Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 2, 213–221
DOI: 10.7546/nntdm.2020.26.2.213-221

An identity for vertically aligned entries in Pascal's triangle

Heidi Goodson

Department of Mathematics, Brooklyn College
2900 Bedford Avenue, Brooklyn, NY 11210 USA
e-mail: heidi.goodson@brooklyn.cuny.edu

Received: 30 December 2019 **Accepted:** 16 June 2020

Abstract: The classic way to write down Pascal's triangle leads to entries in alternating rows being vertically aligned. In this paper, we prove a linear relation on vertically aligned entries in Pascal's triangle. Furthermore, we give an application of this relation to morphisms between hyperelliptic curves.
Keywords: Pascal's triangle, Binomial coefficients, Hyperelliptic curves.
2010 Mathematics Subject Classification: 05A10, 11G30.

1 Introduction

We consider entries in row n of Pascal's triangle, where n is any nonnegative integer. It is well known that the i-th entry in this row can be computed as the binomial coefficient $\binom{n}{i}$, where $0 \leq i \leq n$.

The entries in alternating rows of Pascal's triangle are vertically aligned. For example, in Figure 1 below we have circled the entries that are vertically aligned with and above the third entry in Row 11.

In Figure 2 we have circled the entries that are vertically aligned with and above the sixth entry in Row 12. Note that these values are the central binomial coefficients $\binom{2n}{n}$ and are closely related to the ubiquitous Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see, for example, [6, 7]).

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 136 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

Figure 1. Entries vertically aligned with the third entry in Row 11.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 136 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

Figure 2. Entries vertically aligned with the sixth entry in Row 12.

We can describe these entries in the following way. Starting with entry $\binom{n}{r}$ in row n, the elements that are vertically aligned and above are all of the form

$$\binom{n-2k}{r-k},$$

where $1 \leq k \leq r$ and $k \leq \lfloor \frac{n}{2} \rfloor$. For example, the elements that are vertically aligned with and above $\binom{11}{3}$ are $\binom{9}{2}$, $\binom{7}{1}$, $\binom{5}{0}$.

1.1 Interesting observations

Observe that

$$\binom{11}{3} - 11\binom{9}{2} + 44\binom{7}{1} - 77\binom{5}{0} = 0.$$

When $n = 12$ and $i = 6$, we have

$$\binom{12}{6} - 12\binom{10}{5} + 54\binom{8}{4} - 112\binom{6}{3} + 105\binom{4}{2} - 36\binom{2}{1} + 2\binom{0}{0} = 0.$$

The following theorem generalizes these two observations.

2 General formula

Theorem 2.1. *Let n be a nonnegative integer and $0 < r < n$. Then*

$$\sum_{k=0}^{r} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{n-2k}{r-k} = 0.$$

Remark 1. If $r > \lfloor n/2 \rfloor$, as is the case when our elements are to right of the vertical line through the middle of Pascal's Triangle, there will be some values of k for which $n - 2k < r - k$. But recall that $\binom{m}{i} = 0$ whenever $0 \leq m < i$ (see, for example, [2, Section 1.9]). Thus, terms for which $0 \leq n - 2k < r - k$ do not contribute to the sum in Theorem 2.1.

If $n - 2k < 0$, then $\binom{n-2k}{r-k}$ is no longer 0. However in this case, we have $n - k < k$ and, therefore, $\binom{n-k}{k} = 0$ instead. Hence, all terms for which $r > \lfloor n/2 \rfloor$ do not contribute to the sum in Theorem 2.1.

Proof of Theorem 2.1. The following proof starts with an identity attributed to E.H. Lockwood. For any $n \geq 1$,

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} \tag{1}$$

(see, for example, [2, Section 9.8]).

We separate the $k = 0$ term from the summation to get

$$x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}. \tag{2}$$

The Binomial Theorem tells us that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + y^n + \sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r \tag{3}$$

Substituting this expression for $(x+y)^n$ into equation (2) yields

$$x^n + y^n = x^n + y^n + \sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k}.$$

Hence,

$$\sum_{r=1}^{n-1} \binom{n}{r} x^{n-r} y^r + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = 0. \quad (4)$$

Thus, when combining the two sums, the coefficient of each $x^{n-r}y^r$ term must equal 0. We expand the second summand in order to identify all terms of the form $x^{n-r}y^r$. The Binomial Theorem tells us that, for each k,

$$(x+y)^{n-2k} = \sum_{j=0}^{n-2k} \binom{n-2k}{j} x^{n-2k-j} y^j.$$

Hence,

$$(xy)^k (x+y)^{n-2k} = \sum_{j=0}^{n-2k} \binom{n-2k}{j} x^{n-k-j} y^{j+k}. \quad (5)$$

