Number of tuples with a given least common multiple

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Abstract: In this paper, for a given natural number \( n \), we count the number of \( k \)-tuples \((x_1, x_2, \ldots, x_k) \in \mathbb{N}^k\) with certain conditions such that \( \text{lcm}(x_1, x_2, \ldots, x_k) = n \). In the process, we derived different arithmetic functions.

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1 Introduction and preliminaries

Let \( n = \prod_{i=1}^{m} p_i^{\alpha_i} \) be the prime factorization of positive integer \( n \) and

\[ \Delta_n(k) = \{ (x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \mid \forall j, x_j \mid n \}. \]

For a given \((x_1, x_2, \ldots, x_k) \in \Delta_n(k)\) we associate a \( k \)-tuple \((\beta_1, \beta_2, \ldots, \beta_k)\) such that \( p_i^{\beta_j} \mid x_j \) for each \( p_i \leq i \leq m \). Since \( 0 \leq \beta_j \leq \alpha_i \), the total number of possible \( k \)-tuples \((x_1, x_2, \ldots, x_k)\) corresponding to \( p_i \) is \((\alpha_i + 1)^k\). Thus \( |\Delta_n(k)| = \prod_{i=1}^{m} (\alpha_i + 1)^k \). If \( k = 1 \), then \( \Delta_n(1) \subseteq \mathbb{N} \) is exactly the set of positive divisors of \( n \). We denote \( |\Delta_n(1)| \) as \( \tau(n) \), number of positive divisors of \( n \). Hence \( |\Delta_n(k)| = \tau(n)^k \).
The following result, also proved by O. Bagdasar [2] we are giving the proof for the sake of completeness. For elementary properties of divisor function, $\text{lcm}$, $\text{gcd}$ refer any one of the following [1, 3, 6, 7].

**Theorem 1.1** ([2]). Let $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ and

$$A_n(k) = \{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k | \text{lcm}(x_1, x_2, \ldots, x_k) = n\}.$$

Then $|A_n(k)| = \prod_{i=1}^{m} (\alpha_i + 1) - (\alpha_i)^k$.

**Proof.** First note that $A_n(k) \subseteq \Delta_n(k)$. In order to have $\text{lcm}(x_1, x_2, \ldots, x_k) = n$, at least one of $x_j$ should be equal to $p_i^{\alpha_i}$. Corresponding to each $p_i$, the number of elements in $\Delta_n(k) \setminus A_n(k)$ are $\alpha_i^k$. Thus the total number of valid cases for $p_i$ is $(\alpha_i + 1) - (\alpha_i)^k$. Hence the result follows from the product rule.

**Example 1.2.** If $n = 12$ and $k = 2$, then we have $A_{12}(2) = \{(1, 12), (2, 12), (3, 4), (3, 12), (4, 3), (4, 6), (4, 12), (6, 4), (6, 12), (12, 1), (12, 2), (12, 3), (12, 4), (12, 6), (12, 12)\}$ and

$$|A_{12}(2)| = ((2 + 1)^2 - 2^2)((1 + 1)^2 - 1^2).$$

Let $P_k$ denote the product of first $k$ primes. For example $P_1 = 2, P_2 = 6, P_3 = 30$. The sequence whose $n$-th term is $P_n$ is called **primorial** and $P_n$ is called $n$-th primorial number.

**Corollary 1.3.** Let $P_n$ be the $n$-th primorial number. Then $|A_{P_n}(k)| = (2^k - 1)^n$.

It is easy to see that $\Delta_n(k)$ and $A_n(k)$ are multiplicative functions in $n$. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is multiplicative if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$.

Let $S_n(k) = \{(x_1, x_2, \ldots, x_k) \in A_n(k) | 1 < x_1 < \cdots < x_k < n\}$. Then it is clear that $S_{p^t}(k)$ is an empty set, where $p$ is a prime number and $t \in \mathbb{N}$. Further, $2 \leq k \leq \tau(n) - 2$. For example

$$S_{30}(2) = \{(2, 15), (3, 10), (5, 6), (6, 10), (6, 15), (10, 15)\}.$$

Our goal is to find out $|S_n(k)|$ for a given $n$ and $k$. Before stating main result, we state and prove a couple of results.

**Lemma 1.4.** Let $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ and $B_n(k) = \{(x_1, x_2, \ldots, x_k) \in A_n(k) | \forall i \; x_i < n\}$. Then

$$|B_n(k)| = |A_n(k)| - (\tau(n)^k - (\tau(n) - 1)^k).$$

**Proof.** Let $\Delta_n^*(k) = \{(x_1, x_2, \ldots, x_k) \in \Delta_n(k) | x_j = n \text{ for some } j\}$. Then $\Delta_n^*(k) \subseteq A_n(k)$ and $B_n(k) = A_n(k) \setminus \Delta_n^*(k)$. It is easy to see that $|\Delta_n(k) \setminus \Delta_n^*(k)| = (\tau(n) - 1)^k$. Hence the result follows.

