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Bi-unitary multiperfect numbers, II

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Dedicated to the memory of Prof. D. Suryanarayana

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Abstract: A divisor d of a positive integer n is called a unitary divisor if $\gcd(d, n/d) = 1$; and d is called a bi-unitary divisor of n if the greatest common unitary divisor of d and n/d is unity. The concept of a bi-unitary divisor is due to D. Surynarayana (1972). Let $\sigma^{**}(n)$ denote the sum of the bi-unitary divisors of n. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis (1987) proved that there are no odd bi-unitary multiperfect numbers. The present paper is Part II in a series of papers on even bi-unitary multiperfect numbers. In the first part we found all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 3$ and u is odd; the only one being $n = 120$. In this second part we find all bi-unitary triperfect numbers in the cases $a = 4$ and $a = 5$. For $a = 4$ the only one is $n = 2160$, and for $a = 5$ they are $n = 672$, $n = 10080$, $n = 528800$ and $n = 22932000$.
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1 Introduction

Throughout this paper, all lower case letters denote positive integers; p and q denote primes. The letters u, v and w are reserved for odd numbers.

A divisor d of n is called a unitary divisor (written $d\|n$) if $\gcd(d, n/d) = 1$. A divisor d of n is called a *bi-unitary* divisor if $(d, n/d)^{**} = 1$, where $(a, b)^{**}$ stands for the greatest common unitary divisor of a and b. The concept of a bi-unitary divisor is due to D. Suryanarayana (cf. [4]). Let $\sigma^{**}(n)$ denote the sum of bi-unitary divisors of n. The function $\sigma^{**}(n)$ is multiplicative, that is, $\sigma^{**}(1) = 1$ and $\sigma^{**}(mn) = \sigma^{**}(m)\sigma^{**}(n)$ whenever $(m, n) = 1$.

The concept of a bi-unitary perfect number was introduced by C. R. Wall [5]; a positive integer n is called a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. C. R. Wall [5] proved that there are only three bi-unitary perfect numbers; namely 6, 60 and 90.

A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis [1] proved that there are no odd bi-unitary multiperfect numbers. Our present paper is Part II in a series of papers on even bi-unitary multiperfect numbers. In Part I (see [2]), we found all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 3$. In fact, we proved that if $1 \leq a \leq 3$ and $n = 2^a u$ is a bi-unitary triperfect number, then $a = 3$ and $n = 120 = 2^3.3.5$.

In this Part II, we go through the cases $a = 4$ and $a = 5$. In Theorem 3.1 we prove that if $n = 2^4 u$ is a bi-unitary triperfect number, then $n = 2160 = 2^4.3^3.5$, and in Theorem 4.1 we prove that if $n = 2^5 u$ is a bi-unitary triperfect number, then $n = 672 = 2^5.3.7$, $n = 10080 = 2^5.3^2.5.7$, $n = 528800 = 2^5.3.5^2.13$ or $n = 22932000 = 2^5.3^2.5^3.7^2.13$. This shows that the case $a = 4$ yields one bi-unitary triperfect number, and the case $a = 5$ yields four bi-unitary triperfect numbers.

For a general account on various perfect-type numbers, we refer to [3].

2 Preliminaries

We assume that the reader has Part I available (see [2]). We, however, recall Lemmas 2.1 and 2.2 from Part I, because they are also important here.

Lemma 2.1. (I) *If α is odd, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} > \frac{\sigma^{**}(p^{\alpha+1})}{p^{\alpha+1}}$$

for any prime p.

(II) *For any $\alpha \geq 2\ell - 1$ and any prime p,*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} \geq \left(\frac{1}{p-1}\right)\left(p - \frac{1}{p^{2\ell}}\right) - \frac{1}{p^\ell} = \frac{1}{p^{2\ell}}\left(\frac{p^{2\ell+1}-1}{p-1} - p^\ell\right).$$

(III) *If p is any prime and α is a positive integer, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} < \frac{p}{p-1}.$$

Remark 2.1. (I) and (III) of Lemma 2.1 are mentioned in C. R. Wall [5]; (II) of Lemma 2.1 has been used by him [5] without explicitly stating it.

Lemma 2.2. *Let $a > 1$ be an integer not divisible by an odd prime p and let α be a positive integer. Let r denote the least positive integer such that $a^r \equiv 1 \pmod{p^\alpha}$; then r is usually denoted by $ord_{p^\alpha}\, a$. We have the following properties.*

(i) *If r is even then $s = r/2$ is the least positive integer such that $a^s \equiv -1 \pmod{p^\alpha}$. Also, $a^t \equiv -1 \pmod{p^\alpha}$ for a positive integer t if and only if $t = su$, where u is odd.*

(ii) *If r is odd then $p^\alpha \nmid a^t + 1$ for any positive integer t.*

Remark 2.2. Let a, p, r and $s = r/2$ be as in Lemma 2.2 ($\alpha = 1$). Then $p|a^t - 1$ if and only if $r|t$. If t is odd and r is even, then $r \nmid t$. Hence $p \nmid a^t - 1$. Also, $p|a^t + 1$ if and only if $t = su$, where u is odd. In particular if t is even and s is odd, then $p \nmid a^t + 1$. In order to check the divisibility of $a^t - 1$ (when t is odd) by an odd prime p, we can confine to those p for which $ord_p\, a$ is odd. Similarly, for examining the divisibility of $a^t + 1$ by p when t is even we need to consider primes p with $s = ord_p\, a/2$ even.

3 Bi-unitary triperfect numbers of the form $n = 2^4 u$

In this section we find all bi-unitary triperfect numbers n with $2^4 \| n$.

Theorem 3.1. *If n is a bi-unitary triperfect number with $2^4 \| n$, then $n = 2160 = 2^4.3^3.5$.*

Proof. Let $n = 2^4 u$ be a bi-unitary triperfect number so that

$$\sigma^{**}(n) = 3n.$$

Since $\sigma^{**}(2^4) = 27$, we obtain after simplification,

$$2^4.u = 9.\sigma^{**}(u), \tag{3.1}$$

and hence $3^2|u$. Let $u = 3^b.v$, where $b \geq 2$ and v is prime to 2.3. Hence

$$n = 2^4.3^b.v, \tag{3.1a}$$

and substituting $u = 3^b.v$ in (3.1), we get

$$2^4.3^{b-2}.v = \sigma^{**}(3^b).\sigma^{**}(v), \tag{3.1b}$$

where v has no more than three odd prime factors. (3.1c)

The rest of the proof depends on the following Lemmas:

Lemma 3.1. *Let $n = 2^4.3^b.v$, where $b \geq 2$ and $(v, 2.3) = 1$.*

(a) If $b = 2$, then n is not a bi-unitary triperfect number.

(b) If $b = 3$ and n is a bi-unitary perfect number, then $n = 2160 = 2^4.3^3.5$.

Proof. Proof of (a). Let $b = 2$. Suppose that n is a bi-unitary triperfect number so that (3.1a) and (3.1b) hold. From (3.1b) we get $2^4.v = 10.\sigma^{**}(v)$ and this implies $5|v$. Let $v = 5^c.w$. Hence

$$n = 2^4.3^2.5^c.w, \tag{3.2a}$$

and

$$2^3.5^{c-1}.w = \sigma^{**}(5^c).\sigma^{**}(w), \tag{3.2b}$$

where

$$w \text{ has no more than two odd prime factors;} \tag{3.2c}$$

also w is prime to $2.3.5$.

If $c = 1$, from (3.2b) we get, $2^3.w = 6.\sigma^{**}(w)$ so that $3|w$. But this false.

Let $c = 2$. From (3.2b), we have

$$2^2.5.w = 13.\sigma^{**}(w), \tag{3.3}$$

so that $13|w$.

Let $w = 13^d.w'$, where $(w', 2.3.5.13) = 1$. From (3.2a) and (3.2b), we obtain

$$n = 2^4.3^2.5^c.13^d.w', \tag{3.3a}$$

and

$$2^2.5.13^{d-1}.w' = \sigma^{**}(13^d).\sigma^{**}(w'), \tag{3.3b}$$

where

$$w' \text{ has at most one odd prime factor.} \tag{3.3c}$$

We can assume that $w' = p^e$, where $p \geq 7$. Hence from (3.3a), $n = 2^4.3^2.5^2.13^d.p^e$. We have, by Lemma 2.1,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{10}{9}.\frac{26}{25}.\frac{13}{12}.\frac{7}{6} = 2.464583333 < 3,$$

a contradiction.

Hence $c = 2$ is not possible. We may assume that $c \geq 3$.

We obtain a contradiction in the case $b = 2$ by examining the factors of $\sigma^{**}(5^c)$.

Let c be odd so that

$$\sigma^{**}(5^c) = \frac{5^{c+1}-1}{4} = \frac{(5^t-1)(5^t+1)}{4} \quad \left(t = \frac{c+1}{2} \geq 2\right).$$

If t is even, then $4|\sigma^{**}(5^c)$. From (3.2b), it follows that $w = p^d$, where $p \geq 7$. From (3.2a), $n = 2^4.3^2.5^c.p^d$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{10}{9}.\frac{5}{4}.\frac{7}{6} = 2.734375 < 3,$$

a contradiction.

Let t be odd so that $t \geq 3$. Following the same procedure adopted in Lemma 3.3 of [2], we can show that $\frac{5^t-1}{4}$ is divisible by a prime $p \geq 29$ and $p|w$. We obtain a contradiction as in (3.7) of [2].

The case when c is odd is complete.

Let c be even so that $c = 2k$. Hence

$$\sigma^{**}(5^c) = \left(\frac{5^k - 1}{4}\right).(5^{k+1} + 1).$$

If k is even then $4|\sigma^{**}(5^c)$. We proceed exactly as in the case when $t = \frac{c+1}{2}$ was even to obtain a contradiction. If k is odd we obtain a contradiction by imitating the case when $t = \frac{c+1}{2}$ was odd.

This finishes the case that c is even and also the case $b = 2$.

Thus $b = 2$ is not possible. That is, when $b = 2$, n cannot be a bi-unitary triperfect number.

This completes the proof of (a) of Lemma 3.1.

Proof of (b). Let n be a bi-unitary perfect number so that (3.1a) and (3.1b) hold. Let $b = 3$. Since $\sigma^{**}(3^3) = 40 = 2^3.5$, taking $b = 3$ in (3.1b), we get

$$2.3.v = 5.\sigma^{**}(v), \tag{3.4}$$

so that $5|v$. Also, from (3.4), v must be a prime power. Hence $v = 5^c$ and so from (3.1a) ($b = 3$) and (3.4),

$$n = 2^4.3^3.5^c, \tag{3.4a}$$

and

$$2.3.5^{c-1} = \sigma^{**}(5^c). \tag{3.4b}$$

If $c \geq 2$, then from (3.4b), $5|\sigma^{**}(5^c)$, which is false. Hence $c = 1$ and (3.4b) is satisfied. Thus $n = 2^4.3^3.5 = 2160$ is a bi-unitary triperfect number.

This completes the proof of (b) of Lemma 3.1.

Proof of Lemma 3.1 is complete. □

Lemma 3.2. *Let $n = 2^4.3^b.v$, where $b \geq 4$ and $(v, 2.3) = 1$.*

(a) If b is odd or $4|b$, then n cannot be a bi-unitary triperfect number.

