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Saalschütz' theorem and Rising binomial coefficients – Type 2

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Abstract: This paper extends some work of Leonard Carlitz on rising binomial coefficients and hypergeometric series in the context of a result of Louis Saalschütz which has animated further work in a number of branches of mathematics as well as physics.

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1 Introduction

Saalschütz' original paper [20], though built on the work of others in the nineteenth century, has continued to be a source of new ideas and applications in a variety of settings [1, 8]. For example, combinatorics [9, 11], special functions [12, 17, 23], even in the context of enrichment work in teaching [3]. Here we build on an idea of Carlitz [23] to consider a relation with rising binomial coefficients and hypergeometric series [14, 17]. Extending ideas and generalizing results thus opens up new avenues of cross-fertilization of ideas [10, 24].

2 Rising binomial coefficients

There are two types of rising binomial coefficients in the literature. We start with Carlitz' definition of the q-series analogue of the binomial coefficients [6]:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}} \tag{2.1}$$

in which

$$(q)_n = (1-q)(1-q^2)\dots(1-q^n). \tag{2.2}$$

The rising factorial analogues of the binomial coefficients are then attained in a similar fashion. The rising factorial of n can be given by the notation

$$n^{\overline{r}} = n(n+1)\dots(n+r-1) \tag{2.3}$$

which is an r permutation of $(n + r - 1)$ things. In contrast, the falling factorial of n is then given by

$$n^{\underline{r}} = n(n-1)\dots(n-r+1) \tag{2.4}$$

which is equivalent to $P(n,r)$, an r permutation of n distinct things [18]. We have chosen to use these forms of the Pochhammer symbols because the notation is suggestive [13]. These two factorials occupy a central position in the finite difference calculus [19] because

$$\nabla x^{\overline{n}} = n x^{\overline{n-1}},$$

and

$$\nabla x^{\underline{n}} = n x^{\underline{n-1}},$$

for the shift operator

$$\nabla P(n,r) = P(n,r) - P(n-1,r) = rP(n-1,r-1).$$

Thus, the rising binomial coefficients Type 1 are defined as [22]:

$$\begin{bmatrix} n \\ k \end{bmatrix}^a = \frac{a^{\overline{n}}}{a^{\overline{k}} a^{\overline{n-k}}}$$

so that

$$\begin{bmatrix} i \\ j \end{bmatrix}^1 = \binom{i}{j}.$$

Type 2 rising binomial coefficients have been defined [21] as inversions of Type 1:

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a = \frac{n^{\overline{a}}}{k^{\overline{a}}(n-k)^{\overline{a}}} \tag{2.5}$$

Some of the properties arise from various extensions of the Bernoulli numbers [15], including the Rising factorial coefficients – Type 2.

3 Saalschütz' formulas

Carlitz [7] used the formula of Saalschütz (in the present notation)

$$\sum_{k=0}^{n} \frac{(-n)^{\overline{k}} a^{\overline{k}} b^{\overline{k}}}{k!\, c^{\overline{k}} d^{\overline{k}}} = \frac{(c-a)^{\overline{n}} (c-b)^{\overline{n}}}{c^{\overline{n}} (c-a-b)^{\overline{n}}} \tag{3.1}$$

in which

$$c + d = -n + a + b + 1,$$

In order to prove Dixon's theorem

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^3 = (-1)^n\frac{(3n)!}{(n!)^3} \tag{3.2}$$

and some of MacMahon's results [16]. Carlitz also used the q-analogue of Saalschütz' theorem to give an elegant proof of an identity due to Maitland Wright [25].

If we let $b = n - 1$, then Saalschütz' formula becomes

$$\sum_{k=0}^{n}(-1)^k(n-1)^{\overline{k}}\binom{n}{k}\left\langle{a \atop c}\right\rangle^k = \frac{\left\langle{2c-a-n+1 \atop c}\right\rangle^n}{\left\langle{2c-a-n+1 \atop c-a}\right\rangle^n} \tag{3.3}$$

