Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 1, 216–224
DOI: 10.7546/nntdm.2020.26.1.216-224

Some properties of (p, q)-Fibonacci-like and (p, q)-Lucas numbers

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Received: 21 July 2019 **Revised:** 2 February 2020 **Accepted:** 5 February 2020

Abstract: In this paper, we present the investigation of products of (p, q)-Fibonacci-like and (p, q)-Lucas numbers. It also reveals some properties on the products of (p, q)-Fibonacci-like and (p, q)-Lucas numbers.

Keywords: Fibonacci numbers, (p, q)-Fibonacci-like numbers, (p, q)-Lucas numbers.
2010 Mathematics Subject Classification: 11B39.

1 Introduction

For p and q positive real numbers, the generalization of Fibonacci sequence $\{F_{p,q,n}\}_{n\geq 0}$ can be defined as $F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}$ for $n \geq 2$ with the initial conditions $F_{p,q,0} = 0$ and $F_{p,q,1} = 1$. The sequence $\{F_{p,q,n}\}_{n\geq 0}$ is called the (p, q)-*Fibonacci sequence*. Each term in the (p, q)-Fibonacci sequence is called the (p, q)-*Fibonacci number*. In a similar way, the generalization of Lucas sequence $\{L_{p,q,n}\}_{n\geq 0}$ can be defined as $L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}$ for $n \geq 2$ with the initial conditions $L_{p,q,0} = 2$ and $L_{p,q,1} = p$, this sequence is called the (p,q)-***Lucas sequence***. Similar to the (p,q)-Fibonacci numbers, each term in the (p,q)-Lucas sequence is called (p,q)-***Lucas number***. The well-known Binet's formulas for the (p,q)-Fibonacci numbers and the (p,q)-Lucas numbers are given by

$$F_{p,q,n} = \frac{R_1^n - R_2^n}{R_1 - R_2}, \ \ L_{p,q,n} = R_1^n + R_2^n,$$

where

$$R_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \ \ R_2 = \frac{p - \sqrt{p^2 + 4q}}{2}$$

are roots of the characteristic equation $R^2 - pR - q = 0$, see [1, 2, 3, 4, 5]. Then, we have $R_1 + R_2 = p$, $R_1 - R_2 = \sqrt{p^2 + 4q}$ and $R_1 R_2 = -q$.

In 2015, Suvarnamani and Tatong [1] showed some results of the (p,q)-Fibonacci numbers by using the Binet's formula. The year after, Suvarnamani [2] proved some properties of (p,q)-Lucas numbers by using Binet's formula and he [3] studied on the odd and even terms of the (p,q)-Fibonacci numbers and the (p,q)-Lucas numbers by using Binet's formulas. In 2017, he [4] studied some properties of (p,q)-Fibonacci numbers by using the matrix methods. And Suvarnamani and Tatong [5] showed some properties of products of (p,q)-Fibonacci and (p,q)-Lucas numbers.

The Fibonacci-like sequence $\{S_n\}_{n\geq 0}$ can be defined as $S_n = S_{n-1} + S_{n-2}$ for $n \geq 2$ with the initial conditions $S_0 = S_1 = 2$. In 2010, Singh, Sikhwal and Bhatnagar [6] proved Binet's formula and generating function of Fibonacci-like sequence. They showed some of their properties using Binet's formula. Next, Gupta, Singh and Sikhwal [7] studied some properties of generalized Fibonacci-like sequence. In 2016, Wani, Rathore and K. Sisodiya [8] showed some properties of Fibonacci-like sequence. Recently, Suvarnamani [9] studied some properties of the generalized (p,q)-Fibonacci-like number.

In this paper, we present the investigation of products of (p,q)-Fibonacci-like and (p,q)-Lucas numbers.

Definition 1.1 ([9])**.** *For p and q positive real numbers, the (p,q)-Fibonacci-like sequence $\{S_{p,q,n}\}_{n\geq 0}$ is defined by $S_{p,q,n} = pS_{p,q,n-1} + qS_{p,q,n-2}$ for $n \geq 2$ with the initial conditions $S_{p,q,0} = 2$ and $S_{p,q,1} = 2p$. And the (p,q)-Fibonacci-like number is the each term of the (p,q)-Fibonacci-like sequence.*

That is,

$$\{S_{p,q,n}\}_{n\geq 0} = \{2, 2p, 2p^2 + 2q, 2p^3 + 4pq, 2p^4 + 6p^2q + 2q^2, 2p^5 + 8p^3q + 6pq^2, \ldots\}.$$

For $p = q = 1$, we get the Fibonacci-like sequence, that is

$$\{S_{1,1,n}\}_{n\geq 0} = \{2, 2, 4, 6, 10, 16, 26, \ldots\}.$$

For $p = 1$, By the recurrence relation and the initial conditions, then we have

$$S_{1,q,n} = F_{1,q,n} + L_{1,q,n}. \quad (1)$$

Lemma 1.2 ([9]). *The Binet's formulas of (p,q)-Fibonacci-like numbers are given by*

$$S_{p,q,n} = 2\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}; n \geq 0$$

where

$$R_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad R_2 = \frac{p - \sqrt{p^2 + 4q}}{2}$$

are roots of the characteristic equation $R^2 - pR - q = 0$.