(b) Let $b = 2k$ and k be odd. If n is a bi-unitary triperfect number then $5 \nmid n$.

Proof. We return to the equations (3.1a) and (3.1b), in which $b \geq 4$. We obtain a contradiction by considering $\sigma^{**}(3^b)$.

Proof of (a). Let b be odd so that

$$\sigma^{**}(3^b) = \frac{3^{b+1} - 1}{2} = \frac{(3^t - 1)(3^t + 1)}{2} \quad \left(t = \frac{b+1}{2}\right).$$

Let t be even. Since $t = \frac{b+1}{2}$ is even $4|b + 1$. Hence $80 = 3^4 - 1|3^{b+1} - 1$. It follows that $\sigma^{**}(3^b)$ is divisible by 5 and 8. From (3.1b), $8|\sigma^{**}(3^b)$ implies that v cannot have more than one odd prime factor and $5|\sigma^{**}(3^b)$ implies that $v = 5^c$. Hence from (3.1a) and (3.1b), we have

$$n = 2^4.3^b.5^c, \quad (b \geq 4) \tag{3.5a}$$

and

$$2^4.3^{b-2}.5^c = \sigma^{**}(3^b).\sigma^{**}(5^c). \tag{3.5b}$$

From (3.5b), $5|\sigma^{**}(3^b)$. This implies either $5|3^t-1$ or $5|3^t+1$ but not both.

Assume that $5|3^t-1$. Then $5 \nmid 3^t+1$. Thus $\frac{3^t+1}{2} > 1$, odd and not divisible by 3 or 5. This cannot happen from (3.5b) since $\frac{3^t+1}{2}|\sigma^{**}(3^b)$.

Let $5|3^t+1$. Hence $5 \nmid 3^t-1$. Also, from (3.5b), $16 \nmid 3^t-1$. Since t is even, we have $8|3^t-1$; hence $8\|3^t-1$. Hence $\frac{3^t-1}{8}$ is odd, > 1 and not divisible by 3 or 5; since $\frac{3^t-1}{8}|\sigma^{**}(3^b)$, this cannot happen in view of (3.5b).

Thus the case t even cannot occur.

Let t be odd. In this case $4\|3^t+1$ and $2\|3^t-1$ so that $4\|\sigma^{**}(3^b)$. It follows from (3.1b) that

$$v \text{ cannot have more than two odd prime factors.} \tag{3.5c}$$

Note that $5|3^t+1$ if and only if $t = 2u$, u being odd. In particular t must be even. Since t is odd, $5 \nmid 3^t+1$; also, $11 \nmid 3^t+1$ for any positive integer t.

Thus $\frac{3^t+1}{4}$ is odd, > 1 and not divisible by $3, 5$, and 11. Suppose $7 \nmid 3^t+1$. Then $\frac{3^t+1}{4}$ should be divisible by an odd prime $q \notin \{3, 5, 7, 11\}$. Since $q|\frac{3^t+1}{4}|\sigma^{**}(3^b)$, from (3.1b), it follows that $q|v$ and $q \geq 13$.

Suppose that $7|3^t+1$. We prove that $\frac{3^t+1}{4}$ cannot be divisible by 7 alone. On the contrary let us assume that $\frac{3^t+1}{4} = 7^\alpha$, where α is a positive integer. If $\alpha \geq 2$, then $7^2|3^t+1$. But this is if and only if $t = 21u$. Thus $7^2|3^t+1$ implies $3^{21}+1|3^t+1$. We have $3^{21}+1 = 2^2.7^2.43.547.2269$, so that $43|\frac{3^{21}+1}{4}|\frac{3^t+1}{4} = 7^\alpha$, which is not possible. Thus $\alpha = 1$ and hence $\frac{3^t+1}{4} = 7$ or $t = 3$. Hence $b = 5$.

We now show that $b = 5$ is not admissible.

We have $\sigma^{**}(3^5) = \frac{3^6-1}{2} = 2^2.7.13$. Taking $b = 5$ in (3.1b), we get

$$2^2.3^3.v = 7.13.\sigma^{**}(v). \tag{3.5d}$$

From (3.5d), 7 and 13 divide v. Let $v = 7^c.13^d$. Now from (3.5d), we get after simplification

$$2^2.3^3.7^{c-1}.13^{d-1} = \sigma^{**}(7^c).\sigma^{**}(13^d). \tag{3.6}$$

If c is odd or $4|c$ then $8|\sigma^{**}(7^c)$. This is not possible from (3.6).

Let $c = 2k$, where k is odd. We have

$$\sigma^{**}(7^c) = \left(\frac{7^k-1}{6}\right).(7^{k+1}+1).$$

Consider the factor $7^{k+1}+1$. Since $2\|7^{k+1}+1$, $\frac{7^{k+1}+1}{2}$ is odd and trivially > 1. It is not divisible by 3 and not divisible by 7 trivially; $13|7^{k+1}+1$ if and only if $k+1 = 6u$ (u odd), and $7^6+1 = 2.5^2.13.181$. Hence $13|7^{k+1}+1$ implies that $5|7^6+1|7^{k+1}+1|\sigma^{**}(7^c)$. This is not possible from (3.6). So $13 \nmid 7^{k+1}+1$. Thus $\frac{7^{k+1}+1}{2}|\sigma^{**}(7^c)$ is not divisible by 2 or 3 or 7 or 13. This cannot happen from (3.6). This contradiction shows that $b = 5$ is not possible.

This proves that $\frac{3^t+1}{4}$ is divisible by an odd prime $q \neq 7$. Clearly $q \geq 13$ and $q|v$.

Thus we have proved that we can always find an odd prime $q|\frac{3^t+1}{4}$ and $q|v$ with $q \geq 13$.

We shall now turn our attention to the factor $3^t - 1$, where t is odd. First of all $2\|3^t-1$. Also, $5|3^t-1 \iff 4|t$ and $7|3^t-1 \iff 6|t$. In particular t should be even. Since t is odd, 3^t-1 is not divisible by 5 or 7.

Now, $\frac{3^t-1}{2}$ is odd, > 1 and not divisible by $3, 5, 7$ and 11 if we assume that $11 \nmid 3^t - 1$. Hence $\frac{3^t-1}{2}$ should be divisible a prime $p \geq 13$ and $p|v$ by (3.1b).

We may assume that $11|3^t-1$. This is if and only if $5|t$. Hence $3^5-1|3^t-1$. Since $3^5-1 = 2.11^2$, we have $11^2|3^t-1$.

We now show that $\frac{3^t-1}{2}$ is not divisible by 11 alone. On the contrary, let $\frac{3^t-1}{2} = 11^\alpha$, where $\alpha \geq 2$. If $\alpha \geq 3$, then $11^3|3^t-1$; this is equivalent to $55|t$. In particular, $11|t$ and so $3^{11}-1|3^t-1$. But $3^{11}-1 = 2.23.3851$. Hence $23|\frac{3^t-1}{2} = 11^\alpha$. This is impossible. Hence $\alpha = 2$, so that $\frac{3^t-1}{2} = 11^2$ or $t = 5$, so that $b = 9$.

We now prove that $b = 9$ is not admissible. We have $\sigma^{**}(3^9) = \frac{3^{10}-1}{2} = 2^2.11^2.61$. Taking $b = 9$ in (3.1b), we get after simplification, $2^2.3^7.v = 11^2.61.\sigma^{**}(v)$; it follows that 11 and 61 divide v. By (3.5c), $v = 11^c.61^d$. We already proved that $q|v$, where $q|\frac{3^t+1}{4}$. Since $\frac{3^t-1}{2}$ and $\frac{3^t+1}{4}$ are relatively prime, $q \notin \{11, 61\}$. This is a contradiction to $q|v = 11^c.61^d$. Thus $b = 9$ is not admissible.

Hence $\frac{3^t-1}{2}$ must be divisible by an odd prime say $p \neq 11$. It follows that $p \notin \{3, 5, 7, 11\}$ and so $p \geq 13$. From (3.1b), clearly $p|v$. As p and q are factors of two relatively prime numbers, $p \neq q$. We can assume that $p \geq 13$ and $q \geq 17$. By (3.5c), $v = p^c.q^d$. Hence from (3.1a),$n = 2^4.3^b.p^c.q^d$. We have by Lemma 2.1,

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{13}{12}.\frac{17}{16} = 2.913574219 < 3,$$

a contradiction.

The case $t = \frac{b+1}{2}$ is odd is complete.

Let b be even so that $b = 2k$. Then

$$\sigma^{**}(3^b) = \left(\frac{3^k-1}{2}\right).(3^{k+1}+1).$$

Let k be even. This is same as $4|b$. Then $8|3^k-1$ and $4|3^{k+1}+1$. Hence $16|\sigma^{**}(3^b)$. From (3.1b), it follows that $v = 1$ and hence from the same equation we obtain $2^4.3^{b-2} = \sigma^{**}(3^b)$, which is not possible since $b \geq 4$ implies $3|\sigma^{**}(3^b)$ and this is false.

In all the cases we ended up with a contradiction. Hence n cannot be a bi-unitary perfect number.

The proof of (a) of Lemma 3.2 is complete.

Proof of (b). Let k be odd. We prove that n in (3.1a) and (3.1b) is not divisible by 5.

Let n be as in (3.1a) and assume that $5|n$. Hence $v = 5^c.w$, where $(w, 2.3.5) = 1$; substituting this into (3.1a) and (3.1b) we get

$$n = 2^4.3^b.5^c.w, \quad (b \geq 4) \tag{3.6a}$$

and

$$2^4.3^{b-2}.5^c.w = \sigma^{**}(3^b).\sigma^{**}(5^c).\sigma^{**}(w), \tag{3.6b}$$

where

$$w \text{ cannot have more than two odd prime factors.} \tag{3.6c}$$

The case $b = 4$ falls under $b = 2k$, where k is even. We already obtained a contradiction in this case. Hence we may assume that $b \geq 6$. By Lemma 2.1, we have $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{1066}{729}$ $(b \geq 5)$ and $\frac{\sigma^{**}(5^c)}{5^c} \geq \frac{19406}{15625}$, $(c \geq 5)$. Hence for $c \geq 5$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{27}{16}.\frac{1066}{729}.\frac{19406}{15625} = 3.064710519 > 3,$$

a contradiction.

So $c \geq 5$ does not hold and hence $1 \leq c \leq 4$.

Let $c = 1$. Then (3.6a) and (3.6b) reduce to

$$n = 2^4.3^b.5.w, \quad (b \geq 6) \tag{3.7a}$$

and

$$2^3.3^{b-3}.5.w = \sigma^{**}(3^b).\sigma^{**}(w), \tag{3.7b}$$

where w cannot have more than two odd prime factors.

From Lemma 2.1, for $b \geq 7$, $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{9760}{6561}$. Hence for $b \geq 7$, from (3.7a),

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{27}{16}.\frac{9760}{6561}.\frac{6}{5} = 3.012345679 > 3,$$

a contradiction.

Hence $b \leq 6$. Since already $b \geq 6$, we have $b = 6$. We now show that $b = 6$ is not admissible when $c = 1$. The relevant equations are (3.7a) and (3.7b).

