Some modular considerations regarding odd perfect numbers

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Abstract: Let $p^km^2$ be an odd perfect number with special prime $p$. In this article, we provide an alternative proof for the biconditional that $\sigma(m^2) \equiv 1 \pmod{4}$ holds if and only if $p \equiv k \pmod{8}$. We then give an application of this result to the case when $\sigma(m^2)/p^k$ is a square.

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1 Introduction

Let $\sigma(z)$ denote the sum of the divisors of $z \in \mathbb{N}$, the set of positive integers. Denote the deficiency [5] of $z$ by $D(z) = 2z - \sigma(z)$, and the sum of the aliquot divisors [6] of $z$ by $s(z) = \sigma(z) - z$. Note that we have the identity $D(z) + s(z) = z$.

If $n$ is odd and $\sigma(n) = 2n$, then $n$ is said to be an odd perfect number [8]. Euler proved that an odd perfect number, if one exists, must have the form $n = p^km^2$, where $p$ is the special prime satisfying $p \equiv k \equiv 1 \pmod{4}$ and $\gcd(p,m) = 1$. 
Chen and Luo [2] gave a characterization of the forms of odd perfect numbers \( n = p^k m^2 \) such that \( p \equiv k \pmod{8} \). Starni [7] proved that there is no odd perfect number decomposable into primes all of the type \( \equiv 1 \pmod{4} \) if \( n = p^k m^2 \) and \( p \not\equiv k \pmod{8} \). Starni used a congruence from Ewell [3] to prove this result.

Note that, in general, since \( m^2 \) is a square, we get \( \sigma(m^2) \equiv 1 \pmod{2} \).

This paper provides an alternative proof for Theorem 3.3, equation 3.1 in Chen and Luo’s article titled “Odd multiperfect numbers” [2]:

**Theorem 1.1.** Let \( n = \pi^\alpha M^2 \) be an odd 2-perfect number, with \( \pi \) prime, \( \gcd(\pi, M) = 1 \) and \( \pi \equiv \alpha \equiv 1 \pmod{4} \). Then

\[
\sigma(M^2) \equiv 1 \pmod{4} \iff \pi \equiv \alpha \pmod{8}.
\]

The method presented in this paper may potentially be used to extend the arguments to consider \( \sigma(m^2) \) modulo 8.

## 2 Preliminaries

Starting from the fundamental equality

\[
\frac{\sigma(m^2)}{p^k} = \frac{2m^2}{\sigma(p^k)}
\]

(which follows from the facts that \( \sigma(n) = 2n \), \( \sigma \) is multiplicative, and \( \gcd(p^k, \sigma(p^k)) = 1 \)), one can derive

\[
\frac{\sigma(m^2)}{p^k} = \frac{2m^2}{\sigma(p^k)} = \gcd(m^2, \sigma(m^2)),
\]

so that we ultimately have

\[
\frac{D(m^2)}{s(p^k)} = \frac{2m^2 - \sigma(m^2)}{\sigma(p^k) - p^k} = \gcd(m^2, \sigma(m^2))
\]

and

\[
\frac{s(m^2)}{D(p^k)/2} = \frac{\sigma(m^2) - m^2}{p^k - \frac{\sigma(p^k)}{2}} = \gcd(m^2, \sigma(m^2)),
\]

whereby we obtain

\[
\frac{D(p^k)D(m^2)}{s(p^k)s(m^2)} = 2.
\]

Note that we also have the following equation:

\[
\frac{2D(m^2)s(m^2)}{D(p^k)s(p^k)} = \left( \gcd(m^2, \sigma(m^2)) \right)^2.
\]  

\((*)\)

Lastly, notice that we can easily get

\[
\sigma(p^k) \equiv k + 1 \equiv 2 \pmod{4}
\]

(since \( p \equiv k \equiv 1 \pmod{4} \)) so that it remains to consider the possible equivalence classes for \( \sigma(m^2) \) modulo 4. Since \( \sigma(m^2) \) is odd, we only need to consider two.

