Multifarious results for q-Hermite-based Frobenius-type Eulerian polynomials

Waseem Ahmad Khan[1], Idrees Ahmad Khan[2], Mehmet Acikgoz[3] and Ugur Duran[4]

[1] Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University
Al Khobar, 31952, Saudi Arabia
e-mail: `wkhan1@pmu.edu.sa`

[2] Department of Mathematics, Integral University
Lucknow-226026, India
e-mail: `khanidrees077@gmail.com`

[3] Department of Mathematics, University of Gaziantep
TR-27310 Gaziantep, Turkey
e-mail: `acikgoz@gantep.edu.tr`

[4] Department of the Basic Concepts of Engineering, Iskenderun Technical University
TR-31200 Hatay, Turkey
e-mail: `mtdrnugur@gmail.com`

Abstract: In this paper, a new class of q-Hermite-based Frobenius-type Eulerian polynomials is introduced by means of generating function and series representation. Several fundamental formulas and recurrence relations for these polynomials are derived via different generating methods. Furthermore, diverse correlations including the q-Apostol-Bernoulli polynomials, the q-Apostol-Euler poynoomials, the q-Apostol-Genocchi polynomials and the q-Stirling numbers of the second kind are also established by means of the their generating functions.
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1 Introduction

The subject of q-calculus started appearing in the nineteenth century due to its applications in various fields of mathematics, physics and engineering. One of the most popular studies in q-calculus is the q-extension of the some special polynomials such as Bernoulli, Eulerian, Genocchi, Euler polynomials (*cf.* [1–23] and the references cited therein). The classical Eulerian polynomials are firstly considered by Leonhard Euler in his *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques* in 1749 (first printed in 1765) in which he defined a method of computing values of the zeta function at negative integers. Then, the foregoing polynomials have been studied and investigated extensively until now, *cf.* [3,4,12,13]. Moreover, the Frobenius-type Eulerian polynomials are introduced and studied in [22]. Kim [9] studied new q-extensions of Euler numbers and polynomials and investigated some properties of symmetries of these q-Euler polynomials by using q-derivatives and q-integrals. Kim [10] introduced a Daehee constant, the so-called q-extension of the Napier constant, and considered the Daehee formula associated with the q-extensions of trigonometric functions, and then derived the q-extensions of sine and cosine functions from this Daehee formula and the q-calculus related to the q-extensions of sine and cosine functions. In this paper, we perform to give a novel class of q-generalization of the Hermite-based Frobenius-type Eulerian polynomials and to derive multifarious correlation, implicit summation formula, identities, explicit formulas and recurrence relations for the mentioned polynomials by means of the series manipulation methods. Furthermore, we investigate some correlations covering the q-Apostol–Bernoulli polynomials, q-Apostol-Euler polynomials, q-Apostol-Genocchi polynomials and q-Stirling numbers of the second kind for the q-Hermite-based Frobenius-type Eulerian polynomials. The definitions and notations of q-calculus reviewed here are taken from the references [5–10, 14–17, 20]:

The q-analogue of the shifted factorial $(a)_n$ is given by

$$(a;q)_0 = 1, \ (a;q)_n = \prod_{m=0}^{n-1}(1 - q^m a), \ \ n \in \mathbb{N}.$$

The q-analogues of a complex number a and of the factorial function are given by

$$[a]_q = \frac{1 - q^a}{1 - q}, \ q \in \mathbb{C}; 0 < |q| < 1; a \in \mathbb{C},$$

and

$$[n]_q! = \prod_{m=1}^{n}[m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1 - q)^n}; \ \ n \in \mathbb{N}, q \in \mathbb{C}; 0 < |q| < 1$$

with $[0]_q! = 1$. The Gauss q-binomial coefficient $\binom{n}{k}_q$ is given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \ k = 0, 1, \ldots, n.$$

The q-analogue of the function $(x + y)_q^n$ is given by

$$(x + y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} x^{n-k} y^k, n \in \mathbb{N}_0. \tag{1.1}$$

The q-analogues of exponential function are given by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x; q)_{\infty}}, 0 < |q| < 1; |x| < |1-q|^{-1} \tag{1.2}$$

and

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (-(1-q)x; q)_{\infty}, 0 < |q| < 1; x \in \mathbb{C}. \tag{1.3}$$

Moreover, the functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x) = e_q(x), D_q E_q(x) = E_q(qx), \tag{1.4}$$

where the q-derivative $D_q f(x)$ of a function f at a point $0 \neq x \in \mathbb{C}$ is defined as follows:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}, 0 < |q| < 1.$$