We have $\sigma^{**}(3^6) = 1066 = 2.13.41$. Taking $b = 6$ in (3.7b), we get

$$2^2.3^3.5.w = 13.41.\sigma^{**}(w). \tag{3.7c}$$

From (3.7c) we see that w is divisible by 13 and 41. Hence $w = 13^d.41^e$. From (3.7a), we have

$$n = 2^4.3^6.5.13^d.41^e, \tag{3.8a}$$

and

$$2^3.3^3.5.13^{d-1}.41^{e-1} = \sigma^{**}(13^d).\sigma^{**}(41^e). \tag{3.8b}$$

Also, by Lemma 2.1, for $d \geq 3$, $\frac{\sigma^{**}(13^d)}{13^d} \geq \frac{30772}{28561}$. Hence for $d \geq 3$, from (3.8a), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{27}{16}.\frac{1066}{729}.\frac{6}{5}.\frac{30772}{28561} = 3.190340363 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

Taking $d = 1$ in (3.8b), we see that 7 divides its left-hand side which is not true. Taking $d = 2$ in (3.8b), since $\sigma^{**}(13^2) = 170$, it follows that 17 divides the left-hand side of (3.8b). This is false. Therefore, $b = 6$ is not admissible.

This completes the case $c = 1$. So $c = 1$ is not possible.

Let $c = 2$. Since $\sigma^{**}(5^2) = 26 = 2.13$, taking $c = 2$ in (3.6b), we infer that $13|w$. Writing $w = 13^d.w'$, from (3.6a) and (3.6b), we obtain

$$n = 2^4.3^b.5^2.13^d.w', \tag{3.9a}$$

and

$$2^3.3^{b-2}.5^2.13^{d-1}.w' = \sigma^{**}(3^b).\sigma^{**}(13^d).\sigma^{**}(w'), \tag{3.9b}$$

where w' cannot have more than one odd prime factor.

We recall that we are dealing with the case $b = 2k$, where k is odd and $k \geq 3$.

Consider the factor $3^{k+1} + 1$ of $\sigma^{**}(3^b)$. Since $k + 1$ is even, $2||3^{k+1} + 1$ and $3^{k+1} + 1$ is not divisible by 7 and 19.

For any positive integer t, 3^t+1 is not divisible by $11, 13$ and 23. This is applicable to $3^{k+1}+1$ also.

Suppose $17|3^{k+1} + 1$. This is if and only if $k + 1 = 8u$. Hence $3^8 + 1|3^{k+1} + 1$. Also, $3^8 + 1 = 2.7.193$. It follows that $3^{k+1} + 1$ a factor of $\sigma^{**}(3^b)$ is divisible by 17 and 193. From (3.9b) it follows that w' is divisible by 17 and 193. However, w' cannot have more than one odd prime factor. Thus $17 \nmid 3^{k+1} + 1$.

It follows from the above discussion that $\frac{3^{k+1}+1}{2}$ is odd, > 1 and not divisible by any prime in $[3, 23]$ if $5 \nmid 3^{k+1} + 1$. If $q|\frac{3^{k+1}+1}{2}$, then $q \geq 29$. From (3.9b), $q|w'$ and so $w' = q^e$; we now prove that this holds good when $5|3^{k+1} + 1$ also.

Suppose $5|3^{k+1} + 1$. We prove that $\frac{3^{k+1}+1}{2}$ is not divisible by 5 alone. If this is not so, then we must have $\frac{3^{k+1}+1}{2} = 5^\alpha$. If $\alpha \geq 2$, then $5^2|3^{k+1} + 1$; this is if and only if $k + 1 = 10u$. Hence $3^{10} + 1|3^{k+1} + 1$. Also, $3^{10} + 1 = 2.5^2.1181$. Thus $1181|\frac{3^{k+1}+1}{2} = 5^\alpha$. This is impossible. Hence $\alpha = 1$ and so $k = 1$. But $k \geq 3$. Hence $\frac{3^{k+1}+1}{2}$ must be divisible by an odd prime $q \neq 5$ so that $q \geq 29$ as before. Also, $q|w'$ and $w' = q^e$.

From (3.9a), $n = 2^4.3^b.5^2.13^d.q^e$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{26}{25}.\frac{13}{12}.\frac{29}{28} = 2.953727679 < 3,$$

a contradiction.

Hence $c = 2$ is not admissible.

Let $c = 3$. We have $\sigma^{**}(5^3) = 156 = 2^2.3.13$. Taking $c = 3$ in (3.6b), we get

$$2^2.3^{b-3}.5^3.w = 13.\sigma^{**}(3^b).\sigma^{**}(w), \tag{3.9c}$$

and w cannot have more than one odd prime factor. From the above equation (3.9c), $13|w$ and hence $w = 13^d$. From (3.6a), we have $n = 2^4.3^b.5^3.13^d$ and so

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{27}{16}.\frac{1066}{729}.\frac{156}{125} = 3.079555556 > 3,$$

a contradiction. In the above we used that for $b \geq 5$, $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{1066}{729}$. Hence $c = 3$ is not possible.

Let $c = 4$. We have $\sigma^{**}(5^4) = 756 = 2^2.3^3.7$. Taking $c = 4$ in (3.6b), we obtain

$$2^2.3^{b-5}.5^4.w = 7.\sigma^{**}(3^b).\sigma^{**}(w). \tag{3.9d}$$

It follows from (3.9d) that $7|w$ and $w = 7^d$. Hence from (3.6a) and (3.9d), we get

$$n = 2^4.3^b.5^4.7^d, \quad (b \geq 6) \tag{3.10a}$$

and

$$2^2.3^{b-5}.5^4.7^{d-1} = \sigma^{**}(3^b).\sigma^{**}(7^d). \tag{3.10b}$$

By Lemma 2.1, for $d \geq 3$, $\frac{\sigma^{**}(7^d)}{7^d} \geq \frac{2752}{2401}$. We can use $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{1066}{729}$, since $b \geq 5$. Hence for $d \geq 3$, from (3.10a), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{27}{16}.\frac{1066}{729}.\frac{756}{625}.\frac{2752}{2401} = 3.42114519 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

Let $d = 1$. Since $\sigma^{**}(7) = 8$, taking $d = 1$ in (3.10b), we find that 2^4 divides the right-hand side of (3.10b) while its left-hand side is divisible unitarily by 2^2.

Let $d = 2$. We have $\sigma^{**}(7^2) = 50 = 2.5^2$. Taking $d = 2$ in (3.10b), after simplification, $2.3^{b-5}.5^2.7 = \sigma^{**}(3^b)$ and from this it follows that $3|\sigma^{**}(3^b)$ (since $b \geq 6$) which is false.

Hence $5 \nmid n$. The proof of (b) of Lemma 3.2 is complete.

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let* $n = 2^4.3^b.v$ *be given as in* (3.1a) *with* $b = 2k$*, where* k *is odd and* $k \geq 3$.
(I) Suppose that $7|n$ *so that* $n = 2^4.3^b.7^c.w$, $(b \geq 6)$ *and* $(w, 2.3.7) = 1$. *Then we have the following:*

(a) If c *is odd or* $4|c$*, then* n *is not a bi-unitary triperfect number.*

(b) If $c = 2\ell$*, where* ℓ *is odd and* n *is a bi-unitary triperfect number, then* n *is divisible by two distinct primes* p' *and* q' *: (i)* $p'|\frac{7^\ell-1}{6}, p' > 131$ *and (ii)* $q'|\frac{7^{\ell+1}+1}{2}, q' > 131$.
(II) If n *is a bi-unitary triperfect number then* $7 \nmid n$.

Proof. Proof of (I). Let n be as given in (3.1a) and assume that n is a bi-unitary triperfect number. Since $5 \nmid n$ by Lemma 3.2 and $7|n$, $v = 7^c.w$, where $(w, 2.3.5.7) = 1$ Hence from (3.1a) and (3.1b), we get

$$n = 2^4.3^b.7^c.w \quad (b \geq 6) \tag{3.11a}$$

and

$$2^4.3^{b-2}.7^c.w = \sigma^{**}(3^b).\sigma^{**}(7^c)\sigma^{**}(w), \tag{3.11b}$$

where

$$w \text{ cannot have more than two odd prime factors.} \tag{3.11c}$$

We consider $\sigma^{**}(7^c)$ and obtain a contradiction.
Proof of (a). If c is odd or $4|c$, then $8|\sigma^{**}(7^c)$. It follows from (3.1b) that its both sides should be unitarily divisible by 2^4. Hence $w = 1$ and so from (3.11a), $n = 2^4.3^b.7^c$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{7}{6} = 2.953125 < 3,$$

a contradiction. Hence n cannot be a bi-unitary triperfect number.

Proof of (b). Let $c = 2\ell$, where ℓ is odd. We have

$$\sigma^{**}(7^c) = \left(\frac{7^\ell - 1}{6}\right).(7^{\ell+1} + 1).$$

If $\ell = 1$, then $c = 2$. Since $\sigma^{**}(7^2) = 50$, taking $c = 2$ in (3.11b), we find that $5|w$. But w is prime to 5. Hence we may assume that $\ell \geq 3$.

(i) We now consider $7^\ell - 1$, given that ℓ is odd and ≥ 3.

(A) First of all, $2\|7^\ell - 1$ since ℓ is odd; also, $3|7^\ell - 1$. We may note that $27|7^\ell - 1$ if and only if $9|\ell$. Assume that $27||7^\ell - 1$. Hence $7^9 - 1|7^\ell - 1$. Also, $7^9 - 1 = 2.3^3.19.37.1063$. Hence $\frac{7^\ell-1}{6}$ is divisible by $19, 37$ and 1063. Thus $\sigma^{**}(7^c)$ is divisible by these three primes which divide w. This contradicts (3.11c). Thus $27 \nmid 7^\ell - 1$. We shall examine the divisibility by 9 later.

If the interval $[3, 2520]$ is replaced by $[3, 131]$ in Lemma 2.4 (a) of Part I (see [2]), it reduces to the following:

(B) If $p \in [3, 131] - \{3, 19, 37\}$, $ord_p 7$ is odd and $p|7^\ell - 1$, then we can find an odd prime $p'|\frac{7^\ell-1}{6}$ and $p' > 131$.

If $37|7^\ell - 1$, then $9|\ell$. Hence $7^9 - 1|7^\ell - 1$. Also, $7^9 - 1 = 2.3^3.19.37.1063$. If $p' = 1063$, then $p'|\frac{7^\ell-1}{6}$ and $p' > 131$. Hence the statement in (B) can be reduced to the following:

(C) If $p \in [3, 131] - \{3, 19\}$, $ord_p 7$ is odd and $p|7^\ell - 1$, then we can find an odd prime $p'|\frac{7^\ell-1}{6}$ and $p' > 131$.

Let

$$S_7' = \{p|7^\ell - 1 \ : ord_p 7 \text{ is odd and } \ p \in [3, 131] - \{3, 19\}\}.$$

If S_7' is non-empty, then (i) of Lemma 3.3 (a) is true. We may assume that S_7' is an empty set. This means that $p \nmid 7^\ell - 1$ whenever $p \in [3, 131] - \{3, 19\}$ and $ord_p 7$ is odd; trivially $7^\ell - 1$ is not divisible by 7. Thus:

(D) $\frac{7^\ell-1}{6}$ is not divisible by any prime in $[3, 131]$ except possibly $p = 3$ or $p = 19$; (we may recall that $p \nmid \frac{7^\ell-1}{6}$ if $ord_p 7$ is even).