We ask: Which equivalence class of \( \sigma(m^2) \) modulo 4 makes Equation \((*)\) untenable?
3 Discussion and results

We know that the answer to the question we posed in the previous section must somehow depend on the equivalence class of $p$ and $k$ modulo 8, but as we only know that $p \equiv k \equiv 1 \pmod{4}$, we need to consider the following cases separately and thereby prove the corresponding results:

Remark 3.1. Suppose that $n = p^k m^2$ is an odd perfect number with special prime $p$. We claim the truth of the following propositions, which we will need to treat separately later:

1. If $p \equiv k \equiv 1 \pmod{8}$, then $\sigma(m^2) \equiv 3 \pmod{4}$ is impossible.

2. If $p \equiv 1 \pmod{8}$ and $k \equiv 5 \pmod{8}$, then $\sigma(m^2) \equiv 1 \pmod{4}$ is impossible.

3. If $p \equiv 5 \pmod{8}$ and $k \equiv 1 \pmod{8}$, then $\sigma(m^2) \equiv 1 \pmod{4}$ is impossible.

4. If $p \equiv k \equiv 5 \pmod{8}$, then $\sigma(m^2) \equiv 3 \pmod{4}$ is impossible.

First, we prove the following lemmas:

Lemma 3.2. Suppose that $n = p^k m^2$ is an odd perfect number with special prime $p$.

1. If $p \equiv 1 \pmod{8}$, then $\sigma(p^k) \equiv k + 1 \pmod{8}$.

2. If $p \equiv 5 \pmod{8}$ and $k \equiv 1 \pmod{8}$, then $\sigma(p^k) \equiv 6 \pmod{8}$.

3. If $p \equiv 5 \pmod{8}$ and $k \equiv 5 \pmod{8}$, then $\sigma(p^k) \equiv 2 \pmod{8}$.

Proof. Let $n = p^k m^2$ be an odd perfect number with special prime $p$. It follows that $p \equiv 1 \pmod{4}$.

We consider two cases:

Case 1: $p \equiv 1 \pmod{8}$ We obtain

$$\sigma(p^k) = \sum_{i=0}^{k} p^i \equiv 1 + \sum_{i=1}^{k} p^i \equiv 1 + \sum_{i=1}^{k} 1^i \equiv k + 1 \pmod{8},$$

as desired.

Case 2: $p \equiv 5 \pmod{8}$ We get

$$\sigma(p^k) = \sum_{i=0}^{k} p^i \equiv \sum_{i=0}^{k} 5^i \equiv \begin{cases} 6 \pmod{8}, & \text{if } k \equiv 1 \pmod{8} \\ 2 \pmod{8}, & \text{if } k \equiv 5 \pmod{8} \end{cases}$$

This completes the proof.

Lemma 3.3. Suppose that $n = p^k m^2$ is an odd perfect number with special prime $p$.

1. If $p \equiv 1 \pmod{8}$ and $k \equiv 1 \pmod{8}$, then $D(p^k) \equiv 0 \pmod{8}$.

2. If $p \equiv 1 \pmod{8}$ and $k \equiv 5 \pmod{8}$, then $D(p^k) \equiv 4 \pmod{8}$.

3. If $p \equiv 5 \pmod{8}$ and $k \equiv 1 \pmod{8}$, then $D(p^k) \equiv 4 \pmod{8}$.
4. If \( p \equiv 5 \pmod{8} \) and \( k \equiv 5 \pmod{8} \), then \( D(p^k) \equiv 0 \pmod{8} \).

**Proof.** The proof is trivial and follows directly from Lemma 3.2, using the formula \( D(p^k) = 2p^k - \sigma(p^k) \).

**Lemma 3.4.** Suppose that \( n = p^k m^2 \) is an odd perfect number with special prime \( p \).

1. If \( p \equiv 1 \pmod{8} \) and \( k \equiv 1 \pmod{8} \), then \( s(p^k) \equiv 1 \pmod{8} \).
2. If \( p \equiv 1 \pmod{8} \) and \( k \equiv 5 \pmod{8} \), then \( s(p^k) \equiv 5 \pmod{8} \).
3. If \( p \equiv 5 \pmod{8} \) and \( k \equiv 1 \pmod{8} \), then \( s(p^k) \equiv 1 \pmod{8} \).
4. If \( p \equiv 5 \pmod{8} \) and \( k \equiv 5 \pmod{8} \), then \( s(p^k) \equiv 5 \pmod{8} \).