For any two arbitrary functions $f(x)$ and $g(x)$, the q-derivative operator D_q satisfies the following product and quotient relations:

$$D_{q,x}(f(x)g(x)) = f(x)D_{q,x}g(x) + g(qx)D_{q,x}f(x) \tag{1.5}$$

and

$$D_{q,x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{q,x}f(x) - f(qx)D_{q,x}g(x)}{g(x)g(qx)}. \tag{1.6}$$

The Apostol-type q-Bernoulli polynomials $B_{n,q}^{(\alpha)}(x, y; \lambda)$ of order α, the Apostol-type q-Euler polynomials $E_{n,q}^{(\alpha)}(x, y; \lambda)$ of order α and the Apostol-type q-Genocchi polynomials $G_{n,q}^{(\alpha)}(x, y; \lambda)$ of order α are defined by means of the following generating function (see [7, 8, 14–17]):

$$\left(\frac{t}{\lambda e_q(t) - 1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x, y; \lambda)\frac{t^n}{n!}, (|t + \log \lambda|) < 2\pi, 1^{\alpha} = 1, \tag{1.7}$$

$$\left(\frac{2}{\lambda e_q(t) + 1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x, y; \lambda)\frac{t^n}{n!}, (|t + \log \lambda|) < \pi, 1^{\alpha} = 1 \tag{1.8}$$

and

$$\left(\frac{2t}{\lambda e_q(t) + 1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x, y; \lambda)\frac{t^n}{n!}, |t + \log \lambda|) < \pi, 1^{\alpha} = 1. \tag{1.9}$$

Clearly, we have

$$B_{n,q}^{(\alpha)}(\lambda) = B_{n,q}^{(\alpha)}(0, 0; \lambda), E_{n,q}^{(\alpha)}(\lambda) = E_{n,q}^{(\alpha)}(0, 0; \lambda), G_{n,q}^{(\alpha)}(\lambda) = G_{n,q}^{(\alpha)}(0, 0; \lambda).$$

The Frobenius-type Eulerian polynomials $A_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating function as follows (see [22]):

$$\left(\frac{1 - \lambda}{e^{t(\lambda - 1)} - \lambda}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} A_n^{(\alpha)}(x; \lambda)\frac{t^n}{n!}, \tag{1.10}$$

where λ is a complex number with $\lambda \neq 1$. The number A_n is given by

$$A_n^{(\alpha)}(\lambda) = A_n^{(\alpha)}(0; \lambda)$$

are called the Frobenius-type Eulerian numbers (see [21, 22]). Clearly, we have

$$A_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} A_m^{(\alpha)}(\lambda) x^{n-m}. \tag{1.11}$$

In recent days, a new type of q-Hermite polynomials are considered in [16, 20], which is a particular member of the q-Appell family [1]. The q-Appell polynomials are defined by means of the following generating function:

$$\frac{1}{g_q(t)} e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}(x) \frac{t^n}{[n]_q!}, \; A_{n,q} = A_{n,q}(0).$$

Definition 1.1. The continuous q-Hermite polynomials $H_{n,q}^{(s)}(x)$ are defined by

$$e_q\left(xt - \frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} H_{n,q}^{(s)}(x) \frac{t^n}{[n]_q!}, \; (0 < q < 1, 0 \neq s \in \mathbb{R}) \tag{1.12}$$

where $H_{n,q}^{(s)} = H_{n,q}^{(s)}(0)$ are the continuous q-Hermite numbers defined by

$$e_q\left(\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} H_{n,q}^{(s)}(0) \frac{t^n}{[n]_q!}.$$

2 q-Hermite-based Frobenius-type Eulerian polynomials $_H A_{n,q}^{(\alpha,s)}(x, y; \lambda)$

In this section, we define q-Hermite-based Frobenius-type Eulerian polynomials $(qHbFtEp)$ $_H A_{n,q}^{(\alpha,s)}(x, y; \lambda)$ by means of the generating function and series representation. Certain relations for these polynomials are also derived by using various identities. We now ready to start in conjunction with the following definition.

Definition 2.1. Let $q, \lambda \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$. The generalized q-Hermite-based Frobenius-type Eulerian polynomials $_H A_{n,q}^{(\alpha,s)}(x, y; \lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)^{\alpha} e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt) = \sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha,s)}(x, y; \lambda) \frac{t^n}{[n]_q!}. \tag{2.1}$$

When $x = y = s = 0$ in (2.1), $_H A_n^{(\alpha,0)}(0, 0; \lambda) := A_n^{(\alpha)}(\lambda)$ are called the n-th Frobenius-type Eulerian numbers of order α.

