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A new explicit formula for the Bernoulli numbers in terms of the Stirling numbers of the second kind

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Abstract: Let B_r denote the Bernoulli numbers, and $S(r,k)$ denote the Stirling numbers of the second kind. We prove the following explicit formula

$$B_{r+1} = \sum_{k=0}^{r} \frac{(-1)^{k-1}\, k!\, S(r,k)}{(k+1)(k+2)}.$$

To the best of our knowledge, the formula is new.
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1 Introduction

Definition 1.1. The *Bernoulli numbers* B_n can be defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 1.2. The *Stirling number of the second kind*, denoted by $S(n,m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

There are many explicit formulas known for the Bernoulli numbers [1,3,5–10,13,14]. For example, all of the formulas below express the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind:

$$B_r = \sum_{k=0}^{r} \frac{(-1)^k\, k!\, S(r,k)}{k+1}, \tag{1}$$

$$B_r = (-1)^r \sum_{k=1}^{r} \frac{(-1)^{k-1}\,(k-1)!\, S(r,k)}{k+1},$$

$$B_r = \frac{r}{1-2^r} \sum_{k=1}^{r-1} \frac{(-1)^k\, k!\, S(r-1,k)}{2^{k+1}},$$

$$B_{r+1} = \frac{(-1)^r\,(r+1)\,2^{r-1}}{2^{r+1}-1} \sum_{k=1}^{r} \frac{(-1)^k\, k!\, S(r,k)}{k+1}\, 2^{-2k}\binom{2k}{k},$$

$$B_r = \sum_{i=0}^{r} (-1)^i \frac{\binom{r+1}{i+1}}{\binom{r+i}{i}} S(r+i,i),$$

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \left(\sum_{k=0}^{r} \frac{(-1)^k\, S(r,k)}{k+1} \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right)$$

where $x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$ denotes the rising factorial, and (E_r) denotes the *Euler numbers* defined by the following generating function:

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!}\, t^n.$$

The formula (1) has been generalized in [4,11]. In the following section, we derive a new explicit formula for the Bernoulli numbers in terms of the Stirling numbers of the second kind.

2 Main result

Our main result is the following.

Theorem 2.1. *Let r be any non-negative integer. Then we have*

$$B_{r+1} = \sum_{k=0}^{r} \frac{(-1)^{k-1}\, k!\, S(r,k)}{(k+1)(k+2)}.$$

Proof. We begin with the following result [2, formula 3.2.1.6]

$$(s-1)\zeta(s) = -\int_0^{\infty} \frac{\mathrm{Li}_s(-x)}{(1+x)^2}\, dx, \tag{2}$$

where $\zeta(s)$ is the Riemann zeta function, and $\mathrm{Li}_s(-x)$ is the polylogarithm function.

Letting $s = -r$, a negative integer, in the equation (2) we arrive at

$$\int_0^{\infty} \frac{\mathrm{Li}_{-r}(-x)}{(1+x)^2}\, dx = (r+1)\zeta(-r) = -B_{r+1}.$$

We use the following representation from the note [12]

$$\mathrm{Li}_{-r}(-x) = \sum_{k=0}^{r} k!\, S(r,k) \left(\frac{1}{1+x}\right)^{k+1} (-x)^k$$

to conclude that

$$\begin{aligned} B_{r+1} &= -\int_0^\infty \frac{\mathrm{Li}_{-r}(-x)}{(1+x)^2}\, dx \\ &= -\int_0^\infty \left(\sum_{k=0}^{r} \frac{(-1)^k\, k!\, x^k}{(1+x)^{k+3}} S(r,k)\right) dx \\ &= -\sum_{k=0}^{r} \frac{(-1)^k\, k!\, S(r,k)}{(k+1)(k+2)}. \end{aligned}$$

This completes the proof. □

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