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On the quantity $I(q^k) + I(n^2)$ where $q^k n^2$ is an odd perfect number

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Abstract: In this note, we pursue an approach started in the M. Sc. thesis of the author and thereby attempt to produce stronger bounds for the sum $I(q^k) + I(n^2)$, where $q^k n^2$ is an odd perfect number with special prime q and $I(x)$ is the abundancy index of the positive integer x.
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1 Introduction

Let X be a positive integer. We denote the sum of the divisors of X by

$$\sigma(X) = \sum_{d \mid X} d.$$

We also denote the deficiency of X by

$$D(X) = 2X - \sigma(X),$$

the sum of the aliquot divisors of X by

$$s(X) = \sigma(X) - X,$$

and the abundancy index of X by $I(X) = \sigma(X)/X$.

Note that both σ and I are multiplicative. That is, if $\gcd(A, B) = 1$, then we have

$$\sigma(AB) = \sigma(A)\sigma(B)$$

and

$$I(AB) = I(A)I(B).$$

If $\gcd(C, D) > 1$, then we have

$$\sigma(CD) < \sigma(C)\sigma(D)$$

and

$$I(CD) < I(C)I(D),$$

so that in general we have the inequalities

$$\sigma(YZ) \leq \sigma(Y)\sigma(Z)$$

and

$$I(YZ) \leq I(Y)I(Z)$$

for σ and I. Equality holds if and only if $\gcd(Y, Z) = 1$. Lastly, note that although the deficiency function D is not multiplicative, it is in general true that the inequality

$$D(YZ) \leq D(Y)D(Z)$$

holds whenever $\gcd(Y, Z) = 1$, per a result in Dris [5].

If m is odd and $\sigma(m) = 2m$, then m is called an odd perfect number. Euler proved that an odd perfect number, if one exists, must have the form $m = q^k n^2$ where q is the special prime satisfying $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. Note that we have

$$\sigma(q^k)\sigma(n^2) = \sigma(q^k n^2) = \sigma(m) = 2m = 2q^k n^2$$

so that we obtain

$$\frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} = \gcd(n^2, \sigma(n^2)) = \frac{D(n^2)}{s(q^k)} = \frac{2s(n^2)}{D(q^k)}.$$

Descartes and Frenicle conjectured that $k = 1$ always holds. More recently, Sorli predicts $k = 1$ after testing large numbers with eight distinct prime factors for perfection. To date, no proof of the Descartes–Frenicle–Sorli Conjecture on odd perfect numbers is known, although various equivalent conditions have been derived by Dris [4], and Dris and Tejada [2].

In this note, we pursue an approach started in the M. Sc. thesis by Dris [7] and thereby attempt to produce stronger bounds for the sum $I(q^k) + I(n^2)$. Currently, we know by Dris [6] that

$$\frac{57}{20} < I(q^k) + I(n^2) < 3$$

and that these bounds are best-possible.

We also know that

$$\frac{q+1}{q} \leq I(q^k) < \frac{q}{q-1} < \frac{2(q-1)}{q} < I(n^2) \leq \frac{2q}{q+1},$$

from which we get

$$\left(I(q^k) - \frac{q}{q-1}\right)\left(I(n^2) - \frac{q}{q-1}\right) < 0$$

and

$$\left(I(q^k) - \frac{q+1}{q}\right)\left(I(n^2) - \frac{q+1}{q}\right) \geq 0.$$

Using the fact that $I(q^k)I(n^2) = I(q^k n^2) = 2$, we obtain

$$\frac{2(q-1)}{q} + \frac{q}{q-1} < I(q^k) + I(n^2) \leq \frac{2q}{q+1} + \frac{q+1}{q}.$$

Notice that the lower bound equals

$$L(q) = \frac{2(q-1)}{q} + \frac{q}{q-1} = \frac{3q^2 - 4q + 2}{q(q-1)} = 3 - \frac{q-2}{q(q-1)}$$

and that the upper bound equals

$$U(q) = \frac{2q}{q+1} + \frac{q+1}{q} = \frac{3q^2 + 2q + 1}{q(q+1)} = 3 - \frac{q-1}{q(q+1)}.$$

Equality holds in $L(q) < I(q^k) + I(n^2) \leq U(q)$ if and only if the Descartes–Frenicle–Sorli Conjecture on odd perfect numbers holds.

2 On $D(q^k)D(n^2) = 2s(q^k)s(n^2)$

From the Introduction, we have the equation

$$\frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} = \gcd(n^2, \sigma(n^2)) = \frac{D(n^2)}{s(q^k)} = \frac{2s(n^2)}{D(q^k)},$$

from which we obtain

$$D(q^k)D(n^2) = 2s(q^k)s(n^2).$$

We begin with a proof of the following lemma.

Lemma 2.1. *If $m = q^k n^2$ is an odd perfect number with special prime q, then*

$$D(q^k)D(n^2) = 2s(q^k)s(n^2) = \frac{2n^2(q^k-1)(q^{k+1}-2q^k+1)}{(q-1)(q^{k+1}-1)}.$$

Proof. Let $m = q^k n^2$ be an odd perfect number with special prime q.

