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Infinite series involving Fibonacci numbers and the Riemann zeta function

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Abstract: Two new closed forms for infinite series involving Fibonacci numbers and the Riemann zeta function are derived using standard methods from complex analysis. Also, expressions for the companion series with Lucas numbers are presented.
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1 Introduction

Fibonacci numbers F_n and Lucas numbers L_n are defined for $n \geq 0$ as $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$ with initial conditions $F_0 = 0, F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. The Binet forms are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n,$$

where α and β are roots of the quadratic equation $x^2 - x - 1 = 0$, i.e. $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The sequences $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ possess many beautiful properties. They are indexed in the On-Line Encyclopedia of Integer Sequences [8] with entries A000045 and A000032, respectively.

The Riemann zeta function $\zeta(s)$, $s \in \mathbb{C}$, is defined by the series (see [1,6] or [7]):

* Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s},\quad \Re(s)>1.$$

The analytical continuation to all $s\in\mathbb{C}$ with $\Re(s)>0, s\neq 1$, is given by

$$\zeta(s)=(1-2^{1-s})^{-1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^s}.$$

For more information about $\zeta(s)$ consult the textbooks [6] and [7]. Some recent results involving the Riemann zeta function are contained in the articles [2] and [3]. In [4] (problem proposal and solution) the author presents the following two new infinite series evaluations involving the Riemann zeta function at positive even integer arguments and Fibonacci (Lucas) numbers

$$\sum_{n=1}^{\infty}\zeta(2n)\frac{F_{2n}}{5^n}=\frac{\pi}{2\sqrt{5}}\tan\left(\frac{\pi}{2\sqrt{5}}\right), \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\zeta(2n)\frac{L_{2n}}{5^n}=\frac{\pi}{2\sqrt{5}}\tan\left(\frac{\pi}{2\sqrt{5}}\right)+1. \tag{1.2}$$

Moreover, in [5] the author derives the following beautiful identity involving $\zeta(s)$ at odd integer arguments

$$\sum_{n=1}^{\infty}\zeta(2n+1)\frac{F_{2n}}{5^n}=\frac{1}{2}. \tag{1.3}$$

The goal of this short article is to evaluate exactly two additional infinite series involving F_n and $\zeta(n)$. To prove the results, we will work with generating functions and some series evaluations applying methods from complex analysis. In addition, we will apply properties of the digamma function $\psi(z), z\in\mathbb{C}$. Recall that [1] $\psi(z)$ is the first logarithmic derivative of the Gamma function, i.e.,

$$\psi(z)=(\ln\Gamma(z))'=\frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z)$ is the complex gamma function. The digamma function possesses the following properties:

$$\psi(z+1)=\psi(z)+\frac{1}{z}, \tag{1.4}$$

$$\psi(z+1)=-\gamma+\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{n+z}\Big),\qquad z\neq -1,-2,\dots, \tag{1.5}$$

and the reflection property

$$\psi(1-z)-\psi(z)=\pi\cot(\pi z), \tag{1.6}$$

where $\cot(z)$ is the complex cotangent and γ is the famous Euler–Mascheroni constant

$$\gamma=\lim_{n\to\infty}\Big(\sum_{k=1}^{n}\frac{1}{k}-\ln n\Big)=0,5772156649...$$

These properties will be employed in the proofs below.

2 Results

The first main result of this paper are the following evaluations involving the zeta function at even integer arguments and odd (scaled) Fibonacci (Lucas) numbers:

Theorem 2.1. *We have the following identity*

$$\sum_{n=1}^{\infty}\zeta(2n)\frac{F_{2n-1}}{5^{\frac{2n-1}{2}}}=\frac{1}{2}-\beta=\frac{\sqrt{5}}{2}. \tag{2.1}$$

The companion identity with Lucas numbers is

$$\sum_{n=1}^{\infty}\zeta(2n)\frac{L_{2n-1}}{5^{\frac{2n-1}{2}}}=\pi\tan\Big(\frac{\pi}{2\sqrt{5}}\Big)-\frac{\sqrt{5}}{2}. \tag{2.2}$$

The second achievement of this paper are the following identities:

Theorem 2.2. *We have*

$$\sum_{n=2}^{\infty}(\zeta(n)-1)F_{n-1}=\frac{1}{5}\Big(5+\sqrt{5}\pi\tan\Big(\frac{\sqrt{5}\pi}{2}\Big)\Big). \tag{2.3}$$

The corresponding Lucas series can be evaluated as

$$\sum_{n=2}^{\infty}(\zeta(n)-1)L_{n-1}=3-\sum_{n=1}^{\infty}\frac{n}{(n+2)(n^2+5n+5)}. \tag{2.4}$$

The author did not succeed in deriving a closed form for (2.4). The analysis below shows that an equivalent expression is given by

$$\sum_{n=2}^{\infty}(\zeta(n)-1)L_{n-1}=3-\pi\tan\Big(\frac{\sqrt{5}\pi}{2}\Big)+2\alpha\sum_{n=1}^{\infty}\frac{1}{n(n-\alpha)}. \tag{2.5}$$

An evaluation of the sum on the RHS of (2.4) in terms of radicals, known constants and/or elementary functions seems be difficult to establish. It could be possible using more advanced methods than employed here. It would be pleasing to see this problem solved.