**Example 1.5.** Let $n = 12$ and $k = 2$. Then $B_{12}(2) = \{(3, 4), (4, 3), (4, 6), (6, 4)\}$.

$$|B_{12}(2)| = 15 - (6^2 - (6 - 1)^2).$$

The following result follows from Corollary 1.3 and $|B_{P_n}(2)| = |A_{P_n}(2)| - (2^{2n} - (2^n - 1)^2)$. 

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Corollary 1.6. Let $P_n$ be the $n$-th primorial number. Then
\[ |B_{P_n}(2)| = 3^n - 2^{n+1} + 1 = 2\left\{ \frac{n+1}{3} \right\}, \]
where $\left\{ \frac{n}{k} \right\}$ is the Stirling numbers of the second kind.

Lemma 1.7. Let $n = \prod_{i=1}^{m} p_i^{a_i}$ and $C_n(k) = \{ (x_1, x_2, \ldots, x_k) \in B_n(k) | \forall i, x_i > 1 \}$. Then
\[ |C_n(k)| = \left( \sum_{i=0}^{k} (-1)^i \binom{k}{i} |A_n(k-i)| \right) - \left( (\tau(n) - 1)^k - (\tau(n) - 2)^k \right). \]

Proof. We have that $|C_n(k)| = |B_n(k)|$ is the number of tuples in $B_n(k)$ that contain 1. Using the principle of inclusion and exclusion, we get that number of tuples in $B_n(k)$ that contain 1 is
\[ \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |B_n(k-i)| \]
\[ = \left( \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) - \left( \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} t^{k-i} \right) + \left( \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} (t-1)^{k-i} \right) \]
\[ = \left( \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) + \left( \sum_{i=1}^{k} (-1)^{i} \binom{k}{i} t^{k-i} \right) - \left( \sum_{i=1}^{k} (-1)^{i} \binom{k}{i} (t-1)^{k-i} \right) \]
\[ = \left( \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) + (t-1)^k - (t-1)^k - (t-2)^k - (t-1)^k \]
\[ = \left( \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) - (t^k - 2(t-1)^k + (t-2)^k). \]

Therefore, $|C_n(k)|$
\[ = |B_n(k)| - \left( \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} |A_n(k-i)| \right) + (t^k - 2(t-1)^k + (t-2)^k) \]
\[ = |A_n(k)| - (t^k - (t-1)^k) + \left( \sum_{i=1}^{k} (-1)^{i} \binom{k}{i} |A_n(k-i)| \right) + (t^k - 2(t-1)^k + (t-2)^k) \]
\[ = \left( \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} |A_n(k-i)| \right) - ((t-1)^k - (t-2)^k). \]

This completes the proof. \qed

Example 1.8. Since $C_{12}(2) = \{ (3, 4), (4, 3), (4, 6), (6, 4) \}$, we have $|C_{12}(2)| = 4$. One can verify that $|C_{12}(2)| = (A_{12}(2) - 2A_{12}(1) + A_{12}(0)) - (5^2 - 4^2)$.

When $k = 2$, we have that $B_n(2) = C_n(2)$. Thus from Corollary 1.6 we have $|C_{P_n}(2)| = 2\left\{ \frac{n+1}{3} \right\}$.  

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2 Main results

Theorem 2.1. Let \( n = \prod_{i=1}^{m} p_i^{\alpha_i} \). Then

\[
|S_n(k)| = \binom{\tau(n) - 2}{k} + \sum_{d \mid n, d \neq 1} \left( \mu(d) \binom{\tau\left(\frac{n}{d}\right) - 1}{k} \right),
\]

where \( \mu(.) \) denotes well-known Möbius function.

We use the following lemma to prove above theorem. We omit the proof, as it is easy to derive.