Since q is prime and using that

$$\sigma(n^2) = \frac{2q^k n^2}{\sigma(q^k)}$$

and

$$\sigma(q^k) = \frac{q^{k+1}-1}{q-1},$$

we have

$$\begin{aligned}
D(q^k)D(n^2) &= 2s(q^k)s(n^2) = 2(\sigma(q^k) - q^k)(\sigma(n^2) - n^2) \\
&= 2(\sigma(q^k) - q^k)\left(\frac{2q^k n^2}{\sigma(q^k)} - n^2\right) \\
&= 2n^2\left(3q^k - \sigma(q^k) - \frac{2q^{2k}}{\sigma(q^k)}\right) \\
&= 2n^2\left(3q^k - \frac{q^{k+1}-1}{q-1} - \frac{2q^{2k}(q-1)}{q^{k+1}-1}\right) \\
&= 2n^2 \cdot \frac{3q^k(q-1)(q^{k+1}-1) - (q^{k+1}-1)^2 - 2q^{2k}(q-1)^2}{(q-1)(q^{k+1}-1)} \\
&= \frac{2n^2(q^k-1)(q^{k+1}-2q^k+1)}{(q-1)(q^{k+1}-1)}. \qquad \square
\end{aligned}$$

We can now prove the following theorem, using Lemma 2.1.

Theorem 2.2. *If $m = q^k n^2$ is an odd perfect number with special prime q, then*

$$3 - (I(q^k) + I(n^2)) = \frac{(q^k-1)(q^{k+1}-2q^k+1)}{q^k(q-1)(q^{k+1}-1)}.$$

Proof. Let $m = q^k n^2$ be an odd perfect number with special prime q.

By Lemma 2.1, we have

$$D(q^k)D(n^2) = 2s(q^k)s(n^2) = \frac{2n^2(q^k-1)(q^{k+1}-2q^k+1)}{(q-1)(q^{k+1}-1)}.$$

Dividing both sides of

$$2s(q^k)s(n^2) = \frac{2n^2(q^k-1)(q^{k+1}-2q^k+1)}{(q-1)(q^{k+1}-1)}$$

by $2q^k n^2$, we obtain

$$\frac{s(q^k)}{q^k} \cdot \frac{s(n^2)}{n^2} = \frac{(q^k-1)(q^{k+1}-2q^k+1)}{q^k(q-1)(q^{k+1}-1)}.$$

But we also have

$$\begin{aligned}
\frac{s(q^k)}{q^k} \cdot \frac{s(n^2)}{n^2} &= \big(I(q^k) - 1\big)\big(I(n^2) - 1\big) = \big(I(q^k)I(n^2) + 1\big) - \big(I(q^k) + I(n^2)\big) \\
&= 3 - (I(q^k) + I(n^2)).
\end{aligned}$$

This finishes the proof. $\square$

We now attempt to find the global extrema for the expression

$$3 - (I(q^k) + I(n^2)) = \frac{(q^k-1)(q^{k+1}-2q^k+1)}{q^k(q-1)(q^{k+1}-1)},$$

first when the expression is considered as a function of k, and then when the expression is considered as a function of q.

2.1 Global extrema for $f_1(k) = \frac{(q^k-1)(q^{k+1}-2q^k+1)}{q^k(q-1)(q^{k+1}-1)}$

Theorem 2.3. *If $m = q^k n^2$ is an odd perfect number with special prime q, then*

$$\frac{q-1}{q(q+1)} = f_1(1) \leq f_1(k) = 3 - (I(q^k) + I(n^2)) < \frac{q-2}{q(q-1)}.$$

Proof. Let

$$f_1(k) = \frac{(q^k-1)(q^{k+1}-2q^k+1)}{q^k(q-1)(q^{k+1}-1)}.$$

Then, we have

$$f_1{}'(k) = \frac{(q-4)q^{2k+1} + 2q^{k+1} + 2q^{2k} - 1}{q^k(q-1)(q^{k+1}-1)^2} \ln q,$$

which is positive for $k \geq 1$ and $q \geq 5$.

Thus, we see that $f_1(k)$ is increasing for $k \geq 1$.

Since

$$f_1(1) = \frac{q-1}{q(q+1)}$$

and

$$\lim_{k\to\infty} f_1(k) = \lim_{k\to\infty} \frac{(1-\frac{1}{q^k})(1-\frac{2}{q}+\frac{1}{q^{k+1}})}{(q-1)(1-\frac{1}{q^{k+1}})} = \frac{q-2}{q(q-1)},$$

we have

$$\frac{q-1}{q(q+1)} \leq f_1(k) < \frac{q-2}{q(q-1)},$$

with the further result that these bounds are the best possible. □

2.2 Global extrema for $f_2(q) = \frac{(q^k-1)(q^{k+1}-2q^k+1)}{q^k(q-1)(q^{k+1}-1)}$

Theorem 2.4. *If $m = q^k n^2$ is an odd perfect number with special prime q, then*

$$0 < f_2(q) = 3 - (I(q^k) + I(n^2)) \leq f(5) = \frac{(5^k-1)(5^{k+1} - 2\cdot 5^k + 1)}{4\cdot 5^k(5^{k+1}-1)} < \frac{3}{20}.$$

