 3$ or $d = 4$. In these cases 2^7 divides the right-hand side of $(3.29b)$ while 2^6 is a unitary divisor of its left-hand side.

Thus $13 \nmid n$.

(iii) We assume that $19|n$ so that $w = 19^e p'^f q'^g$. From $(3.25a)$ and $(3.25b)$, we have

$$n = 2^6.7^b.17^c.5^d.19^e.p'^f.q'^g, \tag{3.29c}$$

and

$$3.2^6.7^{b-1}.17^{c-1}.5^d.19^e.p'^f.q'^g = \sigma^{**}(7^b)\sigma^{**}(17^c)\sigma^{**}(5^d)\sigma^{**}(19^e)\sigma^{**}(p'^f)\sigma^{**}(q'^g). \tag{3.29d}$$

If $d \geq 7$ and $e \geq 3$, from $(3.29c)$ we obtain (since $b \geq 9,\ c \geq 3$)

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{6723200}{5764801}.\frac{88452}{83521}.\frac{487656}{390625}.\frac{137561}{130321} = 3.026252265 > 3,$$

a contradiction; in the above we used $(3.28c)$, $(3.28d)$,

$$\frac{\sigma^{**}(5^d)}{5^d} \geq \frac{487656}{390625}\ \ (d \geq 7)\ \text{ and }\ \frac{\sigma^{**}(19^e)}{19^e} \geq \frac{137561}{130321}\ \ (e \geq 3).$$

Thus $d \geq 7$ implies that $e = 1$ or $e = 2$.

Let $d \geq 7$. If $e = 1$, then $\sigma^{**}(19^e) = 20$. Hence $4|\sigma^{**}(19^e)$. From $(3.29d)$ it follows that there is a mismatch in powers of two between two sides of $(3.29d)$. If $e = 2$, then $\sigma^{**}(19^e) = 362 = 2.181$. Taking $e = 2$ in $(3.29d)$, we see that 181 divides the left-hand side of $(3.29d)$, which is false.

Hence $d \geq 7$ is not possible so that $1 \leq d \leq 6$.

Taking $d = 1$ in $(3.29c)$, we have $n = 2^6.7^b.17^c.5.19^e.p'^f.q'^g$, and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{6}{5}.\frac{19}{18}.\frac{2521}{2520}.\frac{1201}{1200} = 2.923069382 < 3,$$

a contradiction.

When $d = 2$, $\sigma^{**}(5^d) = 26 = 2.13$. Taking $d = 2$ in $(3.29d)$, we see that 13 divides the left-hand side of it and this is not possible.

We have $\sigma^{**}(5^3) = 2^2.3.13$ and $\sigma^{**}(5^4) = 2^2.3^2.7$. Thus if $d = 3$ or $d = 4$, $4|\sigma^{**}(5^d)$; this results in imbalance in the powers of two between both sides of $(3.29d)$.

Also, $\sigma^{**}(5^5) = 2.3^2.7.31$ and $\sigma^{**}(5^6) = 2.31.313$. Hence if $d = 5$ or $d = 6,\ 31|\sigma^{**}(5^d)$. From $(3.29d)$, it follows that 31 divides the left-hand side of it. This is not possible.

Thus $19 \nmid n$.

(iv) We prove that $23 \nmid n$. On the contrary we assume that $23|n$ and obtain a contradiction. Let $23|n$ and hence $w = 23^e p'^f q'^g$. From $(3.25a)$ and $(3.25b)$, we get,

$$n = 2^6.7^b.17^c.5^d.23^e.p'^f.q'^g, \quad (b \geq 9,\ c \geq 3) \tag{3.30a}$$

and

$$3.2^6.7^{b-1}.17^{c-1}.5^d.23^e.p'^f.q'^g = \sigma^{**}(7^b)\sigma^{**}(17^c)\sigma^{**}(5^d)\sigma^{**}(23^e)\sigma^{**}(p'^f)\sigma^{**}(q'^g). \tag{3.30b}$$

By Lemma 2.1, we have

$$\frac{\sigma^{**}(7^b)}{7^b} \geq \frac{6723200}{5764801} \quad (b \geq 9),$$
$$\frac{\sigma^{**}(17^c)}{17^c} \geq \frac{25641254}{24137569} \quad (c \geq 5),$$
$$\frac{\sigma^{**}(5^d)}{5^d} \geq \frac{487656}{390625} \quad (d \geq 7),$$
$$\frac{\sigma^{**}(23^e)}{23^e} \geq \frac{154752626}{148035889} \quad (e \geq 5).$$

From $(3.30a)$, we have for $c \geq 5,\ d \geq 7,$ and $e \geq 5$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{119}{64}.\frac{6723200}{5764801}.\frac{25641254}{24137569}.\frac{487656}{390625}.\frac{154752626}{148035889} = 3.006276895 > 3,$$

a contradiction.