We note that $19|7^\ell - 1 \Longleftrightarrow 3|\ell \Longleftrightarrow 9|7^\ell - 1$.

Suppose that $19 \nmid 7^\ell - 1$. Then $9 \nmid 7^\ell - 1$. Hence from the discussion in (A), $3\|7^\ell - 1$. Thus $\frac{7^\ell-1}{6}$ is odd, > 1 and not divisible by any prime in $[3, 131]$. Hence every prime factor of $\frac{7^\ell-1}{6}$ is > 131 and divides w. In particular we can find a prime $p'|\frac{7^\ell-1}{6}$, $p'|w$ and $p' > 131$.

Suppose that $19|7^\ell - 1$. Hence $9|7^\ell - 1$ and since $27 \nmid 7^\ell - 1$, we have $9\|7^\ell - 1$. Hence $\frac{7^\ell-1}{18} > 1$, odd and not divisible by 3. We now show that it is possible to find a prime $p'|\frac{7^\ell-1}{18}$ and $p' \neq 19$. Suppose that $\frac{7^\ell-1}{18} = 19^\alpha$, $\alpha \geq 1$. If $\alpha \geq 2$, then $19^2|7^\ell - 1$. But this is if and only if $57|\ell$; hence $7^{57} - 1|7^\ell - 1$. From the factors of $7^{57} - 1$ given in Appendix F of Part I (see [2]), $419|7^{57} - 1$ and so $419|\frac{7^\ell-1}{18} = 19^\alpha$, which is impossible. Hence $\alpha = 1$ so that $\frac{7^\ell-1}{18} = 19$ or $\ell = 3$. We show that this is not possible.

Let $\ell = 3$ and so $c = 6$. We have $\sigma^{**}(7^6) = \left(\frac{7^3-1}{6}\right).(7^4 + 1) = 2.3.19.1201$.

Taking $c = 6$ in (3.11b), after simplification we get

$$2^3.3^{b-3}.7^6.w = 19.1201.\sigma^{**}(3^b).\sigma^{**}(w). \tag{3.12}$$

From (3.12), it follows that w is divisible by 19 and 1201 and so $w = 19^d.(1201)^e$. Substituting this into (3.11a) and (3.12), we get

$$n = 2^4.3^b.7^6.19^d.(1201)^e, \tag{3.12a}$$

and

$$2^3.3^{b-3}.7^6.19^{d-1}.(1201)^{e-1} = \sigma^{**}(3^b).\sigma^{**}(19^d).\sigma^{**}((1201)^e). \tag{3.12b}$$

Since $b \geq 5$, $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{1066}{729}$. Also, $\frac{\sigma^{**}(7^6)}{7^6} = \frac{136914}{117649}$ and for $d \geq 3$, $\frac{\sigma^{**}(19^d)}{19^d} \geq \frac{137200}{130321}$. Therefore, for $d \geq 3$, from (3.12a), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{27}{16}.\frac{1066}{729}.\frac{136914}{117649}.\frac{137200}{130321} = 3.023241107 > 3,$$

a contradiction.

Hence $d = 1$ or $d = 2$.

Taking $d = 1$ in (3.12b), since $\sigma^{**}(19) = 20$, we find that 5 divides its right-hand side while it not so with respect to its left-hand side.

We have $\sigma^{**}(19^2) = 362 = 2.181$. Taking $d = 2$ in (3.12b), we see that 181 divides its left-hand side which is false.

Thus $c = 6$ (or $\ell = 3$) is not admissible. It now follows that $\frac{7^\ell-1}{18}$ is not divisible by 19 alone. Hence we can find a prime $p'|\frac{7^\ell-1}{18}$ and $p' \neq 19$. Thus $\frac{7^\ell-1}{18}$ is divisible by a prime $p' \notin [3, 131]$. Hence $p' > 131$. Since $p'|\frac{7^\ell-1}{18}|\frac{7^\ell-1}{6}|\sigma^{**}(7^c)$, it follows from (3.11b) that $p'|w$ and $p' > 131$.

This proves (i) in part (b) of Lemma 3.3.

(ii) We now prove that $\frac{7^{\ell+1}+1}{2}$ is divisible by an odd prime $q'|w$ with $q' > 131$, when ℓ is odd and ≥ 3.

Replacing the interval $[3, 2520]$ by $[3, 131]$ in Lemma 2.4 (b) in Part I (see [2]), it reduces to the following:

(E) If $q \in [3, 131] - \{5, 13\}$, $s = \frac{1}{2}ord_q 7$ is even and $q|7^{\ell+1} + 1$, then we can find a prime q' such that $q'|\frac{7^{\ell+1}+1}{2}$.

Let

$$T_7' = \{q|7^{\ell+1} + 1 \ : \ q \in [3, 131] - \{5, 13\}, \ s = \frac{1}{2}ord_q 7 \text{ is even}\}.$$

If T_7' is non-empty, then (ii) of Lemma 3.3 holds good. We may assume that T_7' is empty. Since s is not even implies that $q \nmid 7^{\ell+1} + 1$, it follows that (taking in to consideration that $7 \nmid 7^{\ell+1} + 1$ trivially):

(F) $7^{\ell+1} + 1$ is not divisible by any prime q in $[3, 131]$ except possibly $q = 5$ or $q = 13$.

It only remains to discuss divisibility of $7^{\ell+1} + 1$ by 5 and 13.

We may note that $13|7^{\ell+1} + 1 \Longleftrightarrow \ell + 1 = 6u \Longleftrightarrow 181|7^{\ell+1} + 1$. Hence $13|7^{\ell+1} + 1$ implies that 181 also divides $7^{\ell+1} + 1$. Part (b) of Lemma 3.3 which is proved already says that $\frac{7^\ell-1}{6}$ is divisible by an odd prime $p' > 131$ which divides w; since 13 and 181 divide w and so totally three primes divide w; this violates (3.11c). Hence $13 \nmid 7^{\ell+1} + 1$.

If $5 \nmid 7^{\ell+1} + 1$, from (F), every prime factor of $\frac{7^{\ell+1}+1}{2}$ exceeds 131 and is a divisor of w.

Suppose that $5|7^{\ell+1} + 1$. Hence $\ell + 1 = 2u$ so that $7^2 + 1 = 2.5^2|7^{\ell+1} + 1$. Thus $5|7^{\ell+1} + 1 \implies 5^2|7^{\ell+1} + 1$. We prove that $\frac{7^{\ell+1}+1}{2}$ must be divisible by an odd prime $q \neq 5$.

On the other hand, let $\frac{7^{\ell+1}+1}{2} = 5^\alpha$, where $\alpha \geq 2$. If $\alpha \geq 3$, then $5^3 | 7^{\ell+1}+1$; this is if and only if $\ell+1 = 10u$. Hence $7^{10}+1 | 7^{\ell+1}+1$. Also, $7^{10}+1 = 2.5^3.281.4021$. In particular, $281 | \frac{7^{\ell+1}+1}{2} = 5^\alpha$ and this is impossible. Hence $\alpha = 2$ so that $\frac{7^{\ell+1}+1}{2} = 5^2$ or $\ell = 1$. But $\ell \geq 3$. This contradiction shows that we can find an odd prime $q' | \frac{7^{\ell+1}+1}{2}$ and $q' \neq 5$. It follows that $q' \notin [3, 131]$ and hence $q' > 131$. Also, from (3.11b), $q' | w$. Thus (ii) of Lemma 3.3 follows.

This proves (b) of Lemma 3.3 completely.

Proof of (II). Suppose that $7|n$ and n is a bi-unitary triperfect number. Then by I(b) of Lemma 3.3, w is divisible by two primes p and q, where $p \geq 137$ and $q \geq 139$. This implies that $n = 2^4.3^b.7^c.p^d.q^e$, and hence we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{7}{6}.\frac{137}{136}.\frac{139}{138} = 2.99639596 < 3,$$

a contradiction. Hence $7 \nmid n$.

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let* $n = 2^4.3^b.v$, *where* $(v, 2.3) = 1$. *If* $b = 6$ *and* n *is a bi-unitary triperfect number, then* $11 \nmid n$.

Proof. Let $b = 6$ and n be a bi-unitary triperfect number. Hence the equation (3.1b) holds good. We have $\sigma^{**}(3^6) = 13.82 = 2.13.41$. When $b = 6$, it follows from (3.1b) that 13 and 41 divide v.

Suppose that $11|n$ and so $11|v$. Hence v is divisible by $11, 13$ and 41. We can assume that $v = 11^c.13^d.41^e$. Substituting this into (3.1a) and (3.1b), we get

$$n = 2^4.3^6.11^c.13^d.41^e, \qquad (3.13a)$$

and

$$2^3.3^4.11^c.13^{d-1}.41^{e-1} = \sigma^{**}(11^c).\sigma^{**}(13^d).\sigma^{**}(41^e). \qquad (3.13b)$$

From Lemma 2.1,

$$\frac{\sigma^{**}(11^c)}{11^c} \geq \frac{1}{11^8}\left(\frac{11^9-1}{10} - 11^4\right) = \frac{235780128}{214358881}, \quad (c \geq 7),$$
$$\frac{\sigma^{**}(13^d)}{13^d} \geq \frac{1}{13^6}\left(\frac{13^7-1}{12} - 13^3\right) = \frac{5226846}{4826809}, \quad (d \geq 5),$$
$$\frac{\sigma^{**}(41^e)}{41^e} \geq \frac{1}{41^4}\left(\frac{41^5-1}{40} - 41^2\right) = \frac{2894724}{2825761}, \quad (e \geq 3).$$

Hence from (3.13a),

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{27}{16}.\frac{1066}{729}.\frac{235780128}{214358881}.\frac{5226846}{4826809}.\frac{2894724}{2825761} = 3.01085858 > 3,$$

a contradiction.

Hence $c \geq 7,\ d \geq 5$ and $e \geq 3$ cannot hold simultaneously.

We have

$$\sigma^{**}(11) = 12 = 2^2.3;\ \sigma^{**}(11^2) = 2.61;\ \sigma^{**}(11^3) = 2^3.3.61;$$

and

$$\sigma^{**}(11^4) = 2^4.3^3.37; \sigma^{**}(11^5) = 2^2.3^2.7.19.37; \sigma^{**}(11^6) = 2.7.19.7321.$$

Hence when $c = 1, 3, 4$, and 5, $2^2|\sigma^{**}(11^c)$. Taking $c = 1, 3, 4, 5$ successively in (3.13b), we see that 2^4 divides its right-hand side while 2^3 divides its left-hand side unitarily.

When $c = 2,\ 61|\sigma^{**}(11^c)$. Hence from (3.13b) ($c = 2$), 61 divides right-hand side but it does not divide its left-hand side.

When $c = 6,\ 7|\sigma^{**}(11^c)$. Again from (3.13b) ($c = 2$), it follows that 7 is a factor of its right-hand side while it is not so with respect its left-hand side.

Hence the values of $c = 1, 2, 3, 4, 5, 6$ are not admissible.

