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On the equation $\varphi(n) + d(n) = n$ and related inequalities

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Abstract: We study the equation $\varphi(n) + d(n) = n$, and prove related new inequalities.
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1 Introduction

Let $n \geq 1$ be a positive integer, and denote by $\varphi(n)$ the Euler totient function. Let $d(n)$ denote the number of divisors of n. Put $\varphi(1) = d(1) = 1$. It is well-known that for $n > 1$ having the prime factorization $n = p_1^{a_1} \dots p_r^{a_r}$ one has

$$\varphi(n) = p_1^{a_1-1} \dots p_r^{a_r-1}(p_1 - 1) \dots (p_r - 1) = n.\prod_{p|n}\left(1 - \frac{1}{p}\right), \tag{1}$$

where p runs through all prime divisors of n, and $p_1, \dots, p_r$ are distinct primes, with $a_1, \dots, a_r \geq 1$ integers. It is also well-known that

$$d(n) = (a_1 + 1) \dots (a_r + 1). \tag{2}$$

In paper [3], we have proved the following inequality

$$\varphi(n) + d(n) \leq n + 1, \tag{3}$$

for $n \geq 2$, with equality only for $n = 4$ or n being prime.

In fact, (3) was a consequence of the stronger relation (see [3]):

$$\varphi(n) + d(n) \leq n, \tag{4}$$

for any $n \neq 4$, different from prime.

However, the cases of equality for (4) are not studied in [3]. The aim of this note is to consider also the case of equality. Certain related new inequalities will be pointed out, too.

2 Main results

Theorem 1. *The equation $\varphi(n) + d(n) = n$ has the only solutions $n = 8$ and $n = 9$.*

Proof. <u>Case 1.</u> Let n be an even number. Then it is well-known that $\varphi(n) \leq \frac{n}{2}$. Using the relation $d(n) < 2\sqrt{n}$ (see, e.g., [1]), we get

$$\varphi(n) + d(n) < \frac{n}{2} + 2\sqrt{n} \leq n$$

if $2\sqrt{n} \leq \frac{n}{2}$, i.e., $n \geq 16$. Now, for $n < 16$ and even, a simple verification shows that (4) holds true with a strict inequality, except for $n = 8$, when there is equality. Therefore, the only even solution except for $n = 4$ is $n = 8$.

<u>Case 2.</u> Let n be odd and not a prime. Suppose that $\varphi(n) + d(n) = n$ holds true. As for $n \geq 3$, $\varphi(n)$ is even, then $d(n)$ should be an odd number. But from (2) it is immediate that n must be a perfect square, i.e., $n = m^2$. As $\varphi(m^2) = m\varphi(m)$, the equality becomes

$$m\varphi(m) + d(m^2) = m^2. \tag{5}$$

Equality (5) implies that m should be a divisor of $d(m^2)$, i.e.,

$$d(m^2) = k.m, \tag{6}$$

for certain, $k \geq 1$. As $d(N) < 2\sqrt{N}$, we get $d(m^2) < 2m$, implying that one must have $k = 1$ in (6). Equation $d(m^2) = m$ can be also written as

$$(2\alpha_1 + 1) \ldots (2\alpha_s + 1) = q_1^{\alpha_1} \ldots q_s^{\alpha_s}, \tag{7}$$

where $m = q_1^{\alpha_1} \ldots q_s^{\alpha_s}$ is the prime factorization of m.

Now, as m is odd, let q_1 be the least odd prime factor of m, with $q_1 \geq 3$. Since the inequality $3^{\alpha_1} \geq 2\alpha_1 + 1$ holds true, with equality only for $\alpha_1 = 1$, and as $5^{\alpha_2} \geq 2\alpha_2 + 1$, etc., we must have $m = 3$. This finally gives $n = m^2 = 9$, as the single odd solution of the equation. This finishes the proof of Theorem 1. □

Remark 1. From Theorem 1 and relation (4) it follows that

$$\varphi(n) + d(n) < n, \tag{8}$$

for any $n \neq 4, 8, 9$ and n being composite.

Remark 2. From the proof of Theorem 1 we get that (8) holds true for any even number, distinct from $2, 4, 8$, and that (8) holds true for any odd composite number distinct from 9.

Remark 3. Another proof of (4) and Theorem 1 can be obtained by the use of a computer and some known inequalities. This is based on the following lemmas.

Lemma 1. *When $n > 1$ is composite, then*

$$\varphi(n) \leq n - \sqrt{n}. \tag{9}$$

Lemma 2. *For any $n \geq 1262$ one has*

$$d(n) < \sqrt{n}. \tag{10}$$

Lemma 1 is well-known (see, e.g., [1]), while (10) is proved in [2].

Now, by (9) and (10) we get that $\varphi(n) + d(n) < n$ for any $n \geq 1262$, n being composite. By a computer search for $n \leq 1261$, one can deduce (4) and obtain the only solutions $n = 8$ and $n = 9$.

Remark 4. Thus, for any $n \geq 1262$ composite, we get the following refinement of (4):

$$\varphi(n) \leq n - \sqrt{n} < n - d(n).,\tag{11}$$

Remark 5. The inequality (11) can be further improved for squarefull numbers, i.e., for numbers with the property: if $p^a || n$ (where p^a is the greatest prime power of a prime dividing n) , then $a \geq 2$.

Indeed, this is based on the following relation (see [4]):

$$\varphi(n) \leq n - \frac{n}{\gamma(n)},\tag{12}$$

where $\gamma(n) = \prod\limits_{p|n} p$ is the product of the prime divisors of n.

Now, it is immediate that, $\frac{n}{\gamma(n)} \geq \sqrt{n}$, or equivalently $\gamma(n) \leq \sqrt{n}$, i.e., $p_1 \dots p_r \leq p_1^{\frac{\alpha_1}{2}} \dots p_r^{\frac{\alpha_r}{2}}$. This is valid, when $\alpha_1 \geq 2, \dots, \alpha_r \geq 2$, i.e., when n is squarefull.

Thus we get: when $n \geq 1262$ is a squarefull number, one has

$$\varphi(n) \leq n - \frac{n}{\gamma(n)} \leq n - \sqrt{n} < n - d(n).$$

Remark 6. This paper has been motivated by [3]. The results are quite strong to offer the solutions to other equations, too. For example, let $\omega(n)$ denote the number of distinct prime factors of $n > 1$. Let $d^*(n)$ denote the number of unitary divisors of n (see [4]). Then, as $d^*(n) \geq \omega(n) + 1$, we get, by relation (4) that:

$$n + 1 \geq \varphi(n) + d(n) \geq \varphi(n) + d^*(n) \geq \varphi(n) + \omega(n) + 1.$$

Thus, particularly, the equation

$$\varphi(n) + \omega(n) = n$$

has the only solutions as $n =$ prime. Many other equations are studied in our paper [5].

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