Lemma 2.2. Let \( n \geq 2 \), \( G_n(k) = \{(x_1, x_2, \ldots, x_k) | 1 < x_1 < \cdots < x_k < n, \forall i x_i | n \} \) and \( H_n(k) = \{(x_1, x_2, \ldots, x_k) | 1 < x_1 < \cdots < x_k \leq n, \forall i x_i | n \} \). Then

\[
|G_n(k)| = \binom{\tau(n) - 2}{k}, \quad |H_n(k)| = \binom{\tau(n) - 1}{k}.
\]

Proof. Proof of Theorem 2.1. First note that \( S_n(k) \subseteq G_n(k) \). Let \((x_1, \ldots, x_k) \in G_n(k)\). If lcm\((x_1, x_2, \ldots, x_k) = n\), then \((x_1, \ldots, x_k) \in S_n(k)\). Let us assume that lcm\((x_1, \ldots, x_k) = l < n\). Then there exists a prime \( p \) such that \( p | \frac{n}{l} \). Hence \((x_1, \ldots, x_k) \in H_{\frac{n}{p}}(k)\) and for every \( p | n, H_{\frac{n}{p}}(k) \subseteq G_n(k) \). Hence

\[
S_n(k) = G_n(k) \setminus \left( \cup_{p | n} H_{\frac{n}{p}}(k) \right).
\]

Therefore,

\[
|S_n(k)| = |G_n(k)| - \left| \left( \cup_{p | n} H_{\frac{n}{p}}(k) \right) \right|.
\]

Since the prime factors of \( n \) are \( p_1, p_2, \ldots, p_m \), after applying principle of inclusion and exclusion, we get

\[
- \left| \left( \cup_{p | n} H_{\frac{n}{p}}(k) \right) \right| = - \sum_{p_i} \left| H_{\frac{n}{p_i}}(k) \right| + \cdots + (-1)^x \sum_{p_i < \cdots < p_x} \left| H_{\frac{n}{p_1 \cdots p_x}}(k) \right| + \cdots + (-1)^m \left| H_{\frac{n}{p_1 \cdots p_m}}(k) \right|
\]

\[
= \sum_{d \mid n, d \neq 1} (\mu(d) | H_{\frac{n}{d}}(k)|) = \sum_{d \mid n, d \neq 1} \left( \mu(d) \binom{\tau\left(\frac{n}{d}\right) - 1}{k} \right).
\]

Therefore

\[
|S_n(k)| = \binom{\tau(n) - 2}{k} + \sum_{d \mid n, d \neq 1} \left( \mu(d) \binom{\tau\left(\frac{n}{d}\right) - 1}{k} \right).
\]

Example 2.3. For \( n = 30 \) and \( k = 3 \), we have \( S_{30}(3) = \{(2, 3, 5), (2, 3, 10), (2, 3, 15), (2, 5, 6), (2, 5, 15), (2, 6, 10), (2, 6, 15), (2, 10, 15), (3, 5, 6), (3, 5, 10), (3, 6, 10), (3, 6, 15), (3, 10, 15), (5, 6, 10), (5, 6, 15), (5, 10, 15), (6, 10, 15)\}. Hence \( |S_{30}(3)| = 17 \).
Lemma 2.4. Let \( n \geq 2 \) and \( Q_n(k) = \{(x_1, x_2, \ldots, x_k) | 1 \leq x_1 < \cdots < x_k < n, \ \text{lcm}(x_1, \ldots, x_k) = n\} \). Then
\[
|Q_n(k)| = \left(\tau\left(\frac{n}{k}\right) - 1\right) + \sum_{d|n, \ d \neq 1} \left(\mu(d)\left(\left\lfloor \frac{n}{d} \right\rfloor \right)\right).
\]

Proof. If \( 1 < x_1 \), then \( (x_1, \ldots, x_k) \in S_n(k) \). If \( 1 = x_1 \), then \( (x_2, \ldots, x_k) \in S_n(k-1) \). Therefore,
\[
|Q_n(k)| = |S_n(k)| + |S_n(k-1)|
\]
\[
= \left(\tau\left(\frac{n}{k}\right) - 2\right) + \sum_{d|n, \ d \neq 1} \left(\mu(d)\left(\left\lfloor \frac{n}{d} \right\rfloor \right)\right) + \left(\tau\left(\frac{n}{k-1}\right) - 2\right) + \sum_{d|n, \ d \neq 1} \left(\mu(d)\left(\left\lfloor \frac{n}{d-1} \right\rfloor \right)\right)
\]
\[
= \left(\tau\left(\frac{n}{k}\right) - 2\right) + \sum_{d|n, \ d \neq 1} \left(\mu(d)\left(\left\lfloor \frac{n}{d} \right\rfloor \right)\right) + \left(\tau\left(\frac{n}{k-1}\right) - 1\right) + \sum_{d|n, \ d \neq 1} \left(\mu(d)\left(\left\lfloor \frac{n}{d-1} \right\rfloor \right)\right)
\]
\[
= \left(\tau\left(\frac{n}{k}\right) - 1\right) + \sum_{d|n, \ d \neq 1} \left(\mu(d)\left(\left\lfloor \frac{n}{d} \right\rfloor \right)\right).
\]

Example 2.5. For \( n = 12 \) and \( k = 3 \), we have \( Q_{12}(3) = \{(1, 3, 4), (1, 4, 6), (2, 3, 4), (2, 4, 6), (3, 4, 6)\} \).
\[
|Q_{12}(3)| = 10 + (-4 - 1 + 0 + 0 + 0).
\]

Corollary 2.6. Let \( P_n \) denote the \( n \)-th primorial number. Then \( |Q_{P_n}(2)| = \left\{\binom{n+1}{3}\right\} \).