The values of j that yield $x^{n-r}y^r$ terms are $j = r - k$. Note that we must have $k \leq r$, since otherwise $j \leq 0$. Thus, the coefficient of $x^{n-r}y^r$ in equation (5) is

$$\sum_{k=1}^{r} \binom{n-2k}{r-k}.$$

Hence, the sum of the coefficients of the $x^{n-r}y^r$ terms in equation (4) is

$$\sum_{k=0}^{r} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{n-2k}{r-k} = 0,$$

where the $k = 0$ term is $\binom{n}{r}$, which comes from the first summation in equation (4). □

Remark 2. The expressions $\frac{n}{n-k}\binom{n-k}{k}$ that appear in Theorem 2.1 are referred to as the Triangle of coefficients of Lucas (or Cardan) polynomials, denoted $T(n,k)$, in the On-Line Encyclopedia of Integer Sequences [5]. We also recall that the nth Lucas number, L_n, is given by

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k}$$

(see, for example, [2]).

3 Application to hyperelliptic curves

We now give an application of the identity in Theorem 2.1. Work on this application in [1, Section 5.1] is what led the author to discover the identity in Theorem 2.1.

Let K be a field with $\mathrm{char}(K) \neq 2$. A hyperelliptic curve is a compact Riemann surface defined by a nonsingular equation of the form $y^2 = f(x)$, where $f(x) \in K[x]$. The degree of the polynomial $f(x)$ is either $2g + 2$ or $2g + 1$, where g is the genus of the curve. A defining property of hyperelliptic curves is that they have degree two maps with $2g + 2$ branch points onto the projective line $\mathbb{P}^1$ (see, for example, [3, Chapter 3], [4, Chapter 2]).

In the section we will start with genus g hyperelliptic curves C of the form $y^2 = x^{2g+1} + x$. The map

$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \frac{y}{x^a}\right),$$

where $a = \frac{g+1}{2}$, is a nonconstant morphism from C to some curve C'. Note that the curve C' will also be hyperelliptic. We initially define C' to be of the form

$$y^2 = c_d x^d + \cdots + c_{d-i} x^{d-i} + \cdots + c_0$$

and we will apply the transformation of variables given by ϕ to determine the coefficients c_j. Applying the transformation yields

$$\begin{aligned}
\left(\frac{y}{x^a}\right)^2 &= c_d \left(\frac{x^2+1}{x}\right)^d + \cdots + c_{d-i}\left(\frac{x^2+1}{x}\right)^{d-i} + \cdots + c_0 \\
\frac{y^2}{x^{g+1}} &= c_d x^{-d}(x^2+1)^d + \cdots + c_{d-i} x^{i-d}(x^2+1)^{d-i} + \cdots + c_0 \\
y^2 &= c_d x^{g+1-d}(x^2+1)^d + \cdots + c_{d-i} x^{g+1+i-d}(x^2+1)^{d-i} + \cdots + c_0 x^{g+1}.
\end{aligned}$$

Note that the degree of the expression in x will be $g + 1 - d + 2d = g + 1 + d$. In order for ϕ to be a morphism from C to C', this last equation should, in fact, be the equation for the curve C. Hence, we need $c_d = 1$ and $g + 1 + d = 2g + 1$, which implies $d = g$. Consequently,

$$y^2 = x(x^2+1)^g + \cdots + c_{g-i} x^{1+i}(x^2+1)^{g-i} + \cdots + c_0 x^{g+1}. \tag{6}$$

In order to determine the coefficients c_j, we need to expand the right-hand side of the equation and match coefficients with those of C. We now work through two examples to better understand what the coefficients of C' will be.

Example 3.1. *Let $g = 5$, so that C is the hyperelliptic curve $y^2 = x^{11} + x$. From our above work we know that the degree of C' will be 5. Letting*

$$\begin{aligned}
A_1 &= x(x^2+1)^5 \\
&= x^{11} + 5x^9 + 10x^7 + 10x^5 + 5x^3 + x, \\
A_2 &= x^3(x^2+1)^3 \\
&= x^9 + 3x^7 + 3x^5 + x^3, \\
A_3 &= x^5(x^2+1)^1 \\
&= x^7 + x^5,
\end{aligned}$$

we see that $A_1 - 5A_2 + 5A_3 = x^{11} + x$. Hence, ϕ is a morphism from C to $y^2 = x^5 - 5x^3 + 5x$.

