A new explicit formula for the Bernoulli numbers in terms of the Stirling numbers of the second kind

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Abstract: Let $B_r$ denote the Bernoulli numbers, and $S(r,k)$ denote the Stirling numbers of the second kind. We prove the following explicit formula

$$B_{r+1} = \sum_{k=0}^{r} \frac{(-1)^{k-1} k! S(r,k)}{(k+1)(k+2)}.$$ 

To the best of our knowledge, the formula is new.

Keywords: Bernoulli numbers, Stirling numbers of the second kind, Riemann zeta function, Polylogarithm function.

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1 Introduction

Definition 1.1. The Bernoulli numbers $B_n$ can be defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 1.2. The Stirling number of the second kind, denoted by $S(n,m)$, is the number of ways of partitioning a set of $n$ elements into $m$ nonempty sets.
There are many explicit formulas known for the Bernoulli numbers \([1,3,5–10,13,14]\). For example, all of the formulas below express the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind:

\[
B_r = \sum_{k=0}^{r} \frac{(-1)^k k! S(r, k)}{k + 1},
\]

\[
B_r = (-1)^r \sum_{k=1}^{r} \frac{(-1)^{k-1} (k - 1)! S(r, k)}{k + 1},
\]

\[
B_r = \frac{r}{1 - 2^r} \sum_{k=1}^{r-1} \frac{(-1)^k k! S(r - 1, k)}{2^{k+1}},
\]

\[
B_{r+1} = \frac{(-1)^r (r + 1)}{2^{r+1} - 1} \sum_{k=1}^{r} \frac{(-1)^k k! S(r, k)}{k + 1} 2^{-2k} \binom{2k}{k},
\]

\[
B_r = \sum_{i=0}^{r} (-1)^{i+1} \frac{(r+1)}{(i+1)} S(r+i, i),
\]

\[
B_{r+1} = -\frac{r + 1}{4(1 + 2^{r+1}(1 - 2r))} \left( \sum_{k=0}^{r} \frac{(-1)^k S(r, k)}{k + 1} \left( \frac{3}{4} \right)^k + 4^{-r} E_r \right)
\]

where \(x^{(n)} = x(x + 1)(x + 2) \cdots (x + n - 1)\) denotes the rising factorial, and \((E_r)\) denotes the Euler numbers defined by the following generating function:

\[
\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n.
\]

The formula (1) has been generalized in [4,11]. In the following section, we derive a new explicit formula for the Bernoulli numbers in terms of the Stirling numbers of the second kind.

### 2 Main result

Our main result is the following.

**Theorem 2.1.** Let \(r\) be any non-negative integer. Then we have

\[
B_{r+1} = \sum_{k=0}^{r} \frac{(-1)^{k-1} k! S(r, k)}{(k + 1)(k + 2)}.
\]

**Proof.** We begin with the following result [2, formula 3.2.1.6]

\[
(s - 1) \zeta(s) = -\int_{0}^{\infty} \frac{\text{Li}_s(-x)}{(1 + x)^2} dx,
\]

where \(\zeta(s)\) is the Riemann zeta function, and \(\text{Li}_s(-x)\) is the polylogarithm function.

Letting \(s = -r\), a negative integer, in the equation (2) we arrive at

\[
\int_{0}^{\infty} \frac{\text{Li}_{-r}(-x)}{(1 + x)^2} dx = (r + 1) \zeta(-r) = -B_{r+1}.
\]

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We use the following representation from the note [12]
\[
Li_{-r}(-x) = \sum_{k=0}^{r} k! S(r, k) \left( \frac{1}{1 + x} \right)^{k+1} (-x)^k
\]
to conclude that
\[
B_{r+1} = -\int_{0}^{\infty} \frac{Li_{-r}(-x)}{(1 + x)^2} \, dx
\]
\[
= -\int_{0}^{\infty} \left( \sum_{k=0}^{r} \frac{(-1)^k k! x^k}{(1 + x)^{k+3}} S(r, k) \right) \, dx
\]
\[
= -\sum_{k=0}^{r} \frac{(-1)^k k! S(r, k)}{(k + 1)(k + 2)}.
\]
This completes the proof. \(\Box\)

References