**Proof.** The proof is trivial and follows directly from Lemma 3.3, using the formula \( s(p^k) = p^k - D(p^k) \).

**Lemma 3.5.** Suppose that \( n = p^k m^2 \) is an odd perfect number with special prime \( p \).

1. If \( \sigma(m^2) \equiv 1 \pmod{4} \), then \( D(m^2) \equiv 1 \pmod{4} \).
2. If \( \sigma(m^2) \equiv 3 \pmod{4} \), then \( D(m^2) \equiv 3 \pmod{4} \).

**Proof.** The proof is trivial and follows directly from the fact that \( m^2 \equiv 1 \pmod{4} \) (since \( m \) is odd), using the underlying assumptions and the formula \( D(m^2) = 2m^2 - \sigma(m^2) \).

**Lemma 3.6.** Suppose that \( n = p^k m^2 \) is an odd perfect number with special prime \( p \).

1. If \( \sigma(m^2) \equiv 1 \pmod{4} \), then \( s(m^2) \equiv 0 \pmod{4} \).
2. If \( \sigma(m^2) \equiv 3 \pmod{4} \), then \( s(m^2) \equiv 2 \pmod{4} \).

**Proof.** The proof is trivial and follows directly from Lemma 3.5, using the formula \( s(m^2) = m^2 - D(m^2) \).

We are now ready to prove our main result.

**Theorem 3.7.** Suppose that \( n = p^k m^2 \) is an odd perfect number with special prime \( p \).

1. If \( p \equiv k \equiv 1 \pmod{8} \), then \( \sigma(m^2) \equiv 3 \pmod{4} \) is impossible.
2. If \( p \equiv 1 \pmod{8} \) and \( k \equiv 5 \pmod{8} \), then \( \sigma(m^2) \equiv 1 \pmod{4} \) is impossible.
3. If \( p \equiv 5 \pmod{8} \) and \( k \equiv 1 \pmod{8} \), then \( \sigma(m^2) \equiv 1 \pmod{4} \) is impossible.
4. If \( p \equiv k \equiv 5 \pmod{8} \), then \( \sigma(m^2) \equiv 3 \pmod{4} \) is impossible.
Proof. Let \( n = p^k m^2 \) be an odd perfect number with special prime \( p \).

Notice that the right-hand side of Equation (*) is odd. (Furthermore, it is congruent to 1 modulo 8.)

First, suppose that \( p \equiv k \equiv 1 \pmod{8} \), and assume to the contrary that \( \sigma(m^2) \equiv 3 \pmod{4} \) holds. By Lemma 3.3, \( D(p^k) \equiv 0 \pmod{8} \). By Lemma 3.5, \( D(m^2) \equiv 3 \pmod{4} \). By Lemma 3.4, \( s(p^k) \equiv 1 \pmod{8} \). By Lemma 3.6, \( s(m^2) \equiv 2 \pmod{4} \). Thus, from Equation (*) we obtain (symbolically)

\[
2(4a_1 + 3)(4b_1 + 2) = (8x_1 + 1)(8c_1)(8d_1 + 1),
\]

which does not have any integer solutions.

Next, suppose that \( p \equiv 1 \pmod{8} \) and \( k \equiv 5 \pmod{8} \), and assume to the contrary that \( \sigma(m^2) \equiv 1 \pmod{4} \) holds. By Lemma 3.3, \( D(p^k) \equiv 4 \pmod{8} \). By Lemma 3.5, \( D(m^2) \equiv 1 \pmod{4} \). By Lemma 3.4, \( s(p^k) \equiv 1 \pmod{8} \). By Lemma 3.6, \( s(m^2) \equiv 0 \pmod{4} \). Thus, from Equation (*) we obtain (symbolically)

\[
2(4a_2 + 1)(4b_2) = (8x_2 + 1)(8c_2 + 4)(8d_2 + 5),
\]

which does not have any integer solutions.

Now, suppose that \( p \equiv 5 \pmod{8} \) and \( k \equiv 1 \pmod{8} \), and assume to the contrary that \( \sigma(m^2) \equiv 1 \pmod{4} \) holds. By Lemma 3.3, \( D(p^k) \equiv 4 \pmod{8} \). By Lemma 3.5, \( D(m^2) \equiv 1 \pmod{4} \). By Lemma 3.4, \( s(p^k) \equiv 1 \pmod{8} \). By Lemma 3.6, \( s(m^2) \equiv 0 \pmod{4} \). Thus, from Equation (*) we obtain (symbolically)

\[
2(4a_3 + 1)(4b_3) = (8x_3 + 1)(8c_3 + 4)(8d_3 + 1),
\]

which does not have any integer solutions.