Proof: If $b = n - 1$, then $c + d = -n + a + b + 1$ implies that

$$\sum_{k=}^{n}\frac{(-n)^{\overline{k}}a^{\overline{k}}b^{\overline{k}}}{k!\,c^{\overline{k}}d^{\overline{k}}} = \sum_{k=0}^{n}\frac{(-n)^{\overline{k}}a^{\overline{k}}(n-1)^{\overline{k}}}{k!\,c^{\overline{k}}(a-c)^{\overline{k}}}$$

$$= \sum_{k=0}^{n}\frac{(-n)^{\overline{k}}}{k!}(n-1)^{\overline{k}}\left\langle{a \atop c}\right\rangle^k$$

$$= \sum_{k=0}^{n}(-1)^k(n-1)^{\overline{k}}\binom{n}{k}\left\langle{a \atop c}\right\rangle^k$$

since

$$(-n)^{\overline{k}} = (-1)^k k!\binom{n}{k}$$

from the definition of the rising factorial. On the other side of Equation (3.3) we have

$$\frac{(c-a)^{\overline{n}}(c-b)^{\overline{n}}}{c^{\overline{n}}(c-a-b)^{\overline{n}}} = \frac{(c-a)^{\overline{n}}(c-n+1)^{\overline{n}}}{c^{\overline{n}}(c-a-n+1)^{\overline{n}}}$$

$$= \frac{\left\langle{2c-a-n+1 \atop c}\right\rangle^n}{\left\langle{2c-a-n+1 \atop c-a}\right\rangle^n}$$

if unity is used in the form of the factor $\frac{(2c-a-n+1)^{\overline{n}}}{(2c-a-n+1)^{\overline{n}}}$. □

4 A hypergeometric function

Carlitz proved Saalschütz' formula by induction. At this stage let us replace c by $a + n - c$ and (3.3) becomes

$$\sum_{k=0}^{n}(-1)^k(n-1)^{\overline{k}}\binom{n}{k}\left\langle{a \atop c-n}\right\rangle^k = \frac{(n-c)^{\overline{n}}(a-c+1)^{\overline{n}}}{(a+n-c)^{\overline{n}}(1-c)^{\overline{n}}}. \tag{4.1}$$

From this we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n-c)^{\overline{n}}(a-c+1)^{\overline{n}}}{n!\,(a+n-c)^{\overline{n}}}x^n &= \sum_{n=0}^{\infty}(1-c)^{\overline{n}}\frac{x^n}{n!}\sum_{k=0}^{n}(-1)^k(n-1)^{\overline{k}}\binom{n}{k}\left\langle\begin{matrix}a\\c-n\end{matrix}\right\rangle^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{a^{\overline{k}}(n-1)^{\overline{k}}}{(a+n-c)^{\overline{k}}}\frac{x^k}{k!}(1-c)^{\overline{n-k}}\frac{x^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty}\frac{a^{\overline{k}}(n-1)^{\overline{k}}}{(a+n-c)^{\overline{k}}}\frac{x^k}{k!}(1-x)^{1-c} \\ &= (1-x)^{1-c}F(a,n-1;a+n-c;x)\end{aligned}$$

where F is the common hypergeometric function, and in which we have used

$$\begin{aligned}\frac{(1-c)^{\overline{n}}}{(c-n)^{\overline{k}}} &= \frac{(1-c)(2-c)\dots(n-c)}{(c-n)(c-n+1)\dots(c-n+k-1)} \\ &= (-1)^k(1-c)(2-c)\dots(n-k-c) \\ &= (-1)^k(1-c)^{\overline{n-k}}.\end{aligned}$$

It is customary to prove this type of result by making use of the differential equation of the second order satisfied by $F(a, b, c, x)$.

5 Concluding comments

Finally, we conclude with a quotation from the late Richard Askey, an expert in special functions in general and hypergeometric series in particular [2], because it may whet the appetites of interested readers to delve further into these fascinating series, [26].

> *"Most mathematicians have little or no training in the ways of thought that historians have developed, so it is unrealistic to expect many of them to write papers or books that will satisfy mathematical historians. However, some mathematicians are tempted to write a paper on the history of a topic they have studied for years. I was tempted and did this over ten years ago. I had found a few series identities in papers that had been forgotten, and in one case an important result usually attributed to Saalschütz (1890) had been found by Pfaff (1797) almost one hundred years earlier. Actually, I did not find this paper but read about it in (Jacobi 1848)… There were two reasons I wanted to call attention to Pfaff's paper. One is historical, and should have been of interest to historians... This result was proved by Gauss in his published paper on hypergeometric series (1813). Notice the publication date: Gauss lived in Pfaff's home for a few months in 1797, and this was the year in which Pfaff published … a book, the middle third of which contains the most comprehensive treatment of hypergeometric functions that appeared before Gauss's work. In addition to the published paper mentioned above, Gauss wrote a sequel that was only published posthumously (1866)".*

Askey is perhaps most famous for the development of the Askey–Wilson polynomials (q-analogues of the Wilson polynomials) [5], part of what is now known as the Askey scheme of hypergeometric orthogonal polynomials.

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