Remark 2.1. For $x = y = 0$ in (2.1), $_H A_{n,q}^{(\alpha,s)}(\lambda) = {_H A_{n,q}^{(\alpha,s)}}(0,0;\lambda)$ are the the q-Hermite-based Frobenius-type Eulerian numbers defined by

$$\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)^\alpha e_q\left(\frac{st^2}{1+q}\right) = \sum_{n=0}^\infty {_H A_{n,q}^{(\alpha,s)}}(\lambda)\frac{t^n}{[n]_q!}. \tag{2.2}$$

Theorem 2.1. The following identity for $_H A_{n,q}^{(\alpha,s)}(x,y;\lambda)$

$$(\lambda-1)^{-n} {_H A_{n,q}^{(\alpha,(\lambda-1)^2 s))}}\left((\lambda-1)x, (\lambda-1)^2 y; \lambda\right) = {_H H_{n,q}^{(\alpha,s)}}(x,y;\lambda) \tag{2.3}$$

holds true.

Proof. Consider the generating function (2.1), we have

$$\sum_{n=0}^\infty \frac{{_H A_{n,q}^{(\alpha,(\lambda-1)^2 s)}}\left((\lambda-1)x, (\lambda-1)y; \lambda\right)}{n!}\frac{t^n}{(\lambda-1)^n}$$

$$= \left(\frac{1-\lambda}{e_q(t) - \lambda}\right)^\alpha e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt) = \sum_{n=0}^\infty {_H H_{n,q}^{(\alpha,s)}}(x,y;\lambda)\frac{t^n}{[n]_q!},$$

where $_H H_{n,q}^{(\alpha,s)}(x,y;\lambda)$ are called the q-Hermite based Frobenius polynomials, which is defined by Riyasat and Khan [20] and comparing the coefficients of t^n, we arrive at the required result (2.3). $\qquad\square$

Theorem 2.2. The following summation representation for the q-Hermite-based Frobenius-type Eulerian polynomials $_H A_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of order α

$$_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \sum_{m=0}^n \binom{n}{m}_q A_{n-m,q}^{(\alpha)}(y;\lambda)(\lambda) H_{m,q}^{(s)}(x) \tag{2.4}$$

holds true.

Proof. Using equation (1.10) and (1.12) in the left-hand side of equation (2.1) and then applying the Cauchy product rule and equating the coefficients of same powers of t in both sides of resultant equation, we get representation (2.4). $\qquad\square$

Theorem 2.3. The following summation formulae

$$_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \sum_{m=0}^n \binom{n}{m}_q {_H A_{m,q}^{(\alpha,s)}}(0,0;\lambda)(x+y)_q^{n-m}, \tag{2.5}$$

$$_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \sum_{m=0}^n \binom{n}{m}_q {_H A_{m,q}^{(\alpha,s)}}(0,y;\lambda)x^{n-m} \tag{2.6}$$

and

$$_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \sum_{m=0}^n \binom{n}{m}_q q^{(n-k)(n-k-1)/2} {_H A_{m,q}^{(\alpha,s)}}(x,0;\lambda)y^{n-m} \tag{2.7}$$

hold true for the q-Hermite-based Frobenius-type Eulerian polynomials of order α.

Proof. We suitably use equations (1.1)–(1.3) in generating function (2.1) to get three different forms. Further making use of the Cauchy product rule in the resultant expressions and then comparing the like powers of t in both sides of resultant equation, we find formulas (2.5)–(2.7). □

Theorem 2.4. The following recursive formulas for the q-Hermite-based Frobenius-type Eulerian polynomials $_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of order α holds true:

$$D_{q,x\,H}A_{n,q}^{(\alpha,s)}(x,y;\lambda) = [n]_{q\,H}A_{n-1,q}^{(\alpha,s)}(x,y;\lambda) \tag{2.8}$$

and

$$D_{q,y\,H}A_{n,q}^{(\alpha,s)}(x,y;\lambda) = [n]_{q\,H}A_{n-1,q}^{(\alpha,s)}(x,qy;\lambda). \tag{2.9}$$

Proof. Differentiating generating function (2.1) with respect to x and y with the help of equation (1.4) and then simplifying with the help of the Cauchy product rule formulas (2.8) and (2.9) are obtained. □

Theorem 2.5. The following relation for the q-Hermite-based Frobenius-type Eulerian polynomials $_HA_{n,q}^{(\alpha,s)}(x,y;;\lambda)$ of order α holds true:

$$(2\lambda-1)\sum_{k=0}^{n}\binom{n}{k}_q A_{k,q}(0,y;\lambda)_HA_{n-k,q}^{(s)}(x,0;1-\lambda) = \lambda_HA_{n,q}^{(s)}(x,y;\lambda)-(1-\lambda)_HA_{n,q}^{(s)}(x,y;1-\lambda). \tag{2.10}$$