We have

$$\sigma^{**}(13) = 14 = 2.7;\ \sigma^{**}(13^2) = 170 = 2.5.17; \sigma^{**}(13^3) = 2^2.5.7.17;\ \sigma^{**}(13^4) = 2^2.7^2.157.$$

From (3.13b), it is clear that its left-hand side is neither divisible by 7 or 17. However, $7|\sigma^{**}(13^d)$ when $d = 1, 3, 4$ and $17|\sigma^{**}(13^d)$ when $d = 2$. Hence the values of $d = 1, 2, 3, 4$ are not admissible.

Since $7|\sigma^{**}(41) = 42$ and $29|\sigma^{**}(41^2) = 2.29^2$, taking $e = 1$ and $e = 2$ successively in (3.13b), we see that 7 and 29 have to divide its left-hand side. This is false. Hence $e = 1$ or $e = 2$ cannot occur.

Thus we arrived at a contradiction in all cases by assuming that $11|n$. Hence $11 \nmid n$.

This proves Lemma 3.4. □

Lemma 3.5. *Let* $n = 2^4.3^b.v$, *where* $b = 2k,\ k \geq 3$ *and odd; also,* $(v, 2.3) = 1$. *If* n *is a bi-unitary triperfect number, then we have*

(a) $\frac{3^k-1}{2}$ *is divisible by a prime* $p > 53$ *and* $p|v$,

(b) $3^{k+1} + 1$ *is divisible by a prime* $q > 53$*; also,* $q|v$.

Proof. We assume that n is a bi-unitary triperfect number. Hence (3.1b) holds. Also,

$$\sigma^{**}(3^b) = \left(\frac{3^k - 1}{2}\right).(3^{k+1} + 1).$$

Remark 3.1. By Lemmas 3.2 and 3.3, n and hence v is not divisible by 5 or 7. We can assume that any prime factor of v is at least 11.

Proof of (a).

(I) Since k is odd, $3^k - 1$ is divisible by none of the primes $5, 7, 17, 19, 29, 31, 37, 41, 43, 53$; trivially not divisible by 3. The remaining odd primes up to 53 are 11, 13, 23 and 47.

(II) Suppose $23|3^k - 1$. This is if and only if $11|k$. Hence $3^{11} - 1|3^k - 1$. Also, $3^{11} - 1 = 2.23.3851$. It follows that 23 and 3851 divide $\frac{3^k-1}{2}|\sigma^{**}(3^b)$; from (3.1b), these two primes divide v. By Remark 3.3, we may assume that v is divisible by a prime $y \geq 11$. Hence $n = 2^4.3^b.23^c.(3851)^d.y^d$ and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{23}{22}.\frac{3851}{3850}.\frac{11}{10} = 2.911693588 < 3,$$

a contradiction. Hence $23 \nmid 3^k - 1$.

(III) Suppose $47|3^k - 1$. This is if and only if $23|k$. Hence $3^{23} - 1|3^k - 1$. Also, $3^{23} - 1 = 2.47.1001523179 = 2.p_1.p_2$, say. We use $p_2 \geq 59$. The primes p_1 and p_2 divide $\frac{3^k-1}{2}|\sigma^{**}(3^b)$; from (3.1b), these two primes divide v. If y denotes a possible third prime factor of v, then we have $y \geq 11$. We have $n = 2^4.3^b.p_1^c.p_2^d.y^e$, and hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{47}{46}.\frac{59}{10}.\frac{11}{10} = 2.893954976 < 3,$$

a contradiction. Hence $47 \nmid 3^k - 1$.

(IV) If $\frac{3^k-1}{2}$ is neither divisible by 11 nor by 13, then $\frac{3^k-1}{2} > 1$, odd and not divisible by any prime in $[3, 53]$. Hence each prime factor of is > 53 and is a factor of v. This proves (a) of Lemma 3.5 in this case.

(V) Suppose that $11|\frac{3^k-1}{2}$ and $13 \nmid \frac{3^k-1}{2}$. We may note that $11|3^k - 1$ if and only if $5|k$. Hence $3^5 - 1|3^k - 1$. Also, $3^5 - 1 = 2.11^2$. Thus $11|3^k - 1$ implies that $11^2|3^k - 1$. We claim that $\frac{3^k-1}{2}$ is divisible by a prime $p \neq 11$. If this is not the case, then $\frac{3^k-1}{2} = 11^\alpha, \ (\alpha \geq 2)$. If $\alpha \geq 3$, then $11^3|3^k - 1$. This is if and only if $55|k$; in particular $11|k$. Hence $23|3^{11} - 1|3^k - 1$ (see (II) above).

Thus $23|\frac{3^k-1}{2} = 11^\alpha$, which is impossible. Hence $\frac{3^k-1}{2} = 11^2$ or $3^k = 243$ or $k = 5$. We show that $k = 5$ is not possible.

If $k = 5$, then $b = 10$ and $\sigma^{**}(3^{10}) = \frac{3^{11}-1}{2} = \frac{2.23.3851}{2} = 23.3851$. From (3.1b) it follows that 23 and 3851 are factors of v. If y denotes the possible third prime factor of v so that $y \geq 11$, we have $n = 2^4.3^b.23^c.(3851)^d.y^e$ and hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{23}{22}.\frac{3851}{3850}.\frac{11}{10} = 2.911693588 < 3,$$

a contradiction. Hence $b = 10$ or $k = 5$ is not possible.

It follows that $\frac{3^k-1}{2}$ is divisible by a prime $p \neq 11$; since $13 \nmid \frac{3^k-1}{2}$, $p \neq 13$ also. Hence $p \notin [3, 53]$ so that $p > 53$ and $p|v$. This proves (a) of Lemma 3.5 in this case.

(VI) Suppose $11 \nmid \frac{3^k-1}{2}$ and $13|\frac{3^k-1}{2}$. If 13 alone divides $\frac{3^k-1}{2}$ then $\frac{3^k-1}{2} = 13^\beta$, where $\beta \geq 1$. If $\beta \geq 2$, then $13^2|3^k - 1$; this is if and only if $39|k$. Also, $3^{39} - 1 = 2.13^2.313.6553.7333.797161$. Hence $313|\frac{3^{39}-1}{2}|\frac{3^k-1}{2} = 13^\beta$. This is not possible. Hence $\frac{3^k-1}{2} = 13$ or $k = 3$, so that $b = 6$. We show that $b = 6$ is not possible.

We have $\sigma^{**}(3^6) = 13.82 = 2.13.41 \ (= 1066)$. Taking $b = 6$ in (3.1b), we see that v is divisible by 13 and 41. By Lemma 3.4, $11 \nmid n$ and so $11 \nmid v$. If y denotes the possible third prime factor of v, since it is not divisible by 5 or 7 or 11, then $y \geq 17$. Hence $n = 2^4.3^6.13^c.41^d.y$ and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{1066}{729}.\frac{13}{12}.\frac{41}{40}.\frac{17}{16} = 2.911309438 < 3,$$

a contradiction. Hence $\frac{3^k-1}{2}$ must be divisible by a prime $p \neq 13$. It follows that $p \notin [3, 53]$ so that $p|v$. This proves (a) of Lemma 3.5 in this case.

(VII) Suppose $\frac{3^k-1}{2}$ is divisible by both 11 and 13. We show that $\frac{3^k-1}{2}$ has a prime factor $p \neq 11$ and 13. On the contrary, assume that each prime factor of $\frac{3^k-1}{2}$ is either 11 or 13. This means that $\frac{3^k-1}{2} = 11^\alpha.13^\beta$, where $\alpha \geq 1$ and $\beta \geq 1$. We have $11|3^k - 1 \iff 5|k$ and $13|3^k - 1 \iff 3|k$. Since both 11 and 13 divide $3^k - 1$, it follows that $15|k$. Hence $3^{15} - 1|3^k - 1$.

Also, $3^{15}-1=2.11^2.13.4561$. This implies that $4561|\frac{3^k-1}{2}=11^\alpha.13^\beta$ which is impossible. Hence we can find an odd prime $p|\frac{3^k-1}{2}$ and $p\notin\{11,13\}$. We have $p>53$ and from (3.16), $p|v$.

The proof of (a) of Lemma 3.5 is complete.

Proof of (b). We now prove that $3^{k+1}+1$ has an odd prime factor $q>53$, where $k\geq 3$ and odd.

First of all, $2\|3^{k+1}+1$.

(I) Since $k+1$ is even, $3^{k+1}+1$ is not divisible by $7, 19, 31$ and 43; not divisible by 3 trivially.

(II) For any positive integer t, 3^t+1 is not divisible by $11, 13, 23$ and 47; in particular $3^{k+1}+1$ is not divisible by these primes.

(III) The remaining primes from 3 to 53 are $5, 17, 29$ and 53. It remains to check the divisibility of $3^{k+1}+1$ by these four primes.

We shall discuss the divisibility of 5 at the end.

(IV) Suppose $17|3^{k+1}+1$. This is equivalent to $k+1=8u$. Hence $3^8+1|3^{k+1}+1$. Also, $3^8+1=2.17.193$. Hence 17 and 193 are factors of $3^{k+1}+1|\sigma^{**}(3^b)$. From (3.1b), we have that 17 and 193 divide v. In (a) of the present Lemma 3.5, we already proved that $\frac{3^k-1}{2}$ is divisible by an odd prime $p>53$. Thus v is divisible by $17, 193$ and p. By (3.1c),$v=p^c.17^d.193^e$ and so by (3.1a), $n=2^4.3^b.p^c.17^d.193^e$. Hence we have

$$3=\frac{\sigma^{**}(n)}{n}<\frac{27}{16}.\frac{3}{2}.\frac{59}{58}.\frac{17}{16}.\frac{193}{192}=2.750072085<3,$$

a contradiction.

Hence $17\nmid 3^{k+1}+1$.

(V) Suppose $29|3^{k+1}+1$. This is equivalent to $k+1=14u$. Hence $3^{14}+1|3^{k+1}+1$. Also, $3^{14}+1=2.5.29.16493$. As before, it follows that v is divisible by p, 29 and 16493, where $p|\frac{3^k-1}{2}$ and $p>53$. Hence $n=2^4.3^b.p^c.29^d.16493^e$ and so

$$3=\frac{\sigma^{**}(n)}{n}<\frac{27}{16}.\frac{3}{2}.\frac{59}{58}.\frac{29}{28}.\frac{16493}{16492}=2.667014384<3,$$

a contradiction.

Hence $29\nmid 3^{k+1}+1$.

(VI) Assume that $53|3^{k+1}+1$. This is equivalent to $k+1=26u$. Hence $3^{26}+1|3^{k+1}+1$. Also, $3^{26}+1=2.5.53.4795973261=2.5.p_1.p_2$, say. Then p, p_1 and p_2 divide v, where $p|\frac{3^k-1}{2}$ and $p>53$. Hence $n=2^4.3^b.p^c.p_1^d.p_2^e$. We take $p_2>61$. We have

$$3=\frac{\sigma^{**}(n)}{n}<\frac{27}{16}.\frac{3}{2}.\frac{59}{58}.\frac{53}{52}.\frac{61}{60}=2.668149557<3,$$

a contradiction.

Hence $53\nmid 3^{k+1}+1$.