Hence $c \geq 5,\ d \geq 7$ implies $1 \leq e \leq 4$. Assume that $c \geq 5,\ d \geq 7$.

We have $\sigma^{**}(23) = 24 = 2^3.3,\ \sigma^{**}(23^3) = 2^4.3.5.53$ and $\sigma^{**}(23^4) = 2^6.3^3.13^2$. Hence $2^3|\sigma^{**}(23^e)$ when $e = 1, 3, 4$. From $(3.30b)$, this is not possible as in such a case 2^8 divides its right-hand side, whereas its left-hand side is divisible by 2^6 unitarily.

When $e = 2,\ \sigma^{**}(23^e) = 2.5.53$. Taking $e = 2$ in $(3.30b)$, it follows that 53 should divide its left-hand side. This is not possible.

Thus $c \geq 5,\ d \geq 7$ cannot hold.

Let $c \geq 5$ and $1 \leq d \leq 6$.

When $d = 1$, from $(3.30a)$, $n = 2^6.7^b.17^c.5.23^e.p'^f.q'^g$ and so we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{119}{64}.\frac{7}{6}.\frac{17}{16}.\frac{6}{5}.\frac{23}{22}.\frac{2521}{2520}.\frac{1201}{1200} = 2.895097426 < 3,$$

a contradiction.

When $d = 2,\ 13|\sigma^{**}(5^d)$; in this case from $(3.30b)$, 13 should divide its left-hand side. This is not possible.

When $d = 3$ or $d = 4,\ 4|\sigma^{**}(5^d)$; in these two cases there will be a mismatch of the powers of 2 between its two sides.

When $d = 5$ or $d = 6$, $31|\sigma^{**}(5^d)$. Hence form $(3.30b)$, 31 should divide its left-hand side. This is not possible.

Hence $c \geq 5$ is not possible. So, we must have $c = 3$ or $c = 4$ since $c \geq 3$.

Since $\sigma^{**}(17^3) = 2^2.3^2.5.29$ and $\sigma^{**}(17^4) = 2^2.3^5.7.13$, $4|\sigma^{**}(17^c)$ when $c = 3$ or $c = 4$. In these cases we obtain a contradiction from $(3.30b)$ due to the imbalance of the powers of 2 between its two sides.

Thus $23 \nmid n$.

This proves (III)(C) in all the cases. □

Lemma 3.3. *The number* $n = 2^6 7^b 17^c v$, *where* $b \geq 3$, $5|n$, $3 \nmid n$ *and* $(v, 2.3.7.17) = 1$ *cannot be a bi-unitary triperfect number.*

Proof. Since $5|n$, n is of the form given in $(3.25a)$. Suppose that n is a bi-unitary triperfect number. By (III)(A) and (B) of Lemma 3.2, w is divisible by primes $p' > 2507$ and $q' > 1201$. Let us redesignate p' and q' by p_2 and p_3. Since w cannot have more than three odd prime factors, by (III)(C), a possible third prime factor say p_1 of w will be ≥ 29. Now, the primes p_1, p_2 and p_3 satisfy the hypothesis of Lemma 3.1. Hence n cannot be a bi-unitary triperfect number. □

This completes the proof of (a) of Theorem 3.1 and also the proof of Theorem 3.1. □

4 Concluding remarks

Partial results on bi-unitary triperfect numbers divisible unitarily by 2^7 are obtained. We mention one such result: if n is a bi-unitary triperfect number divisible unitarily by 2^7 and 5^2, then $n = 44553600$. We could fix bi-unitary triperfect numbers divisible unitarily by 2^8; 57657600 is the only such number.

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