Then, we have $R_1 + R_2 = p$, $R_1 - R_2 = \sqrt{p^2 + 4q}$ and $R_1 R_2 = -q$.

2 Main results

In this section, we prove some properties of the products of (p,q)-Fibonacci-like and (p,q)-Lucas numbers.

Theorem 2.1. *For integers k, m such that $k \geq 1$ and $2k \geq m \geq 0$, we have*

1. $S_{p,q,2k+m} L_{p,q,2k+m} = S_{p,q,4k+2m} + 2(-q)^{2k+m}$;
2. $S_{p,q,2k-m} L_{p,q,2k-m} = S_{p,q,4k-2m} + 2(-q)^{2k-m}$.

Proof. Using Binet's formulas, we have

$$\begin{aligned}
S_{p,q,2k+m} L_{p,q,2k+m} &= 2\left(\frac{R_1^{2k+m+1} - R_2^{2k+m+1}}{R_1 - R_2}\right)\left(R_1^{2k+m} + R_2^{2k+m}\right) \\
&= 2\left(\frac{R_1^{4k+2m+1} - R_2^{4k+2m+1} + R_1^{2k+m+1} R_2^{2k+m} - R_1^{2k+m} R_2^{2k+m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+2m+1} - R_2^{4k+2m+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k+m+1} R_2^{2k+m} - R_1^{2k+m} R_2^{2k+m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+2m+1} - R_2^{4k+2m+1}}{R_1 - R_2}\right) + 2\left(R_1 R_2\right)^{2k+m} \\
&= S_{p,q,4k+2m} + 2(-q)^{2k+m}
\end{aligned}$$

and

$$\begin{aligned}
S_{p,q,2k-m} L_{p,q,2k-m} &= 2\left(\frac{R_1^{2k-m+1} - R_2^{2k-m+1}}{R_1 - R_2}\right)\left(R_1^{2k-m} + R_2^{2k-m}\right) \\
&= 2\left(\frac{R_1^{4k-2m+1} - R_2^{4k-2m+1} + R_1^{2k-m+1} R_2^{2k-m} - R_1^{2k-m} R_2^{2k-m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k-2m+1} - R_2^{4k-2m+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k-m+1} R_2^{2k-m} - R_1^{2k-m} R_2^{2k-m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k-2m+1} - R_2^{4k-2m+1}}{R_1 - R_2}\right) + 2\left(R_1 R_2\right)^{2k-m} \\
&= S_{p,q,4k-2m} + 2(-q)^{2k-m}.
\end{aligned}$$

□

Corollary 2.1.1. *For integers k, m such that $k \geq 0$ and $2k \geq m \geq 0$, we have*

1. $S_{1,q,2k+m}F_{1,q,2k+m} = S_{1,q,2k+m}^2 - S_{1,q,4k+2m} - 2(-q)^{2k+m}$;
2. $S_{1,q,2k-m}F_{1,q,2k-m} = S_{1,q,2k-m}^2 - S_{1,q,4k-2m} - 2(-q)^{2k-m}$.

Proof. Using equation (1) and Theorem 2.1, we have

$$\begin{aligned}
S_{1,q,4k+2m} + 2(-q)^{2k+m} &= S_{1,q,2k+m}L_{1,q,2k+m} \\
&= S_{1,q,2k+m}(S_{1,q,2k+m} - F_{1,q,2k+m}) \\
&= S_{1,q,2k+m}^2 - S_{1,q,2k+m}F_{1,q,2k+m} \\
S_{1,q,2k+m}F_{1,q,2k+m} &= S_{1,q,2k+m}^2 - S_{1,q,4k+2m} - 2(-q)^{2k+m}
\end{aligned}$$

and

$$\begin{aligned}
S_{1,q,4k-2m} + 2(-q)^{2k-m} &= S_{1,q,2k-m}L_{1,q,2k-m} \\
&= S_{1,q,2k-m}(S_{1,q,2k-m} - F_{1,q,2k-m}) \\
&= S_{1,q,2k-m}^2 - S_{1,q,2k-m}F_{1,q,2k-m} \\
S_{1,q,2k-m}F_{1,q,2k-m} &= S_{1,q,2k-m}^2 - S_{1,q,4k-2m} - 2(-q)^{2k-m}.
\end{aligned}$$

□

Theorem 2.2. *For integers k, m such that $2k \geq m \geq 1$, we have*

1. $S_{p,q,2k+m}L_{p,q,2k-m} = S_{p,q,4k} + (-q)^{2k-m}S_{p,q,2m}$;
2. $S_{p,q,2k-m}L_{p,q,2k+m} = S_{p,q,4k} - (-q)^{2k-m+1}S_{p,q,2m-2}$.