Proof. We set

$$\frac{(2\lambda-1)}{(e_q^{t(\lambda-1)}-\lambda)(e_q^{t(\lambda-1)}-(1-\lambda))} = \frac{1}{e_q^{t(\lambda-1)}-\lambda} - \frac{1}{e_q^{t(\lambda-1)}-(1-\lambda)}.$$

From the above equation, we see that

$$(2\lambda-1)\frac{(1-\lambda)e_q\left(xt-\frac{st^2}{1+q}\right)(1-(1-\lambda))E_q(yt)}{(e_q^{t(\lambda-1)}-\lambda)(e_q^{t(\lambda-1)}-(1-\lambda))}$$

$$= \frac{(1-\lambda)e_q\left(xt-\frac{st^2}{1+q}\right)\lambda E_q(yt)}{e_q^{t(\lambda-1)}-\lambda} - \frac{(1-\lambda)e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)(1-(1-\lambda))}{e_q^{t(\lambda-1)}-(1-\lambda)},$$

which on using equations (1.10) and (2.1) in both sides, we have

$$(2\lambda-1)\left(\sum_{n=0}^{\infty}{}_HA_{k,q}(0,y;\lambda)\frac{t^k}{[k]_q!}\right)\left(\sum_{n=0}^{\infty}{}_HA_{n,q}^{(s)}(x,0;1-\lambda)\frac{t^n}{[n]_q!}\right)$$

$$= \lambda\sum_{n=0}^{\infty}{}_HA_{n,q}^{(s)}(x,y;\lambda)\frac{t^n}{[n]_q!} - (1-\lambda)\sum_{n=0}^{\infty}{}_HA_{n,q}^{(s)}(x,y;1-\lambda)\frac{t^n}{[n]_q!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.10) follows. □

Theorem 2.6. The following relation for the q-Hermite-based Frobenius-type Eulerian polynomials $_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of order α holds true:

$$\lambda_H A_{n,q}^{(s)}(x,y;\lambda) = \sum_{k=0}^{n} \binom{n}{k}_q {}_H A_{n-k,q}^{(s)}(x,y;\lambda)(1-\lambda)^k - (1-\lambda)H_{n,q}^{(s)}(x,y). \qquad (2.11)$$

Proof. Consider the following identity

$$\frac{\lambda}{(e_q^{t(\lambda-1)} - \lambda)e_q^{t(\lambda-1)}} = \frac{1}{(e_q^{t(\lambda-1)} - \lambda)} - \frac{1}{e_q^{t(\lambda-1)}}.$$

Evaluating the following fraction using above identity, we find

$$\frac{\lambda(1-\lambda)e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt)}{(e_q^{t(\lambda-1)} - \lambda)e_q^{t(\lambda-1)}} = \frac{(1-\lambda)e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt)}{(e_q^{t(\lambda-1)} - \lambda)} - \frac{(1-\lambda)e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt)}{e_q^{t(\lambda-1)}}$$

$$\lambda \sum_{n=0}^{\infty} {}_H A_{n,q}^{(s)}(x,y;\lambda)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} {}_H A_{n,q}^{(s)}(x,y;\lambda)\frac{t^n}{n!} \sum_{k=0}^{\infty}(1-\lambda)^k \frac{t^k}{[k]_q!} - (1-\lambda)\sum_{n=0}^{\infty} H_{n,q}^{(s)}(x,y)\frac{t^n}{[n]_q!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.11) follows. $\qquad\square$

Theorem 2.7. The following relation for the q-Hermite-based Frobenius-type Eulerian polynomials $_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of order α holds true:

$$_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{1-\lambda}\sum_{k=0}^{n} \binom{n}{k}_q \Big[A_{n-k,q}(0,y;\lambda){}_H A_{k,q}^{(\alpha,s)}((\lambda-1)x,0;\lambda)$$

$$-\lambda A_{n-k,q}(0,y;\lambda){}_H A_{k,q}^{(\alpha,s)}(x,0;\lambda)\Big]. \qquad (2.12)$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!}$$

$$= \left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)\left(\frac{e_q^{t(\lambda-1)} - \lambda}{1-\lambda}\right)\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)^\alpha e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt)$$

$$= \frac{1}{1-\lambda}\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)e_q^{(\lambda-1)t}\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)^\alpha e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt)$$

$$- \frac{\lambda}{1-\lambda}\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right)^\alpha e_q\left(xt - \frac{st^2}{1+q}\right)E_q(yt)$$

$$= \frac{1}{1-\lambda}\sum_{n=0}^{\infty} A_{n,q}(0,y;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty} {}_H A_{k,q}^{(\alpha,s)}((\lambda-1)x,0;\lambda)\frac{t^k}{[k]_q!}$$

$$- \frac{\lambda}{1-\lambda}\sum_{n=0}^{\infty} A_{n,q}(0,y;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty} {}_H A_{k,q}^{(\alpha,s)}(x,0;\lambda)\frac{t^k}{[k]_q!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.12) follows. $\qquad\square$