Finally, suppose that \( p \equiv k \equiv 5 \pmod{8} \), and assume to the contrary that \( \sigma(m^2) \equiv 3 \pmod{4} \) holds. By Lemma 3.3, \( D(p^k) \equiv 0 \pmod{8} \). By Lemma 3.5, \( D(m^2) \equiv 3 \pmod{4} \). By Lemma 3.4, \( s(p^k) \equiv 5 \pmod{8} \). By Lemma 3.6, \( s(m^2) \equiv 2 \pmod{4} \). Thus, from Equation (*) we obtain (symbolically)

\[
2(4a_4 + 3)(4b_4 + 2) = (8x_4 + 1)(8c_4)(8d_4 + 5),
\]

which does not have any integer solutions.

This concludes the proof.

\[\square\]

Remark 3.8. To summarize, Theorem 3.7 just states that if \( n = p^k m^2 \) is an odd perfect number with a special prime \( p \), then \( \sigma(m^2) \equiv 1 \pmod{4} \) holds if and only if \( p \equiv k \pmod{8} \). Our argument provides an alternative proof for Theorem 3.3, equation 3.1 in [2] (as reproduced above in Theorem 1.1).

4 An application

Let \( n = p^k m^2 \) be an odd perfect number with special prime \( p \), and let \( \sigma(m^2)/p^k \) be a square. Since \( \sigma(m^2)/p^k \) is odd, it follows that \( \sigma(m^2)/p^k \equiv 1 \pmod{4} \). But it is known that \( p \equiv k \equiv 1 \)
In particular, we know that \( p^k \equiv 1 \pmod{4} \). This implies that \( \sigma(m^2) \equiv 1 \pmod{4} \), if \( \sigma(m^2)/p^k \) is a square. By Theorem 3.7, we know that \( p \equiv k \pmod{8} \).

Moreover, Broughan, Delbourgo, and Zhou proved in [1] (Lemma 8, page 7) that if \( \sigma(m^2)/p^k \) is a square, then \( k = 1 \) holds.

Thus, under the assumption that \( \sigma(m^2)/p^k \) is a square, we have

\[
p \equiv k = 1 \pmod{8}.
\]

This implies that the lowest possible value for the special prime \( p \) is 17.

We state this result as our next theorem.

**Theorem 4.1.** Suppose that \( n = p^k m^2 \) is an odd perfect number with special prime \( p \). If \( \sigma(m^2)/p^k \) is a square, then \( p \geq 17 \).

**Remark 4.2.** Let \( n = p^k m^2 \) be an odd perfect number with special prime \( p \).

Note that if

\[
\frac{\sigma(m^2)}{p^k} = \frac{m^2}{\sigma(p^k)/2}
\]

is a square, then \( k = 1 \) and \( \sigma(p^k)/2 = (p + 1)/2 \) is also a square.

The possible values for the special prime satisfying \( p < 100 \) and \( p \equiv 1 \pmod{8} \) are 17, 41, 73, 89, and 97.

For each of these values:

\[
\begin{align*}
p_1 + 1 &= \frac{17 + 1}{2} = 9 = 3^2, \\
p_2 + 1 &= \frac{41 + 1}{2} = 21, \text{ which is not a square.} \\
p_3 + 1 &= \frac{73 + 1}{2} = 37, \text{ which is not a square.} \\
p_4 + 1 &= \frac{89 + 1}{2} = 45, \text{ which is not a square.} \\
p_5 + 1 &= \frac{97 + 1}{2} = 49 = 7^2.
\end{align*}
\]

A quick way to rule out 41, 73 and 89, as remarked by Ochem [4] over at Mathematics StackExchange, is as follows: “If \( (p + 1)/2 \) is an odd square, then \( (p + 1)/2 \equiv 1 \pmod{8} \), so that \( p \equiv 1 \pmod{16} \). This rules out 41, 73, and 89.”

## 5 Conclusion

Additional tools are required if we are to push the analysis from \( \sigma(m^2) \) modulo 4 to consider \( \sigma(m^2) \) modulo 8. The authors have tried to check Equation (*) by considering \( m^2 \equiv 1 \pmod{8} \), and the various corresponding cases for \( \sigma(m^2) \) modulo 8 (which are determined by Theorem 3.7), but so far all their attempts have not resulted in any contradictions.
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