3 Proofs

3.1 Proof of Theorem 2.1

The next lemma will play a key role in the proof.

Lemma 3.1. *We have*

$$\sum_{n=0}^{\infty}\frac{1}{5n^2+5n+1}=\frac{\pi}{\sqrt{5}}\tan\Big(\frac{\pi}{2\sqrt{5}}\Big). \tag{3.1}$$

Proof: We use the residue theorem to prove the sum identity. First, note that since

$$5(n+1)^2 - 5(n+1) + 1 = 5n^2 + 5n + 1,$$

it follows that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{5n^2+5n+1} &= \sum_{n=-\infty}^{-1} \frac{1}{5n^2+5n+1} + \sum_{n=0}^{\infty} \frac{1}{5n^2+5n+1} \\
&= \sum_{n=1}^{\infty} \frac{1}{5n^2-5n+1} + \sum_{n=0}^{\infty} \frac{1}{5n^2+5n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{5(n+1)^2-5(n+1)+1} + \sum_{n=0}^{\infty} \frac{1}{5n^2+5n+1},
\end{aligned}$$

and therefore

$$\sum_{n=0}^{\infty} \frac{1}{5n^2+5n+1} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{5n^2+5n+1}.$$

Setting $f(z) = 5z^2 + 5z + 1$, we proceed as follows

$$\begin{aligned}
\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{5n^2+5n+1} &= -\frac{\pi}{2} \sum_{z^*} \mathrm{Res}\Big(\frac{\cot(\pi z)}{f(z)} \Big| z^* \text{ is a pole of } 1/f(z) \Big) \\
&= -\frac{\pi}{2\sqrt{5}} \Big(\cot\Big(\frac{\pi\beta}{\sqrt{5}}\Big) - \cot\Big(\frac{\pi\alpha}{\sqrt{5}}\Big) \Big) \\
&= -\frac{\pi}{2\sqrt{5}} \Big(\cot\Big(-\Big(\frac{\pi}{2} - \frac{\pi}{2\sqrt{5}}\Big)\Big) - \cot\Big(-\Big(\frac{\pi}{2} + \frac{\pi}{2\sqrt{5}}\Big)\Big) \Big) \\
&= -\frac{\pi}{2\sqrt{5}} \Big(-\tan\Big(\frac{\pi}{2\sqrt{5}}\Big) - \tan\Big(\frac{\pi}{2\sqrt{5}}\Big) \Big) \\
&= \frac{\pi}{\sqrt{5}} \tan\Big(\frac{\pi}{2\sqrt{5}}\Big).
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Proof of (2.1): From the paper [2] we know that

$$\sum_{n=1}^{\infty} \zeta(2n) \frac{z^{2n}}{n} = \ln \Gamma(1+z) + \ln \Gamma(1-z), \qquad |z| < 1.$$

Differentiating gives immediately

$$\sum_{n=1}^{\infty} \zeta(2n) z^{2n-1} = \frac{1}{2} \Big(\psi(1+z) - \psi(1-z) \Big), \qquad |z| < 1,$$

and we can write

$$
\begin{aligned}
\sum_{n=1}^{\infty}\zeta(2n)\frac{F_{2n-1}}{5^{\frac{2n-1}{2}}} &= \frac{1}{2\sqrt{5}}\Big(\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1+\frac{\beta}{\sqrt{5}}\Big)+\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)\Big)\\
&= \frac{1}{2\sqrt{5}}\Big(\psi\Big(1+\frac{\beta}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)+\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)-\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)\\
&\quad -2\psi\Big(1+\frac{\beta}{\sqrt{5}}\Big)+2\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)\Big)\\
&= \frac{\pi}{2\sqrt{5}}\cot\Big(\pi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)\Big)+\frac{1}{2\sqrt{5}}\Big(\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)-\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)\Big)\\
&\quad +\frac{1}{\sqrt{5}}\Big(\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1+\frac{\beta}{\sqrt{5}}\Big)\Big),
\end{aligned}
$$

where in the last step the reflection principle for the digamma function was used. Next, we observe that

$$\cot\Big(\pi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)\Big)=\tan\Big(\frac{\pi}{2\sqrt{5}}\Big),$$

and

$$\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)=\sqrt{5}\sum_{n=1}^{\infty}\frac{1}{5n^2+5n+1}.$$

In view of Lemma 3.1 we deduce that

$$\sum_{n=1}^{\infty}\zeta(2n)\frac{F_{2n-1}}{5^{\frac{2n-1}{2}}}=\frac{1}{2}+\frac{1}{\sqrt{5}}\Big(\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1+\frac{\beta}{\sqrt{5}}\Big)\Big).$$