Example 3.2. *Now let $g = 6$, so that C is the hyperelliptic curve $y^2 = x^{13} + x$. From our above work we know that the degree of C' will be 6. Letting*

$$
\begin{aligned}
B_1 &= x(x^2+1)^6 \\
&= x^{13} + 6x^{11} + 15x^9 + 2 - x^7 + 15x^5 + 6x^3 + x, \\
B_2 &= x^3(x^2+1)^4 \\
&= x^{11} + 4x^9 + 6x^7 + 4x^5 + x^3, \\
B_3 &= x^5(x^2+1)^2 \\
&= x^9 + 2x^7 + x^5, \\
B_4 &= x^7(x^2+1)^0 \\
&= x^7,
\end{aligned}
$$

we see that $B_1 - 6B_2 + 9B_3 - 2B_4 = x^{13} + x$. *Hence,* ϕ *is a morphism from* C *to the curve* $y^2 = x^6 - 6x^4 + 9x^2 - 2$.

While working on [1, Section 5.1], the author determined (by hand) the curve C' for $g = 11$, obtaining 1, 11, 44, 77, 55, and 11, with alternating signs (see Table 1 below). The author entered this sequence of numbers into the On-line Encyclopedia of Integer Sequences [5] search bar and found that these numbers are the Triangle of coefficients of Lucas (or Cardan) polynomials, $T(n, k)$. The coefficients that appear in Examples 3.1 and 3.2 are also of the form $T(n, k)$. As noted in Remark 2,

$$
T(n,k) = \frac{n}{n-k}\binom{n-k}{k}.
$$

This leads us to the following theorems.

Theorem 3.3. *Let* C *be the hyperelliptic curve* $y^2 = x^{2g+1} + x$ *and let* C' *be the hyperelliptic curve*

$$
y^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k} x^{g-2k}.
$$

Then the map

$$
\phi(x,y) = \left(\frac{x^2+1}{x}, \frac{y}{x^a}\right),
$$

where $a = \frac{g+1}{2}$, *is a nonconstant morphism from* C *to* C'.

We can generalize Theorem 3.3. Let $c \in \mathbb{Q}^*$ be constant and ζ be a primitive g-th root of unity. In the following theorem we work over the field $\mathbb{F} = \mathbb{Q}(\zeta, c^{1/g})$.

Theorem 3.4. *Let* C *be the hyperelliptic curve* $y^2 = x^{2g+1} + cx$ *and let* C_i *be the hyperelliptic curve*

$$
y^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k} \zeta^{ik} c^{k/g} x^{g-2k}
$$

for $i = 0, 1$. *Then the map*

$$
\phi_i(x,y) = \left(\frac{x^2+\zeta^i c^{1/g}}{x}, \frac{y}{x^a}\right),
$$

where $a = \frac{g+1}{2}$, *is a nonconstant morphism from* C *to* C_i.

Theorem 3.3 follows from Theorem 3.4 by letting $c = 1$ and $i = 0$. Furthermore, since

$$\frac{g}{g-k}\binom{g-k}{k} = \left[\binom{g-k}{k} + \binom{g-k-1}{k-1}\right]$$

(see, for example, [2, Section 9.9]), Theorem 3.4 also generalizes Lemma 5.1 in [1] because we are no longer restricting g to be odd. Though the proof of Theorem 3.4 is nearly identical to the proof of Lemma 5.1 in [1], we include it here for the sake of completion.

Proof of Theorem 3.4. Recall Lockwood's identity from equation (1)

$$A^n + B^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(AB)^k(A+B)^{n-2k}.$$

Letting $n = g$, $A = x^2$, and $B = \zeta^i c^{1/g}$ yields

$$x^{2g} + c = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k}\zeta^{ik}c^{k/g}x^{2k}(x^2 + \zeta^i c^{1/g})^{g-2k},$$

since $\zeta^{ig} = 1$. We multiply both sides by x to get

$$x^{2g+1} + cx = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k}\zeta^{ik}c^{k/g}x^{2k+1}(x^2 + \zeta^i c^{1/g})^{g-2k}. \tag{7}$$

We now demonstrate that ϕ_i is indeed a morphism between C and C_i. We apply the transformation of variables to C_i to get

$$\left(\frac{y}{x^a}\right)^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k}\zeta^{ik}c^{k/g}\left(\frac{x^2 + \zeta^i c^{1/g}}{x}\right)^{g-2k}$$

$$y^2 = \sum_{k=0}^{\lfloor g/2 \rfloor} (-1)^k \frac{g}{g-k}\binom{g-k}{k}\zeta^{ik}c^{k/g}x^{2k+1}\left(x^2 + \zeta^i c^{1/g}\right)^{g-2k}$$

$$= x^{2g+1} + cx,$$

where the last equality holds by equation (7). Hence, we have shown that ϕ_i is a morphism from C to C_i. □