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Theorem 2.8. The following recurrence relation for the q-Hermite-based Frobenius-type Eulerian polynomials $_H A_{n,q}^{(\alpha,s)}(x, y; \lambda)$ of order α holds true:

$$_H A_{n+1,q}^{(s)}(x, y; \lambda) = -\left(\frac{2s}{1+q}\right) [n]_{q\,H} A_{n-1,q}^{(s)}(qx, qy; \lambda) + x_H A_{n,q}^{(s)}(x, y; \lambda)$$

$$+ y_H A_{n,q}^{(s)}(qx, qy; \lambda) + \frac{1}{(\lambda-1)^2} \sum_{k=0}^{n} \binom{n}{k}_q {}_H A_{n-k,q}^{(s)}(x, y; \lambda) q^{n-k} A_{k,q}(1, \lambda - 1; \lambda). \quad (2.13)$$

Proof. Taking $\alpha = 1$ and then applying q-derivative on both sides of generating function (2.1), it follows that

$$\sum_{n=0}^{\infty} {}_H A_{n+1,q}^{(s)}(x, y; \lambda) \frac{t^n}{[n]_q!} = (1 - \lambda) D_{q,t} \left(\frac{e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt)}{e_q^{t(\lambda-1)} - \lambda} \right),$$

which on performing differentiation in left-hand side, using formula (1.6), yields

$$\sum_{n=0}^{\infty} {}_H A_{n+1,q}^{(s)}(x, y; \lambda) \frac{t^n}{[n]_q!}$$

$$= (1 - \lambda) \left[\frac{(e_q^{qt(\lambda-1)} - \lambda) D_{q,t} \left(e_q(xt) e_q\left(-\frac{st^2}{1+q}\right) E_q(yt) \right)}{(e_q^{t(\lambda-1)} - \lambda)(e_q^{qt(\lambda-1)} - \lambda)} \right]$$

$$- \frac{e_q(qxt) E_q(yqt) e_q\left(-\frac{sq^2t^2}{1+q}\right) D_{q,t}(e_q^{t(\lambda-1)} - \lambda)}{(e_q^{t(\lambda-1)} - \lambda)(e_q^{qt(\lambda-1)} - \lambda)}$$

$$= -\left(\frac{2s}{1+q}\right)\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right) e_q\left(qxt - \frac{st^2}{1+q}\right) E_q(qyt)t$$

$$+ x\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right) e_q\left(xt - \frac{st^2}{1+q}\right) E_q(yt) + y\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right) e_q\left(qxt - \frac{st^2}{1+q}\right) E_q(qyt)$$

$$+ \frac{1}{(\lambda-1)^2}\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right) e_q\left(qxt - \frac{st^2}{1+q}\right) E_q(qyt)\left(\frac{1-\lambda}{e_q^{t(\lambda-1)} - \lambda}\right) e_q^{(\lambda-1)t},$$

which on making use of the Cauchy product rule in the right-hand side and comparing the coefficients of $\dfrac{t^n}{n!}$ on both sides of the resultant equation gives recurrence relation (2.13). $\square$

3 Summation formulae for q-Hermite-based Frobenius-type Eulerian polynomials

In this section, we provide implicit formulae, Stirling numbers of the second kind and some relationships for q-Hermite-based Frobenius-type Eulerian polynomials of order α related to Apostol-type Bernoulli polynomials, Apostol-type Euler polynomials and Apostol-type Genocchi polynomials. We now begin with the following theorem.

Theorem 3.1. The following summation formulae for q-Hermite-based Frobenius-type Eulerian polynomials $_H A_{n,q}^{(\alpha,s)}(x, y; \lambda)$ of order α holds true:

$$_H A_{k+l,q}^{(\alpha,s)}(z, y; \lambda) = \sum_{n,m=0}^{k,l} \binom{l}{m}_q \binom{k}{n}_q (z - x)^{n+m} {}_H A_{k+l-n-m,q}^{(\alpha,s)}(x, y; \lambda). \tag{3.1}$$