Finally, we can evaluate the last expression as

$$\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1+\frac{\beta}{\sqrt{5}}\Big)=\frac{\sqrt{5}}{\alpha},$$

where we have applied

$$\psi(z)-\psi(z-1)=\frac{1}{z-1},$$

with $z=1+\alpha/\sqrt{5}$ and $\alpha-\beta=\sqrt{5}$. This completes the proof of the first identity. The second part is derived similarly:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\zeta(2n)\frac{L_{2n-1}}{5^{\frac{2n-1}{2}}} &= \frac{1}{2}\Big(\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)+\psi\Big(1+\frac{\beta}{\sqrt{5}}\Big)\Big)\\
&= \frac{\pi}{2}\tan\Big(\frac{\pi}{2\sqrt{5}}\Big)+\frac{1}{2}\Big(\psi\Big(1+\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)\Big)\\
&= \frac{\pi}{2}\tan\Big(\frac{\pi}{2\sqrt{5}}\Big)+\frac{\sqrt{5}}{2}\sum_{n=1}^{\infty}\frac{1}{5n^2+5n+1}.
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

3.2 Proof of Theorem 2.2

The central lemma in this proof is:

Lemma 3.2. *It holds that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n-1} = \frac{1}{5}\Big(5+\sqrt{5}\pi \tan\Big(\frac{\sqrt{5}\pi}{2}\Big)\Big). \tag{3.2}$$

Proof: Once more we use the residue theorem to prove the sum identity. First, note that since

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n-1} = 1+\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1},$$

we have to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1} = \frac{\pi}{\sqrt{5}} \tan\Big(\frac{\sqrt{5}\pi}{2}\Big).$$

We have

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2+3n+1},$$

and since $g(z) = z^2+3z+1 = (z+\alpha^2)(z+\beta^2)$, we conclude that

$$\begin{aligned}
\frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2+3n+1} &= -\frac{\pi}{2}\sum_{z^*} \mathrm{Res}\Big(\frac{\cot(\pi z)}{g(z)}\Big|z^* \text{ is a pole of } 1/g(z)\Big) \\
&= -\frac{\pi}{2\sqrt{5}}\Big(\cot(-\pi\beta^2)-\cot(-\pi\alpha^2)\Big) \\
&= -\frac{\pi}{2\sqrt{5}}\Big(\tan\Big(\frac{\pi}{2}+\pi\beta^2\Big)-\tan\Big(\frac{\pi}{2}+\pi\alpha^2\Big)\Big) \\
&= \frac{\pi}{\sqrt{5}}\tan\Big(\frac{\sqrt{5}\pi}{2}\Big),
\end{aligned}$$

where we used the fact that $\alpha^2-\beta^2=\sqrt{5}$. □

Proof of (2.3): The generating function (see [6])

$$\sum_{n=2}^{\infty}(\zeta(n)-1)z^{n-1} = 1-\gamma-\psi(2-z), \qquad |z|<2,$$

in combination with $2-\alpha=\alpha^{-2}$ and $2-\beta=\alpha^2$ produces

$$\begin{aligned}
\sum_{n=2}^{\infty}(\zeta(n)-1)F_{n-1} &= \frac{1}{\sqrt{5}}\Big(\psi(\alpha^2)-\psi(\beta^2)\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\psi(\alpha+1)-\psi(\beta+1)\Big) \\
&= \sum_{n=1}^{\infty}\frac{1}{n^2+n-1}.
\end{aligned}$$

The result now follows from Lemma 3.2.
For the companion series with Lucas numbers we obtain analogously

$$\begin{aligned}\sum_{n=2}^{\infty}(\zeta(n)-1)L_{n-1} &= 2(1-\gamma)-\Big(\psi(\alpha+1)+\psi(\beta+1)\Big)\\ &= 2-\sum_{n=1}^{\infty}\frac{n-2}{n(n^2+n-1)}\\ &= 3-\sum_{n=3}^{\infty}\frac{n-2}{n(n^2+n-1)}\\ &= 3-\sum_{n=1}^{\infty}\frac{n}{(n+2)(n^2+5n+5)}.\end{aligned}$$

The second expression for the Lucas sum is established as follows:

$$\begin{aligned}\sum_{n=2}^{\infty}(\zeta(n)-1)L_{n-1} &= 2(1-\gamma)-\Big(\psi(\alpha+1)+\psi(\beta+1)\Big)\\ &= 2(1-\gamma)-\Big(\psi(\alpha+1)-\psi(\beta+1)\Big)-2\psi(\beta+1)\\ &= 2(1-\gamma)-\Big(\psi(\alpha+1)-\psi(\beta+1)\Big)-2\Big(\psi(\beta)+\frac{1}{\beta}\Big)\\ &= 3-2\gamma-\pi\tan\Big(\frac{\sqrt{5}\pi}{2}\Big)-2\psi(\beta)\\ &= 3-\pi\tan\Big(\frac{\sqrt{5}\pi}{2}\Big)-2\sum_{n=0}^{\infty}\frac{\beta-1}{(n+1)(n+\beta)}.\end{aligned}$$

This completes the proof of Theorem 2.2. □

4 Conclusion

In this article, the author investigated infinite series involving Fibonacci numbers and the Riemann zeta function. He presented some new closed forms for these series. To prove the results, the residue calculus was combined with properties of the psi function. It is desirable to seek for more such relations. This is left for future research.

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