(VII) If $5\nmid 3^{k+1}+1$, then it follows from (I)–(VI) that $\frac{3^{k+1}+1}{2}$ is not divisible by any prime in $[3,53]$. Hence each prime factor of $\frac{3^{k+1}+1}{2}$ is >53. This is much more than what we stated in (b).

(VII) Suppose $5|3^{k+1}+1$. We show that $\frac{3^{k+1}+1}{2}$ is divisible by a prime $q\neq 5$. If this is not true, then we must have $\frac{3^{k+1}+1}{2}=5^\alpha$, where $\alpha\geq 1$. Let $\alpha\geq 2$. Then $5^2|3^{k+1}+1$. This is if and only if $k+1=10u$. Hence $3^{10}+1|3^{k+1}+1$. Also, $3^{10}+1=2.5^2.1181$. Thus, $1181|\frac{3^{k+1}+1}{2}=5^\alpha$ and this is impossible. Hence $\alpha=1$ and $\frac{3^{k+1}+1}{2}=5$ so that $k=1$. But $k\geq 3$. It follows that

$\frac{3^{k+1}+1}{2}$ must be divisible by an odd prime $q \neq 5$. From (I)–(VI), we conclude that $q \notin [3, 53]$. Hence $q > 53$ and $q|v$ by (3.1b) since q is a factor of $\frac{3^{k+1}+1}{2}|\sigma^{**}(3^b)$.

This completes the proof of (b) of Lemma 3.5, and also the whole Lemma 3.5. □

Lemma 3.6. *Let* $n = 2^4.3^b.v$, *where* $b = 2k,\ k \geq 3$ *and odd; also,* $(v, 2.3) = 1$. *Then* n *cannot be a bi-unitary triperfect number.*

Proof. Assume that n is a bi-unitary triperfect number. We obtain a contradiction. By our assumption n satisfies (3.1b). Hence v cannot have more than three odd prime factors. By Lemma 3.5, two odd primes p and q divide v, where $p|\frac{3^k-1}{6}$ and $q|3^{k+1}+1$; also, p and q exceed 53. We may assume that $p \geq 59$ and $q \geq 61$. By Lemmas 3.2 and 3.3, v is not divisible by 5 and 7. If y denotes the possible third prime factor of v, then we can assume that $y \geq 11$. It follows that $n = 2^4.3^b.p^c.q^d.y^e$ and we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{27}{16}.\frac{3}{2}.\frac{59}{58}.\frac{61}{60}.\frac{11}{10} = 2.879587823 < 3,$$

a contradiction.

This proves Lemma 3.6. □

Completion of proof of Theorem 3.1. Follows from Lemmas 3.1 and 3.6. □

4 Bi-unitary triperfect numbers of the form $n = 2^5 u$

In this section, we find all bi-unitary triperfect numbers n with $2^5 \| n$.

Theorem 4.1. *The only bi-unitary triperfect numbers of the form* $2^5 u$ *(with* u *odd) are*

$$672 = 2^5.3.7;\ 10080 = 2^5.3^2.5.7;\ 1528800 = 2^5.3.5^2.13;\ \textit{and}\ 22932000 = 2^5.3^2.5^3.7^2.13.$$

Proof. Let $n = 2^5 u$ be a bi-unitary triperfect number, where u is odd. Since $\sigma^{**}(n) = 3n$ and $\sigma^{**}(2^5) = 2^6 - 1 = 63 = 3^2.7$, we obtain after simplification,

$$2^5.u = 3.7.\sigma^{**}(u). \quad (4.1)$$

From (4.1) it is clear that 3 and 7 are factors of u so that $u = 3^b.7^c.v$, where $(v, 2.3.7) = 1$; using this form of u we have

$$n = 2^5.3^b.7^c.v; \quad (4.1a)$$

from (4.1), we obtain

$$2^5.3^{b-1}.7^{c-1}.v = \sigma^{**}(3^b).\sigma^{**}(7^c).\sigma^{**}(v), \quad (4.1b)$$

where $(v, 2.3.7) = 1$ and

$$v \text{ has at most three odd prime factors.} \quad (4.1c)$$

The remaining proof of Theorem 4.1 depends on the following:

Lemma 4.1. *Let* n *be as in* (4.1a). *If* $b = 1$ *and* n *is a bi-unitary triperfect number then* $n = 672 = 2^5.3.7$ *or* $n = 1528800 = 2^5.3.5^2.13$.

Proof. We assume that n is a bi-unitary triperfect number and hence (4.1b) holds. Let $b = 1$. Taking $b = 1$ in (4.1a) and (4.1b), we get

$$n = 2^5.3.7^c.v, \tag{4.2a}$$

and

$$2^3.7^{c-1}.v = \sigma^{**}(7^c).\sigma^{**}(v), \tag{4.2b}$$

where

$$v \text{ has no more than two odd prime factors.} \tag{4.2c}$$

Suppose $c = 1$. Taking $c = 1$ in (4.2a) and (4.2b), we get

$$n = 2^5.3.7.v, \tag{4.3a}$$

and

$$2^3.v = 8.\sigma^{**}(v), \tag{4.3b}$$

so that $v = \sigma^{**}(v)$. Hence $v = 1$ and $n = 2^5.3.7 = 672$ is a bi-unitary triperfect number.

Let $c = 2$. From (4.2b), we get $2^2.7.v = 5^2.\sigma^{**}(v)$; hence $5^2|v$. Let $v = 5^d.w$, where $d \geq 2$ and $(w, 2.3.5.7) = 1$. Thus we have

$$n = 2^5.3.7^2.5^d.w, \quad (d \geq 2) \tag{4.4a}$$

and

$$2^2.7.5^{d-2}.w = \sigma^{**}(5^d).\sigma^{**}(w), \tag{4.4b}$$

where w has at most one odd prime factor.

Suppose $d = 2$. From (4.4b), we obtain

$$2.7.w = 13.\sigma^{**}(w); \tag{4.4c}$$

hence $13|w$. Since w has at most one odd prime factor, we have $w = 13^e$. From (4.4a) and (4.4c), we get

$$n = 2^5.3.7^2.5^2.13^e, \tag{4.5a}$$

and

$$2.7.13^{e-1} = \sigma^{**}(13^e). \tag{4.5b}$$

Clearly, (4.5b) is satisfied when $e = 1$. Hence $n = 2^5.3.7^2.5^2.13 = 1528800$ is a bi-unitary triperfect number.

If $e \geq 2$, from (4.5b) we find that $13|\sigma^{**}(13^e)$ which is false. Thus the case $c = 2$, $d = 2$ and $e \geq 2$ cannot occur.

Let $c = 2$ and $d \geq 3$. For $d \geq 3$, $\frac{\sigma^{**}(5^d)}{5^d} \geq \frac{756}{625}$. From (4.4a), we have $n = 2^5.3.7^2.5^d.w$ and hence for $d \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{63}{32}.\frac{4}{3}.\frac{50}{49}.\frac{756}{625} = 3.24 > 3,$$

a contradiction.

So we may assume that $c \geq 3$; hence $\frac{\sigma^{**}(7^c)}{7^c} \geq \frac{2752}{2401}$. From (4.2a),

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{63}{32}.\frac{4}{3}.\frac{2752}{2401} = 3.008746356 > 3,$$

a contradiction.

The proof of Lemma 4.1 is complete. $\square$

Lemma 4.2. *Let n be as in* (4.1a) *and n be a bi-unitary triperfect number. Let $b = 2$. Then $n = 2^5.3^2.7^c.5^d.w$ and w is prime to* 2.3.5.7.

(i) If $c = 1$, then $d = 1$ and $n = 2^5.3^2.7.5 = 10080$.

(ii) If $c = 2$ then $d \geq 3$; if $d = 3$ then $13 \| n$ and $n = 2^5.3^2.7^2.5^3.13 = 22932000$.

Proof. Since n is assumed to be a bi-unitary triperfect number, the equation (4.1b) holds. Taking $b = 2$ in (4.1b), we obtain

$$2^4.3.7^{c-1}.v = 5.\sigma^{**}(7^c).\sigma^{**}(v). \tag{4.5c}$$

From (4.5c), we have $5|v$. Let $v = 5^d.w$. From (4.1a) and (4.5c), we obtain

$$n = 2^5.3^2.7^c.5^d.w, \tag{4.6a}$$

and

$$2^4.3.7^{c-1}.5^{d-1}.w = \sigma^{**}(7^c).\sigma^{**}(5^d).\sigma^{**}(w), \tag{4.6b}$$

where w has no more than two odd prime factors.

Proof of (i). Let $c = 1$. From (4.6a) and (4.6b), we get

$$n = 2^5.3^2.7.5^d.w, \tag{4.7a}$$

and

$$2.3.5^{d-1}.w = \sigma^{**}(5^d).\sigma^{**}(w). \tag{4.7b}$$

If $w > 1$, it follows that the right-hand side of (4.7b) is divisible by 2^2 while 2 is a unitary divisor of its left-hand side. Hence $w = 1$ and so (4.7a) and (4.7b) reduce to

$$n = 2^5.3^2.7.5^d, \tag{4.7c}$$

and

$$2.3.5^{d-1} = \sigma^{**}(5^d). \tag{4.7d}$$

If $d \geq 2$, from (4.7d), we have $5|\sigma^{**}(5^d)$ and this is not possible. Hence $d = 1$, and (4.7d) is satisfied when $d = 1$. Hence $n = 2^5.3^2.7.5 = 10080$ is a bi-unitary triperfect number.

This completes the proof of (i).

Proof of (ii). Let $c = 2$. Taking $c = 2$ in (4.6a) and (4.6b), we obtain

$$n = 2^5.3^2.7^2.5^d.w, \tag{4.8a}$$

and

$$2^3.3.7.5^{d-3}.w = \sigma^{**}(5^d).\sigma^{**}(w), \tag{4.8b}$$

where w has no more than two odd prime factors and $(w, 2.3.5.7) = 1$.

From the left-hand side of (4.8b), it is clear that $d \geq 3$.

Let $d = 3$. We have $\sigma^{**}(5^3) = 2^2.3.13$. Taking $d = 3$ in (4.8b), we get

$$2.7.w = 13.\sigma^{**}(w). \tag{4.8c}$$

From (4.8c), $13|w$ and $w = 13^e$. From (4.8a) and (4.8c), we obtain

$$n = 2^5.3^2.7^2.5^3.13^e, \tag{4.9a}$$

and

$$2.7.13^{e-1} = \sigma^{**}(13^e). \tag{4.9b}$$

If $e \geq 2$, then from (4.9b) it follows that $13|\sigma^{**}(13^e)$. This is not possible. Hence $e = 1$. This value satisfies (4.9b). Hence $n = 2^5.3^2.7^2.5^3.13 = 22932000$ is a bi-unitary triperfect number. This proves (ii).

The proof of Lemma 4.2 is complete. □

Lemma 4.3. *Let* $n = 2^5.3^b.7^c.5^d.w$, *where* $(w, 2.3.5.7) = 1$. *If* $b = 2$, $c = 2$ *and* $d \geq 4$, *then* n *cannot be a bi-unitary triperfect number.*

Proof. Suppose n is a bi-unitary triperfect number with $b = 2$, $c = 2$ and $d \geq 4$. The relevant equations are (4.8a) and (4.8b) with $d \geq 4$.