Proof. We replace t by $t + w$ and rewrite the generating function (2.1) as

$$\left(\frac{1 - \lambda}{e_q((\lambda - 1)(t + w) - \lambda)}\right)^{\alpha} E_q(y(t + w))e_q\left(\frac{-s(t + u)^2}{1 + q}\right)$$

$$= e_q(-x(t + w)) \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha,s)}(x, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}, \quad \text{(see [18, 19])}. \tag{3.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e_q((z - x)(t + w)) \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha,s)}(x, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}$$

$$= \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha,s)}(z, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty} \frac{[(z - x)(t + w)]^N}{[N]_q!} \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha,s)}(x, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}$$

$$= \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha,s)}(z, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}, \tag{3.4}$$

which on using formula [23, p.52(2)]

$$\sum_{N=0}^{\infty} f(N) \frac{(x + y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n + m) \frac{x^n}{n!} \frac{y^m}{m!}, \tag{3.5}$$

in the left-hand side becomes

$$\sum_{n,m=0}^{\infty} \frac{(z - x)^{n+m} t^n w^m}{[n]_q! [m]_q!} \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha,s)}(x, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}$$

$$= \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha,s)}(z, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}. \tag{3.6}$$

Now replacing k by $k - n$, and l by $l - m$ in the left-hand side of (3.6), we get

$$\sum_{k,l=0}^{\infty} \sum_{n,m=0}^{k,l} \frac{(z - x)^{n+m}}{[n]_q! [m]_q!} {}_H A_{k+l-n-m,q}^{(\alpha,s)}(x, y; \lambda) \frac{t^k}{(k - n)_q!} \frac{w^l}{(l - m)_q!}$$

$$= \sum_{k,l=0}^{\infty} {}_H A_{k+l,q}^{(\alpha)}(z, y; \lambda) \frac{t^k}{[k]_q!} \frac{w^l}{[l]_q!}. \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and w in the above equation, we get the required result. $\qquad\square$

Remark 3.1. By taking $l = 0$ in Eq. (3.1), we immediately deduce the following result.

Corollary 3.1. The following summation formula for q-Hermite-based Frobenius-type Eulerian polynomials ${}_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of order α holds true:

$$
{}_HA_{k+l,q}^{(\alpha,s)}(z,y;\lambda) = \sum_{n=0}^{k} \binom{k}{n}_q (z-x)^n {}_HA_{k-n,q}^{(\alpha,s)}(x,y;\lambda). \tag{3.8}
$$

Remark 3.2. On replacing z by $z + x$ and setting $y = 0$ in Theorem 3.1, we get the following result involving q-Hermite-based Frobenius-type Eulerian polynomials ${}_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of one variable

$$
{}_HA_{k+l,q}^{(\alpha,s)}(z+x;\lambda) = \sum_{n,m=0}^{k,l} \binom{l}{m}_q \binom{k}{n}_q z^{n+m} {}_HA_{k+l-n-m,q}^{(\alpha,s)}(x;\lambda), \tag{3.9}
$$

whereas by setting $z = 0$ in Theorem 3.1, we get another result involving q-Hermite-based Frobenius-type Eulerian polynomials ${}_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of one and two variables

$$
{}_HA_{k+l,q}^{(\alpha,s)}(y;\lambda) = \sum_{n,m=0}^{k,l} \binom{l}{m}_q \binom{k}{n}_q (-x)^{n+m} {}_HA_{k+l-n-m,q}^{(\alpha,s)}(x,y;\lambda). \tag{3.10}
$$

Theorem 3.2. The following summation formulae for q-Hermite-based Frobenius-type Eulerian polynomials ${}_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of order α holds true:

$$
{}_HA_{n,q}^{(\alpha+1,s)}(x,y;\lambda) = \sum_{m=0}^{n} \binom{n}{m}_q A_{n-m,q}(\lambda) {}_HA_{m,q}^{(\alpha,s)}(x,y;\lambda). \tag{3.11}
$$

Proof. From (2.1), we have

$$
\frac{1-\lambda}{e_q((\lambda-1)t)-\lambda} \left(\frac{1-\lambda}{e_q((\lambda-1)t)-\lambda} \right)^\alpha e_q \left(xt - \frac{st^2}{1+q} \right) E_q(yt)
$$
$$
= \frac{1-\lambda}{e_q((\lambda-1)t)-\lambda} \sum_{m=0}^{\infty} {}_HA_{m,q}^{(\alpha,s)}(x,y;\lambda) \frac{t^m}{[m]_q!}
$$

$$
\sum_{n=0}^{\infty} {}_HA_{n,q}^{(\alpha+1,s)}(x,y;\lambda) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} A_{n,q}(\lambda) \frac{t^n}{[n]_q!} \sum_{m=0}^{\infty} {}_HA_{m,q}^{(\alpha,s)}(x,y;\lambda) \frac{t^m}{[m]_q!}.
$$

Now replacing n by $n - m$ and equating the coefficients of t^n leads to formula (3.11). $\square$