Proof. We have \( |Q_{P_n}(2)| = \left(\sum_{d|P_n, \ d \neq 1} \left(\mu(d)\left(\left\lfloor \frac{P_n}{d} \right\rfloor \right)\right)\right) \). Thus
\[
|Q_{P_n}(2)| = \left(\frac{2^n - 1}{2}\right) + \sum_{i=1}^{n} (-1)^i \binom{n}{i} \left(\frac{2^{n-i}}{2}\right)
\]
\[
= \left(\frac{2^n - 1}{2}\right) - \left(\frac{2^n}{2}\right) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(\frac{2^{n-i}}{2}\right)
\]
\[
= \left(\frac{2^n - 1}{2}\right) - \left(\frac{2^n}{2}\right) + \frac{1}{2} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2^{n-i} - 1)
\]
\[
= \left(\frac{2^n - 1}{2}\right) - \left(\frac{2^n}{2}\right) + \frac{1}{2} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (4^{n-i} - 2^{n-i})
\]
\[
= \left(\frac{2^n - 1}{2}\right) - \left(\frac{2^n}{2}\right) + \frac{1}{2} 3^n - \frac{1}{2} = \frac{3^n - 2^{n+1} + 1}{2} = \left\{\binom{n+1}{3}\right\}.
\]

Theorem 2.7. Let \( R_n(k) = \{(x_1, x_2, \ldots, x_k) \in S_n(k) | \ \gcd(x_1, \ldots, x_k) = 1\} \). Then
\[
|R_n(k)| = |S_n(k)| + \sum_{d|n, \ d \neq 1} \mu(d)|Q_n(k)|.
\]

Proof. Let \( (x_1, \ldots, x_k) \in S_n(k) \) and \( \gcd(x_1, \ldots, x_k) = d > 1 \). Then there exists a prime \( p \) such that \( p|d \). Hence we can write \( (x_1, \ldots, x_k) = p^{\left(\frac{x_1}{d}\right)} \cdot \ldots \cdot p^{\left(\frac{x_k}{d}\right)} \) and \( (x_1, \ldots, x_k) \in p Q_{P_n}(k) \). Since for every prime \( p \), \( p Q_{P_n}(k) \subseteq S_n(k) \), \( R_n(k) = S_n(k) \setminus \left(\cup_{p|d} p Q_{P_n}(k)\right) \). Hence we have
\[
|R_n(k)| = |S_n(k)| - \left|\left(\cup_{p|d} p Q_{P_n}(k)\right)\right|.
\]
Let the prime factors of $n$ be $\{p_1, p_2, \ldots, p_m\}$. By applying principle of inclusion and exclusion we get

$$-\left| \bigcup_{p|n} Q_{\frac{n}{p}}(k) \right|$$

$$= -\sum_{p_i} \left| Q_{\frac{n}{p_i}}(k) \right| + \cdots + (-1)^x \sum_{p_i < \ldots < p_x} \left| Q_{\frac{n}{p_i \ldots p_x}}(k) \right| + \cdots + (-1)^n \left| Q_{\frac{n}{p_1 \ldots p_m}}(k) \right|$$

$$= \sum_{d|n, d \neq 1} (\mu(d)|Q_{\frac{n}{d}}(k)|).$$

Therefore

$$|R_n(k)| = |S_n(k)| + \sum_{d|n, d \neq 1} \mu(d)|Q_{\frac{n}{d}}(k)|. \quad \square$$

We noticed that the sequence $|S_{P_n}(2)|$ coincides with the sequence in OEIS: A000392 (https://oeis.org/A000392). The following result establishes the same correspondence.