Table 1 below gives C_i for values of g up to 11 and for $c = 1$. Note that this table expands on the table that appears in [1, Section 5.1].

g	curve C_i
5	$y^2 = x^5 - 5\zeta^i x^3 + 5\zeta^{2i} x$
6	$y^2 = x^6 - 6\zeta^i x^4 + 9\zeta^{2i} x^2 - 2\zeta^{3i}$
7	$y^2 = x^7 - 7\zeta^i x^5 + 14\zeta^{2i} x^3 - 7\zeta^{3i} x$
8	$y^2 = x^8 - 8\zeta^i x^6 + 20\zeta^{2i} x^4 - 16\zeta^{3i} x^2 + 2\zeta^{4i}$
9	$y^2 = x^9 - 9\zeta^i x^7 + 27\zeta^{2i} x^5 - 30\zeta^{3i} x^3 + 9\zeta^{4i} x$
10	$y^2 = x^{10} - 10\zeta^i x^8 + 35\zeta^{2i} x^6 - 50\zeta^{3i} x^4 + 25\zeta^{4i} x^2 - 2\zeta^{5i}$
11	$y^2 = x^{11} - 11\zeta^i x^9 + 44\zeta^{2i} x^7 - 77\zeta^{3i} x^5 + 55\zeta^{4i} x^3 - 11\zeta^{5i} x$

Table 1. Examples of curves C_i from Theorem 3.4

3.1 Higher genus observations

The following corollaries to Theorem 3.3 describe patterns for some of the above coefficients.

Corollary 3.4.1. *For all g, the coefficient of x^{g-2} will always be $-g\zeta^i$.*

Proof. This coefficient corresponds to $k = 1$, which equals

$$(-1)^1 \frac{g}{g-1} \binom{g-1}{1} \zeta^i = -g\zeta^i. \qquad \square$$

Corollary 3.4.2. *When g is even, the lowest degree term will always be $(-1)^{g/2} 2\zeta^{ig/2}$.*

Proof. Note that when g is even, the lowest degree term corresponds to $k = g/2$, which yields x^0. We compute the coefficient to be

$$(-1)^{g/2} \frac{g}{g-g/2} \binom{g-g/2}{g/2} \zeta^{ig/2} = (-1)^{g/2} 2\zeta^{ig/2}. \qquad \square$$

Corollary 3.4.3. *When g is odd, the lowest degree term will always be $(-1)^{(g-1)/2} g x \zeta^{i(g-1)/2}$.*

Proof. When g is odd, the lowest degree term corresponds to $k = (g-1)/2$, which yields x^1. We compute the coefficient to be

$$\begin{aligned} &(-1)^{(g-1)/2} \frac{g}{g-(g-1)/2} \binom{g-(g-1)/2}{(g-1)/2} \zeta^{i(g-1)/2} \\ &= (-1)^{(g-1)/2} \frac{g}{(g+1)/2} \binom{(g-1)/2+1}{(g-1)/2} \zeta^{i(g-1)/2} \\ &= (-1)^{(g-1)/2} g \zeta^{i(g-1)/2}. \end{aligned} \qquad \square$$

Acknowledgements

The author thanks Darij Grinberg for helpful comments on an earlier draft of this paper.

References

[1] Emory, M., Goodson, H., & Peyrot, A. (2018). Towards the Sato–Tate Groups of Trinomial Hyperelliptic Curves. *ArXiv e-prints*, page arXiv:1812.00242, Dec. 2018.

[2] Koshy, T. (2014). *Pell and Pell–Lucas numbers with applications*. Springer, New York.

[3] Miranda, R. (1995). *Algebraic curves and Riemann surfaces*, Volume 5 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

[4] Silverman, J. H. (2009). *The arithmetic of elliptic curves*, Volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, Second edition.

[5] Sloane, N. J. A. (2018). Sequence A034807. *The On-Line Encyclopedia of Integer Sequences*. Available online at: `https://oeis.org/A034807`.

[6] Stanley, R. P. (1999). *Enumerative Combinatorics. Volume 2*, *Cambridge Studies in Advanced Mathematics, Volume 62*. Cambridge University Press, Cambridge.

[7] Stanley, R. P. (2012). *Enumerative Combinatorics. Volume 1*, *Cambridge Studies in Advanced Mathematics, Volume 49*. Cambridge University Press, Cambridge, Second edition.