Theorem 3.3. The following summation formulae for q-Hermite-based Frobenius-type Eulerian polynomials ${}_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ of order α holds true:

$$
{}_HA_{n,q}^{(\alpha,s)}(x+1,y;\lambda) = \sum_{k=0}^{n} \binom{n}{k}_q {}_HA_{k,q}^{(\alpha,s)}(x,y;\lambda).. \tag{3.12}
$$

Proof. Using Definition 2.1, we have

$$\sum_{n=0}^{\infty} {}_H A_n^{(\alpha,s)}(x+1,y;\lambda)\frac{t^n}{n!} = \left(\frac{1-\lambda}{e_q((\lambda-1)t)-\lambda}\right)^{\alpha} e_q\left((x+1)t-\frac{st^2}{1+q}\right)E_q(yt)$$

$$= \left(\frac{1-\lambda}{e_q((\lambda-1)t)-\lambda}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)e_q(t)$$

$$= \left(\sum_{k=0}^{\infty} {}_H A_{k,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^k}{[k]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q {}_H A_{k,q}^{(\alpha,s)}(x,y;\lambda)\right)\frac{t^n}{[n]_q!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.12). $\qquad\square$

The q-Stirling numbers of the second kind are defined by

$$\sum_{n=0}^{\infty} S_{2,q}(k,n)\frac{t^n}{[n]_q!} = \frac{(e_q(t)-1)^k}{[k]_q!}, \quad \text{see [14–17, 20]}.$$

Theorem 3.4. The following relationship

$$ {}_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \sum_{u=0}^{n}\sum_{k=0}^{n}\binom{n}{u}_q [k]_q!\binom{-\alpha}{k}_q (1-\lambda)^k S_{2,q}(k,u)(\lambda-1)^u H_{n-u,q}^{(s)}(x,y) \quad (3.13)$$

holds true.

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!}$$

$$= \left(\frac{1-\lambda}{e_q((\lambda-1)t)-\lambda}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$

$$= e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)\left(1+\frac{e_q((\lambda-1)t)-1}{1-\lambda}\right)^{-\alpha}$$

$$= \sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{e_q((\lambda-1)t)-1}{1-\lambda}\right)^k \sum_{n=0}^{\infty} H_{n,q}^{(s)}(x,y)\frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\sum_{k=0}^{n}\binom{n}{u}_q [k]_q!\binom{-\alpha}{k}(1-\lambda)^k S_{2,q}(k,u)(\lambda-1)^u H_{n-u,q}^{(s)}(x,y)\right)\frac{t^n}{[n]_q!}$$

On comparing the coefficients of t^n in both sides, we get (3.13). $\qquad\square$

Theorem 3.5. The following relationship

$$ {}_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \sum_{u=0}^{n}\sum_{k=0}^{n}\binom{n}{u}_q [k]_q!\binom{\alpha+k-1}{k} S_{2,q}(k,u)(\lambda-1)^{u+k} H_{n-u,q}^{(s)}(x,y) \quad (3.14)$$

holds true.

Proof. By using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!} = \left(\frac{1-\lambda}{e_q((\lambda-1)t)-\lambda}\right)^{\alpha} e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)$$

$$= e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)\left(1+\frac{e_q((\lambda-1)t)-1}{1-\lambda}\right)^{-\alpha}$$

$$= \sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k-1}{k}\left(\frac{e_q((\lambda-1)t)-1}{1-\lambda}\right)^k \sum_{n=0}^{\infty}H_{n,q}^{(s)}(x,y)\frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\sum_{k=0}^{n}\binom{n}{u}_q[k]_q!\binom{\alpha+k-1}{k}S_{2,q}(k,u)(\lambda-1)^{u+k}H_{n-u,q}^{(s)}(x,y)\right)\frac{t^n}{[n]_q!}$$

Comparing the coefficients of t^n in both sides, we arrive at the required result (3.14). $\qquad\square$

Theorem 3.6. The following relation between the q-Hermite-based Frobenius-type Eulerian polynomials $_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ and Apostol-type q-Bernoulli polynomials $B_{n,q}(x;\lambda)$ holds true:

$$_HA_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\binom{n+1}{k}_q\left(\lambda\sum_{r=0}^{k}\binom{k}{r}_q B_{k-r,q}(x;\lambda)-B_{k,q}(x;\lambda)\right)$$

$$\times {}_HA_{n-k+1,q}^{(\alpha,s)}(0,y;\lambda). \tag{3.15}$$

Proof. Consider the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)\frac{t^n}{[n]_q!}$$