**Theorem 2.8.** Let $P_n$ be the $n$-th primorial number. Then

$$|S_{P_n}(2)| = \left\{ \frac{n + 1}{3} \right\}.$$

**Proof.** We have

$$|S_{P_n}(2)|$$

$$= \left( 2^n - 2 \right) + \left( \sum_{i=1}^{n} (-1)^i \binom{n}{i} \left( \frac{2^{n-i} - 1}{2} \right) \right)$$

$$= (2^n - 1)(2^n - 3) + \left( \sum_{i=1}^{n} (-1)^i \binom{n}{i} (2^{n-i} - 1)(2^{n-i-1} - 1) \right)$$

$$= -2(2^{n-1} - 1) + (1 - 0)^0 \left( \binom{n}{0} (2^n - 1)(2^{n-1} - 1) + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (2^{n-i} - 1)(2^{n-i-1} - 1) \right)$$

$$= -2(2^{n-1} - 1) + \left( \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2^{n-i} - 1)(2^{n-i-1} - 1) \right)$$

$$= -2(2^{n-1} - 1) + \left( \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( \frac{1}{2} 4^{n-i} - \frac{3}{2} 2^{n-i} + 1 \right) \right)$$

$$= -2(2^{n-1} - 1) + \frac{1}{2} \left( \sum_{i=0}^{n} (-1)^i \binom{n}{i} (4^{n-i}) \right) - \frac{3}{2} \left( \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2^{n-i}) \right) + \left( \sum_{i=0}^{n} (-1)^i \binom{n}{i} \right)$$

$$= -2(2^{n-1} - 1) + \frac{3^n - 3}{2}$$

$$= \frac{3^n - 2^{n+1} + 1}{2}$$

$$= \left\{ \frac{n + 1}{3} \right\}. \quad \square$$
Theorem 2.9. Let \( n = \prod_{i=1}^{m} p_i^{\alpha_i}, k \leq m \) and \( F_n(k) = \{ (x_1, x_2, \ldots, x_k) \in A_n(k) | x_i \neq 1 \text{ and } \gcd(x_i, x_j) = 1 \} \). Then \( |F_n(k)| = \left\{ \begin{array}{l} m \\ k! \end{array} \right\} \).

Proof. Let \( f : \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, k\} \) be an onto function. Let \( f(i) \) denote the position of prime power \( p_i^{\alpha_i} \) in the \( k \)-tuple. Since \( f \) is onto every entry in the \( k \)-tuple is a non-unit. Therefore, the number of onto functions is equal to the number of required \( k \)-tuples. The number of onto functions from a set of size \( m \) to a set of size \( k \) is given by \( \left\{ \begin{array}{l} m \\ k \end{array} \right\} k! \). Hence \( |F_n(k)| = \left\{ \begin{array}{l} m \\ k \end{array} \right\} k! \).

Example 2.10. \( F_{210}(3) = \{ (2, 3, 35), (2, 5, 21), (2, 7, 15), (2, 15, 7), (2, 21, 5), (2, 35, 3), (3, 2, 35), (3, 5, 14), (3, 7, 10), (3, 10, 7), (3, 14, 5), (3, 35, 2), (5, 2, 21), (5, 3, 14), (5, 6, 7), (5, 7, 6), (5, 14, 3), (5, 21, 2), (6, 5, 7), (6, 7, 5), (7, 2, 15), (7, 3, 10), (7, 5, 6), (7, 6, 5), (7, 10, 3), (7, 15, 2), (10, 3, 7), (10, 7, 3), (14, 3, 5), (14, 5, 3), (15, 2, 7), (15, 7, 2), (21, 2, 5), (21, 5, 2), (35, 2, 3), (35, 3, 2) \} \). Hence \( |F_{210}(3)| = 36 \).

Corollary 2.11. Let \( F'_n(k) = \{ (x_1, x_2, \ldots, x_k) \in F_n(k) | x_1 < \cdots < x_k < n \} \).

Proof. Each tuple in \( F'_n(k) \) corresponds to \( k! \) tuples in \( F_n(k) \). Hence
\[
|F'_n(k)| = \frac{|F_n(k)|}{k!} = \left\{ \begin{array}{l} m \\ k \end{array} \right\} k!.
\]

Example 2.12. \( F'_{210}(3) = \{ (2, 3, 35), (2, 5, 21), (2, 7, 15), (3, 5, 14), (3, 7, 10), (5, 6, 7) \} \). Hence \( |F_{210}(3)| = 6 \). It is easy to verify that \( R_n(2) = F'_n(2) \).

3 Conclusion

In this article, for a given natural numbers \( n \) and \( k \), we derived different arithmetic functions of the form \( f_n(k) \) which count the numbers elements in \( \mathbb{N}^k \) satisfying few conditions such that whose lcm is \( n \). We associate these functions with Stirling numbers of the second kind for certain values of \( n \) and \( k \). In future we will work on applications of these functions on the multiplicative representation of integers studied in [5, 8] in particular, Theorem 2.9. One can also explore sequences obtained by iterating these functions as studied in the recent paper [4].

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