We have $\sigma^{**}(5^4) = 2^2.3^3.7$. Hence $3^3|\sigma^{**}(5^4)$. Taking $d = 4$ in (4.8b), we find that 3^3 divides its left-hand side; but it is divisible unitarily by 3. This contradiction shows that $d = 4$ is not admissible. Hence we may assume that $d \geq 5$.

We obtain a contradiction by analyzing the factors of $\sigma^{**}(5^d)$ in (4.8b). We distinguish the following cases:

<u>Case 1.</u> Let d be odd. Hence

$$\sigma^{**}(5^d) = \frac{5^{d+1}-1}{4} = \frac{(5^t-1)(5^t+1)}{4} \quad \left(t = \frac{d+1}{2}\right).$$

Since $d \geq 5$, we have $t \geq 3$.

(a) Let t be even. Hence $8|5^t - 1$ and consequently, $4|\frac{5^t-1}{2}|\sigma^{**}(5^d)$. It now follows from (4.8b), that w can have at most one odd prime factor. We wish to show that $\frac{5^t-1}{2}$ has an odd prime factor $p \geq 29$; and then from (4.8b), $p|w$. Hence $w = p^e$. This leads to a contradiction since $n = 2^5.3^2.7^2.5^d.p^e$ and therefore

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{63}{32}.\frac{10}{9}.\frac{50}{49}.\frac{5}{4}.\frac{29}{28} = 2.889827806 < 3, \tag{4.9d}$$

a contradiction.

(I) First we observe that $8\|5^t-1$. If $16|5^t-1$, then $8|\frac{5^t-1}{2}|\sigma^{**}(5^d)$ and from (4.8b), we find that $w=1$. Hence (4.8b) reduces to $2^3.3.7.5^{d-3}=\sigma^{**}(5^d)$, and since $d\geq 5$, this implies that $5|\sigma^{**}(5^d)$ which is false. Thus $8\|5^t-1$.

(II) Suppose $7|5^t-1$. This is if and only if $6|t$. Hence $5^6-1|5^t-1$. Since $9|5^6-1$, we also have $9|\frac{5^t-1}{2}|\sigma^{**}(5^d)$. From (4.8b) it follows that $3|w$ but w is prime to 3. This contradiction proves that $7\nmid 5^t-1$.

(III) Clearly, $3|5^t-1$. It may be noted that $9|5^t-1 \Longleftrightarrow 6|t \Longleftrightarrow 7|5^t-1$. Since it is proved in (II) above that $7\nmid 5^t-1$, then $9\nmid 5^t-1$. Thus $3\|5^t-1$.

(IV) Suppose $11|5^t-1$. This is equivalent to $5|t$. Hence $5^5-1|5^t-1$. Also, $5^5-1=2^2.11.71$. It follows that $\frac{5^t-1}{2}|\sigma^{**}(5^d)$ is divisible by 11 and 71. From (4.8b) these primes should divide w. But in the present case namely $t=\frac{d+1}{2}$ is even, w cannot have more than one odd prime factor. Hence $11\nmid 5^t-1$.

(V) Suppose $13|5^t-1$. This is if and only if $4|t$. Hence $16|5^4-1|5^t-1$. In (I) above we proved that $16\nmid 5^t-1$. Thus $13\nmid 5^t-1$.

(VI) Assume that $19|5^t-1$. This is if and only if $9|t$. Hence $5^9-1|5^t-1$. Also, $5^9-1=2^2.19.31.829$ so that $\frac{5^t-1}{2}|\sigma^{**}(5^d)$ is divisible by three primes $19, 31$ and 829 which divide w by (4.8b). This cannot happen as w has no more than one odd prime factor. Thus $19\nmid 5^t-1$.

(VII) Finally, suppose $23|5^t-1$. This is if and only if $22|t$. We have $5^{22}-1|5^t-1$ and $5^{22}-1=2^3.3.23.67.5281.12207031$. Hence $\frac{5^t-1}{2}|\sigma^{**}(5^d)$ is divisible by four odd primes and these four primes divide w by (4.8b). This cannot happen. Hence $23\nmid 5^t-1$.

Further since $8\|5^t-1$, $\frac{5^t-1}{8}$ is odd and also >1. From (I)–(VII), it follows that each prime factor of $\frac{5^t-1}{8}$ is odd and >23 or ≥ 29. Certainly $\frac{5^t-1}{8}>1$ is divisible by a prime $p\geq 29$. Since $p|\frac{5^t-1}{8}|\frac{5^t-1}{2}|\sigma^{**}(5^d)$, it follows from (4.8b) that $p|w$.

As mentioned in the beginning of (a) of Case 1, this would lead to a contradiction indicated in (4.9d).

(b) Let t be odd (already $t\geq 3$).

We show that we can find primes $p,\ q,\ p\neq q,\ p|\frac{5^t-1}{4},\ q|\frac{5^t+1}{6},\ p,q|w$ and $p,\ q>23$.

(I) Since t is odd, $4\|5^t-1$ and 5^t-1 is not divisible by $3, 5, 7, 13, 17$ and 23.

(II) Suppose $11|5^t-1$. This is equivalent to $5|t$. Hence $5^5-1|5^t-1$. Also, $5^5-1=2^2.11.71$. Hence $71|\frac{5^t-1}{4}$. It is true in this case that $\frac{5^t-1}{4}$ is divisible by a prime $p>23$ (here $p=71$). So we may assume that $11\nmid 5^t-1$.

(III) Suppose $19|5^t-1$. This is equivalent to $9|t$. Consequently $5^9-1|5^t-1$. Also, $5^9-1=2^2.19.31.829$. It follows that the primes $19, 31$ and 829 divide w by (4.8b). This cannot happen as w cannot have more than two odd prime factors. Hence $19\nmid 5^t-1$.

From (I)–(III), it follows that $\frac{5^t-1}{4}$ is odd, >1 and not divisible by any prime in $[3,23]$. Let $p|\frac{5^t-1}{4}$. Then $p\geq 29$ and $p|w$ by (4.8b).

We now consider the factor 5^t+1, where t is odd.

(IV) Since t is odd, $2\|5^t+1$ and $3|5^t+1$. Also, since 9 cannot divide the left-hand side of (4.8b), we have $9\nmid 5^t+1$. Hence $3\|5^t+1$.

(V) Suppose $7|5^t+1$. This is equivalent to $t=3u$. Hence $5^3+1|5^t+1$. Also, $5^3+1=2.3^2.7$. Hence $9|5^t+1$. From (IV) above this is not so. Hence $7\nmid 5^t+1$.

(VI) For any positive integer t, $11 \nmid 5^t + 1$ and $19 \nmid 5^t + 1$.

(VII) Suppose $13 | 5^t + 1$. This is equivalent to $t = 2u$. Also, since t is odd, $13 \nmid 5^t + 1$.

(VIII) Suppose $17 | 5^t + 1$. This is if and only if $t = 8u$. So t must be even. Since t is odd,$17 \nmid 5^t + 1$.

(IX) Suppose $23 | 5^t + 1$. This is if and only if $t = 11u$. Hence $5^{11} + 1 | 5^t + 1$. Also, $5^{11} + 1 = 2.3.23.67.5281$. Hence $\sigma^{**}(5^d)$ is divisible by three primes $23, 67$ and 5281 which also divide w by (4.8b). This cannot happen. Hence $23 \nmid 5^t + 1$.

It follows from (IV)–(IX) that $\frac{5^t+1}{6}$ is odd, > 1 and not divisible by any prime in $[3, 23]$. Let $q | \frac{5^t+1}{6} | 5^t + 1 | \sigma^{**}(5^d)$. Then $q \geq 29$ and $q | w$ by (4.8b). Since $\frac{5^t-1}{4}$ and $\frac{5^t+1}{6}$ are relatively prime it follows that $p \neq q$. Without loss of generality, we may assume that $p \geq 29$ and $q \geq 31$. Also, $w = p^e.q^f$. From (4.8a), we have $n = 2^5.3^2.7^2.5^d.p^e.q^f$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{63}{32}.\frac{10}{9}.\frac{50}{49}.\frac{5}{4}.\frac{29}{28}.\frac{31}{30} = 2.9861554 < 3,$$

a contradiction.

We have completed Case 1 (d odd.) Thus n given in Lemma 4.3 cannot be a bi-unitary triperfect number if d is odd.

Case 2. Let d be even so that $d = 2k$. We may assume that $k \geq 3$ since $d \geq 5$. We have

$$\sigma^{**}(5^d) = \left(\frac{5^k - 1}{4}\right).(5^{k+1} + 1) \quad (k \geq 3).$$

(a) Let k be even. Then $8 | 5^k - 1$ and $2 | 5^{k+1} + 1$. Hence $4 | \sigma^{**}(5^d)$. It follows from (4.8b) that w cannot have more than one odd prime factor. As in (a) of Case 1, $16 \nmid 5^k - 1$; hence $8 \| 5^k - 1$ and we can find an odd prime $p | \frac{5^k-1}{8}$ and $p | w$ such that $p \geq 29$. In a similar manner, we obtain a contradiction (we simply have to replace k by t and proceed as in (a) of Case 1).

(b) Let k be odd. Here also we follow (b) of Case 1, treating k as t. We have $4 \| 5^k - 1$ so that $\frac{5^k-1}{4}$ is odd. This fraction is > 1 since $k \geq 3$. Exactly as in (b) of Case 1, $\frac{5^k-1}{4}$ is divisible by an odd prime $p | w$ and $p \geq 29$.

We now consider $5^{k+1} + 1$. We wish to show that $\frac{5^{k+1}+1}{2}$ is divisible by a prime $p > 23$.

(I) $2 \| 5^{k+1} + 1$; since $k + 1$ is even, $5^{k+1} + 1$ is not divisible by $3, 7$ and 23.

(II) Since for any positive integer t, $5^t + 1$ is not divisible by 11 or 19, the same holds good for $5^{k+1} + 1$ also.

(III) Suppose $17 | 5^{k+1} + 1$. This is if and only if $k + 1 = 8u$. Hence $5^8 + 1 | 5^{k+1} + 1$. Also, $5^8 + 1 = 2.17.11489$. It follows that $q = 11489$ divides $\frac{5^{k+1}+1}{2} | \sigma^{**}(5^d)$. Trivially $q > 23$ and from (4.8b), q divides w. This is what we wished to prove. We may assume that $17 \nmid 5^{k+1} + 1$.

(IV) Thus from (I), (II) and (III), $\frac{5^{k+1}+1}{2}$ is odd, > 1 and not divisible by any prime in $[3, 23]$ except 13. If $13 \nmid 5^{k+1} + 1$, then it would follow that $\frac{5^{k+1}+1}{2}$ is not divisible by any prime in $[3, 23]$. Hence every prime factor of $\frac{5^{k+1}+1}{2}$ is ≥ 29 and from (4.8b) all prime factors of $\frac{5^{k+1}+1}{2}$ also divide w. That there is an odd prime $q | \frac{5^{k+1}+1}{2}$ and $q | w$ with $q \geq 29$ is true.