$$= \left(\frac{1-\lambda}{e_q(t(\lambda-1))-\lambda}\right)^{\alpha}e_q\left(xt-\frac{st^2}{1+q}\right)E_q(yt)\left(\frac{t}{\lambda e_q(t)-1}\right)\left(\frac{\lambda e_q(t)-1}{t}\right)$$

$$= \frac{1}{t}\left(\lambda\sum_{n=0}^{\infty}{}_HA_{n,q}^{(\alpha,s)}(0,y;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}B_{k,q}(x;\lambda)\frac{t^k}{[k]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\right.$$

$$\left.-\sum_{n=0}^{\infty}{}_HA_{n,q}^{(\alpha,s)}(0,y;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}B_{k,q}(x;\lambda)\frac{t^k}{[k]_q!}\right). \tag{3.16}$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.16), assertion (3.15) follows. $\qquad\square$

Theorem 3.7. The following relation between the q-Hermite-based Frobenius-type Eulerian polynomials $_HA_{n,q}^{(\alpha,s)}(x,y;\lambda)$ and Apostol-type Euler polynomials $E_{n,q}(x;\lambda)$ holds true:

$$_HA_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}_q\left(\lambda\sum_{r=0}^{k}\binom{k}{r}_q E_{k-r,q}(x;\lambda)+E_{k,q}(x;\lambda)\right)$$

$$\times {}_HA_{n-k,q}^{(\alpha,s)}(0,y;\lambda). \tag{3.17}$$

Proof. Consider the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha)}(x,y;\lambda) \frac{t^n}{[n]_q!}$$

$$= \left(\frac{1-\lambda}{e_q(t(\lambda-1))-\lambda} \right)^{\alpha} e_q \left(xt - \frac{st^2}{1+q} \right) E_q(yt) \left(\frac{2}{\lambda e_q(t)+1} \right) \left(\frac{\lambda e_q(t)+1}{2} \right)$$

$$= \frac{1}{2} \left(\lambda \sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha,s)}(0,y;\lambda) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} E_{k,q}(x;\lambda) \frac{t^k}{[k]_q!} \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} \right.$$

$$\left. + \sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha,s)}(0,y;\lambda) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} E_{k,q}(x;\lambda) \frac{t^k}{[k]_q!} \right). \tag{3.18}$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.18), assertion (3.17) follows. $\qquad\square$

Theorem 3.8. The following relation between the q-Hermite-based Frobenius-type Eulerian polynomials ${}_H A_{n,q}^{(\alpha,s)}(x,y;\lambda)$ and Apostol-type Genocchi polynomials $G_{n,q}(x;\lambda)$ holds true:

$$_H A_{n,q}^{(\alpha,s)}(x,y;\lambda) = \frac{1}{2[n+1]_q} \sum_{k=0}^{n+1} \binom{n+1}{k}_q \left(\lambda \sum_{r=0}^{k} \binom{k}{r}_q G_{k-r,q}(x;\lambda) + G_{k,q}(x;\lambda) \right)$$

$$\times {}_H A_{n-k+1,q}^{(\alpha,s)}(0,y;\lambda). \tag{3.19}$$

Proof. Consider the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha)}(x,y;\lambda) \frac{t^n}{[n]_q!}$$

$$= \left(\frac{1-\lambda}{e_q(t(\lambda-1))-\lambda} \right)^{\alpha} e_q \left(xt - \frac{st^2}{1+q} \right) E_q(yt) \left(\frac{2t}{\lambda e_q(t)+1} \right) \left(\frac{\lambda e_q(t)+1}{2t} \right)$$

$$= \frac{1}{2t} \left(\lambda \sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha,s)}(0,y;\lambda) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} G_{k,q}(x;\lambda) \frac{t^k}{[k]_q!} \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} \right.$$

$$\left. + \sum_{n=0}^{\infty} {}_H A_{n,q}^{(\alpha,s)}(0,y;\lambda) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} G_{k,q}(x;\lambda) \frac{t^k}{[k]_q!} \right). \tag{3.20}$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.20), assertion (3.19) follows. $\qquad\square$

4 Conclusion

In the present paper, we have initially considered a novel kind of the Hermite-based Frobenius-type Eulerian polynomials based on q-numbers and then have provided multifarious correlation, implicit summation formula, identities, explicit formulas and recurrence relations for the aforementioned polynomials by means of the series manipulation methods and some special techniques. The aforegoing polynomials involves a lot of well known special polynomials in literature.

Furthermore, we have given several relationships covering the q-Apostol–Bernoulli polynomials, q-Apostol-Euler polynomials, q-Apostol-Genocchi polynomials and q-Stirling numbers of the second kind associated with the q-Hermite-based Frobenius-type Eulerian polynomials.

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