If $k = 1$, then

$$0 < f_2(q) = 3 - (I(q^k) + I(n^2)) \leq \frac{2}{15}.$$

Proof. Let

$$f_2(q) = \frac{(q^k-1)(q^{k+1}-2q^k+1)}{q^k(q-1)(q^{k+1}-1)}.$$

We obtain, with some help of WolframAlpha,

$$f_2{}'(q) = \frac{-(q^{k+1} - kq + k - q)(q^{2k+1}(q-4) + 2q^{k+1} + 2q^{2k} - 1)}{q^{k+1}(q-1)^2(q^{k+1}-1)^2}.$$

Here, let

$$g(q) = q^{k+1} - kq + k - q.$$

Then,

$$g'(q) = (q^k - 1)(k+1) > 0.$$

Hence, $g(q)$ is increasing and

$$g(q) \geq g(5) = 5^{k+1} - 4k - 5 > 0.$$

It follows that $f_2(q)$ is strictly decreasing for $q \geq 5$. Since $\lim_{q \to \infty} f(q) = 0$, we have

$$0 < f_2(q) \leq f(5) = \frac{(5^k - 1)(5^{k+1} - 2 \cdot 5^k + 1)}{4 \cdot 5^k (5^{k+1} - 1)} < \frac{3}{20}.$$

But we have

$$f_2(q) = 3 - \left(I(q^k) + I(n^2)\right).$$

This implies that

$$\frac{57}{20} < 3 - \left(\frac{(5^k - 1)(5^{k+1} - 2 \cdot 5^k + 1)}{4 \cdot 5^k (5^{k+1} - 1)}\right) \leq I(q^k) + I(n^2) < 3,$$

with the further result that these bounds are the best possible. (Note that, when $k = 1$, we have the slightly stronger bound $43/15 \leq I(q^k) + I(n^2) < 3$.) $\square$

3 Concluding remarks and further research

Notice that, although we were unsuccessful in improving the bounds for $I(q^k) + I(n^2)$ (and therefore, we were unable to obtain either nontrivial lower or upper bounds for q; see the paper by Dris [3] for more information), we were able to extract useful information on the common value for

$$D(q^k)D(n^2) = 2s(q^k)s(n^2)$$

and thereby get additional data about the quantity

$$3 - (I(q^k) + I(n^2)).$$

For example, from a finding in Dris [3], it is known that the improved bound $q > 5$ is equivalent to the improved lower bound

$$I(q^k) + I(n^2) > \frac{43}{15}.$$

Thus, if we assume that $q = 5$, then tackling the two remaining cases separately, we obtain

- **Case 1.** If $q = 5$ and $k = 1$, then

$$I(q^k) + I(n^2) = I(q) + I(n^2) = \frac{3q^2 + 2q + 1}{q(q+1)} = I(5) + \frac{2}{I(5)} = \frac{43}{15}.$$

- **Case 2.** If $q = 5$ and $k > 1$, then using the facts that $q^k \neq 5^5$ (see the paper by Cohen and Sorli [1]), and $k \equiv 1 \pmod 4$, we obtain $5^9 \mid q^k$, so that

$$I(5^9) \leq I(q^k) < \frac{5}{4} < \frac{8}{5} < I(n^2) \leq \frac{2}{I(5^9)},$$

where

$$I(5^9) = \frac{2441406}{1953125} = 1.249999872,$$

from which we get

$$\frac{57}{20} < I(q^k) + I(n^2) = I(5^k) + I(n^2) \leq I(5^9) + \frac{2}{I(5^9)},$$

where

$$I(5^9) + \frac{2}{I(5^9)} = \frac{6794928894043}{2384185546875} \approx 2.85000003584.$$

Unfortunately, **Case 1** does not yield a fruitful result (as it is currently unknown whether $5/3$ is an abundancy index or otherwise), and while **Case 2** appears promising, it also does not add to our existing knowledge, as attempting to solve the resulting inequality

$$3 - \left(\frac{(5^k - 1)(5^{k+1} - 2 \cdot 5^k + 1)}{4 \cdot 5^k (5^{k+1} - 1)}\right) \leq I(q^k) + I(n^2) \leq \frac{6794928894043}{2384185546875}$$

only implies that $k \geq 9$ and nothing more.

On a final note: Notice that

$$I(q^k) + I(n^2) = I(q^k) + \frac{2}{I(q^k)} = I(n^2) + \frac{2}{I(n^2)},$$

which somehow leaves the impression that a consideration of the rational function

$$Q(M) = M + \frac{2}{M} = \frac{M^2 + 2}{M}$$

may be in order. The author is not well-versed nor an expert on quadratic forms, but his intuition tells him that perhaps if the rational function $Q(M)$ could be expressed as a quadratic form, then the comprehensive theory of quadratic forms (see Malyshev [8] for a survey of the field) could potentially be used to bear on the problem of getting improved lower and upper bounds for the quantity

$$I(q^k) + I(n^2).$$

We leave this as an open problem for other researchers to investigate.

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