(V) Suppose $13 | 5^{k+1} + 1$. We show that $\frac{5^{k+1}+1}{2}$ must be divisible by an odd prime $q \neq 13$. If this is not so, then we must have $\frac{5^{k+1}+1}{2} = 13^\alpha$, where $\alpha \geq 1$. If $\alpha \geq 2$, $13^2 | 5^{k+1} + 1$. This is if and only if $k + 1 = 26u$. Hence $5^{26} + 1 | 5^{k+1} + 1$. Also, $5^{26} + 1 = 2.13^2.53.8318165204609$. In particular, $53|\frac{5^{k+1}+1}{2}=13^{\alpha}$, which is not possible. Hence $\alpha=1$ so that $\frac{5^{k+1}+1}{2}=13$ or $k=1$. But $k\geq 3$. This proves that we can find an odd prime $q\neq 13$ and $q|\frac{5^{k+1}+1}{2}$. From (I)–(III), it is clear that $q\in[3,23]$ and from (4.18b), $q|w$. Hence $q\geq 29$.

Thus we proved that (i) $\frac{5^k-1}{4}$ is divisible by an odd prime $p|w$ and $p\geq 29$, (ii) $\frac{5^{k+1}+1}{2}$ is divisible by an odd prime $q\geq 29$. Since $\frac{5^k-1}{4}$ and $\frac{5^{k+1}+1}{2}$ are relatively prime $p\neq q$. From (4.8b), $w=p^e.q^f$. Hence form (4.8a), $n=2^5.3^2.7^2.5^d.p^e.q^f$. As in (b) of Case 1, we obtain a contradiction.

This proves Lemma 4.3. □

Lemma 4.4. *Let $n=2^5.3^b.7^c.5^d.w$, where $(w,2.3.5.7)=1$. If $b=2$ and $c\geq 3$, then n cannot be a bi-unitary triperfect number.*

Proof. Assume that n given in Lemma 4.4 is a bi-unitary triperfect number. The relevant equations are (4.6a) and (4.6b).

By Lemma 2.1, we have since $c\geq 3$, $\frac{\sigma^{**}(7^c)}{7^c}\geq\frac{2752}{2401}$. Also, for $d\geq 3$, $\frac{\sigma^{**}(5^d)}{5^d}\geq\frac{756}{625}$. Since $n=2^5.3^2.7^c.5^d.w$, we have for $d\geq 3$,

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{63}{32}.\frac{10}{9}.\frac{2752}{2401}.\frac{756}{625}=3.032816327>3,$$

a contradiction.

Hence $d=1$ or $d=2$.

If $d=1$, we have $n=2^5.3^2.7^c.5.w$ and again

$$3=\frac{\sigma^{**}(n)}{n}\geq\frac{63}{32}.\frac{10}{9}.\frac{2752}{2401}.\frac{6}{5}=3.008746356>3,$$

a contradiction.

Let $d=2$. From (4.6b) $(d=2)$, we obtain

$$2^3.3.7^{c-1}.5.w=13.\sigma^{**}(7^c).\sigma^{**}(w). \tag{4.10c}$$

From (4.10c), we have $13|w$. Hence $w=13^e.w'$, where $(w',2.3.5.7.13)=1$. Now from (4.6a) and (4.10c), we get

$$n=2^5.3^2.7^c.5^2.13^e.w' \quad (c\geq 3), \tag{4.11a}$$

and

$$2^3.3.7^{c-1}.5.13^{e-1}.w'=\sigma^{**}(7^c).\sigma^{**}(13^e).\sigma^{**}(w'), \tag{4.11b}$$

where

$$w' \text{ has no more than one odd prime factor.} \tag{4.11c}$$

By examining the factors of $\sigma^{**}(7^c)$, we arrive at a contradiction.

We distinguish the following cases:

Case 1. Let c be odd. Then $\sigma^{**}(7^c)=\frac{7^{c+1}-1}{6}$. Since $c+1$ is even, $48=7^2-1|7^{c+1}-1$. Hence $8|\sigma^{**}(7^c)$. From (4.11b) we find an imbalance in powers of 2 between its two sides.

Case 2. Let c be even say $c = 2k$. We have

$$\sigma^{**}(7^c) = \left(\frac{7^k - 1}{6}\right).(7^{k+1} + 1).$$

(a) Let k be even. Then $8|7^k-1$ and $8|7^{k+1}+1$. Hence $32|\sigma^{**}(7^c)$. This leads to a contradiction as in Case 1.

(b) Let k be odd. We prove that we can find an odd prime $p|\frac{7^k-1}{6}$, $p|w'$ and $p \geq 29$. If this is done, then by (4.11c), $w' = p^f$ and so $n = 2^5.3^2.7^c.5^2.13^e.p^f$. Hence

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{63}{32}.\frac{10}{9}.\frac{7}{6}.\frac{26}{25}.\frac{13}{12}.\frac{29}{28} = 2.978038194 < 3,$$

a contradiction. This would complete the proof of Lemma 4.4.

(I) Since k is odd, $2\|7^k - 1$ and $7^k - 1$ is divisible by none of the primes $5, 11, 13, 17$ and 23; trivially not divisible by 7.

(II) $3|7^k - 1$. If $27|7^k - 1$, then $9|\frac{7^k-1}{6}|\sigma^{**}(7^c)$. From (4.11b) it follows that $3|w$ which is not true. Hence $27 \nmid 7^k - 1$.

(III) We may note that $9|7^k - 1 \iff 3|k \iff 19|7^k - 1$. If $9 \nmid 7^k - 1$, then $19 \nmid 7^k - 1$ and $3\|7^k - 1$. In this case $\frac{7^k-1}{6}$ is odd and > 1, since $k \geq 3$. Also, $\frac{7^k-1}{6}$ is not divisible by any prime in $[3, 23]$. Hence every prime factor of $\frac{7^k-1}{6}$ is ≥ 29 and also is a factor of w' by (4.11b). This is slightly more than what wanted to prove.

(IV) Suppose $9|7^k - 1$. Hence $9\|7^k - 1$ and $19|7^k - 1$. We have since $k \geq 3$, $\frac{7^k-1}{18} > 1$; also, it is odd and not divisible by 3. We show that $\frac{7^k-1}{18}$ must be divisible by an odd prime $p \neq 19$. If this is not the case, then we have $\frac{7^k-1}{18} = 19^\alpha$, where $\alpha \geq 1$. If $\alpha \geq 2$, then $19^2|7^k - 1$; this is if and only if $57|k$. Hence $7^{57} - 1|7^k - 1$. In Appendix F of Part I (see [2]), factorization of $7^{57} - 1$ is given. It follows that $419|7^{57} - 1|7^k - 1$. Hence $419|\frac{7^k-1}{18} = 19^\alpha$. This is not possible. Hence $\alpha = 1$ and so $\frac{7^k-1}{18} = 19$ or $k = 3$.

We now prove that $k = 3$, that is, $c = 6$ is not possible. We have $\sigma^{**}(7^6) = 2.3.19.1201$. Taking $c = 6$ in (4.11b), we see that 19 and 1201 divide w'. This contradicts (4.11c). Hence $k = 3$ is not admissible.

Thus $\frac{7^k-1}{18}$ is divisible by an odd prime $p \neq 19$. Also, $p \neq 3$. From (I) it is clear that $p \notin [3, 23]$. Also, from (4.11b), $p|w'$.

This completes the proof of Lemma 4.4. □

Lemma 4.5. *Let* $n = 2^5.3^b.7^c.v$, *where* $(v, 2.3.7) = 1$. *If* $b \geq 3$, *then* n *cannot be a bi-unitary triperfect number.*

Proof. Suppose n in Lemma 4.5 (same as n in (4.1a)) is a bi-unitary triperfect number. The relevant equations are (4.1a) and (4.1b) with $b \geq 3$.

By Lemma 2.1, since $b \geq 3$, we have $\frac{\sigma^{**}(3^b)}{3^b} \geq \frac{112}{81}$ and for $c \geq 3$, $\frac{\sigma^{**}(7^c)}{7^c} \geq \frac{2752}{2401}$. Hence for $c \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{63}{32}.\frac{112}{81}.\frac{2752}{2401} = 3.120181406 > 3,$$

a contradiction.

Therefore, $c = 1$ or $c = 2$.

When $c = 1$, we have $n = 2^5.3^b.7.v$, and so

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{63}{32}.\frac{112}{81}.\frac{8}{7} = 3.1111 > 3,$$

a contradiction.

Let $c = 2$. Taking $c = 2$ in (4.1b), we get

$$2^4.3^{b-1}.7.v = 5^2.\sigma^{**}(3^b).\sigma^{**}(v). \tag{4.11d}$$

From (4.11d), we have $5^2|v$. Let $v = 5^d.w$, where $d \geq 2$ and w is prime to 2.3.5.7. From (4.1a) and (4.11d), we have

$$n = 2^5.3^b.7^2.5^d.w, \quad (b \geq 3,\ d \geq 2) \tag{4.12a}$$

and

$$2^4.3^{b-1}.7.5^{d-2}.w = \sigma^{**}(3^b).\sigma^{**}(5^d).\sigma^{**}(w), \tag{4.12b}$$

where w cannot have more than two odd prime factors.

We have by Lemma 2.1, $\frac{\sigma^{**}(5^d)}{5^d} \geq \frac{756}{625}$ $(d \geq 3)$. Hence from (4.12a), for $d \geq 3$,

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{63}{32}.\frac{112}{81}.\frac{50}{49}.\frac{756}{625} = 3.36 > 3,$$

a contradiction.

Hence $d = 2$ since $d \geq 2$. Taking $d = 2$ in (4.12b), we get

$$2^3.3^{b-1}.7.w = 13.\sigma^{**}(3^b).\sigma^{**}(w). \tag{4.12c}$$

From (4.12c), we have $13|w$. Let $w = 13^e.w'$. From (4.12a) and (4.12c), we obtain

$$n = 2^5.3^b.7^2.5^2.13^e.w', \quad (b \geq 3) \tag{4.13a}$$

and

$$2^3.3^{b-1}.7.13^{e-1}.w' = \sigma^{**}(3^b).\sigma^{**}(13^e).\sigma^{**}(w'), \tag{4.13b}$$

where $(w', 2.3.5.7.13) = 1$ and w' cannot have more than one odd prime factor.

By Lemma 2.1, for $e \geq 3$, $\frac{\sigma^{**}(13^e)}{13^e} \geq \frac{30772}{28561}$. Hence for $e \geq 3$, from (4.13a), we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{63}{32}.\frac{112}{81}.\frac{50}{49}.\frac{26}{25}.\frac{30772}{28561} = 3.112527184 > 3,$$

a contradiction.

Hence $e = 1$ or $e = 2$.

If $e = 1$, we have $n = 2^5.3^b.7^2.5^2.13.w'$ and so

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{63}{32}.\frac{112}{81}.\frac{50}{49}.\frac{26}{25}.\frac{14}{13} = 3.111 > 3,$$

a contradiction.

Let $e = 2$. From (4.13b) $(e = 2)$, we get

$$2^3.3^{b-1}.7.13.w' = 170.\sigma^{**}(3^b).\sigma^{**}(w'). \tag{4.13c}$$

From (4.13c), it follows that $5|w'$. But w' is prime to 5. This is a contradiction.

The proof of Lemma 4.5 is complete. □

Completion of proof of Theorem 4.1. Follows from Lemmas 4.1 to 4.5. □

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