Proof. Using Binet's formulas, we have

$$\begin{aligned}
S_{p,q,2k+m}L_{p,q,2k-m} &= 2\left(\frac{R_1^{2k+m+1} - R_2^{2k+m+1}}{R_1 - R_2}\right)\left(R_1^{2k-m} + R_2^{2k-m}\right) \\
&= 2\left(\frac{R_1^{4k+1} - R_2^{4k+1} + R_1^{2k+m+1}R_2^{2k-m} - R_1^{2k-m}R_2^{2k+m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+1} - R_2^{4k+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k+m+1}R_2^{2k-m} - R_1^{2k-m}R_2^{2k+m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+1} - R_2^{4k+1}}{R_1 - R_2}\right) + 2\left(R_1R_2\right)^{2k-m}\left(\frac{R_1^{2m+1} - R_2^{2m+1}}{R_1 - R_2}\right) \\
&= S_{p,q,4k} + (-q)^{2k-m}S_{p,q,2m}
\end{aligned}$$

and

$$\begin{aligned}
S_{p,q,2k-m}L_{p,q,2k+m} &= 2\left(\frac{R_1^{2k-m+1} - R_2^{2k-m+1}}{R_1 - R_2}\right)\left(R_1^{2k+m} + R_2^{2k+m}\right) \\
&= 2\left(\frac{R_1^{4k+1} - R_2^{4k+1} + R_1^{2k-m+1}R_2^{2k+m} - R_1^{2k+m}R_2^{2k-m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+1} - R_2^{4k+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k-m+1}R_2^{2k+m} - R_1^{2k+m}R_2^{2k-m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+1} - R_2^{4k+1}}{R_1 - R_2}\right) - 2\left(R_1R_2\right)^{2k-m+1}\left(\frac{R_1^{2m-1} - R_2^{2m-1}}{R_1 - R_2}\right) \\
&= S_{p,q,4k} - (-q)^{2k-m+1}S_{p,q,2m-2}.
\end{aligned}$$

□

Using equation (1) and Theorem 2.2, we can get the corollary.

Corollary 2.2.1. *For integers k, m such that $2k \geq m \geq 1$, we have*

1. $S_{1,q,2k-m}F_{1,q,2k-m} = S_{1,q,2k+m}S_{1,q,2k-m} - S_{1,q,4k} - (-q)^{2k-m}S_{1,q,2m}$;
2. $S_{1,q,2k-m}F_{1,q,2k+m} = S_{1,q,2k-m}S_{1,q,2k+m} - S_{1,q,4k} + (-q)^{2k-m+1}S_{1,q,2m-2}$.

Theorem 2.3. *For integers $k \geq 1$ and $m \geq 2$, we have*

1. $S_{p,q,2k}L_{p,q,2k+m} = S_{p,q,4k+m} + q^{2k+1}S_{p,q,m-2}$;
2. $S_{p,q,2k+m}L_{p,q,2k} = S_{p,q,4k+m} + q^{2k}S_{p,q,m}$.

Proof. Using Binet's formulas, we have

$$\begin{aligned}
S_{p,q,2k}L_{p,q,2k+m} &= 2\left(\frac{R_1^{2k+1} - R_2^{2k+1}}{R_1 - R_2}\right)\left(R_1^{2k+m} + R_2^{2k+m}\right) \\
&= 2\left(\frac{R_1^{4k+m+1} - R_2^{4k+m+1} + R_1^{2k+1}R_2^{2k+m} - R_1^{2k+m}R_2^{2k+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+m+1} - R_2^{4k+m+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k+1}R_2^{2k+m} - R_1^{2k+m}R_2^{2k+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+m+1} - R_2^{4k+m+1}}{R_1 - R_2}\right) - 2\left(R_1R_2\right)^{2k+1}\left(\frac{R_1^{m-1} - R_2^{m-1}}{R_1 - R_2}\right) \\
&= S_{p,q,4k+m} + q^{2k+1}S_{p,q,m-2}
\end{aligned}$$

and

$$\begin{aligned}
S_{p,q,2k+m}L_{p,q,2k} &= 2\left(\frac{R_1^{2k+m+1} - R_2^{2k+m+1}}{R_1 - R_2}\right)\left(R_1^{2k} + R_2^{2k}\right) \\
&= 2\left(\frac{R_1^{4k+m+1} - R_2^{4k+m+1} + R_1^{2k+m+1}R_2^{2k} - R_1^{2k}R_2^{2k+m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+m+1} - R_2^{4k+m+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k+m+1}R_2^{2k} - R_1^{2k}R_2^{2k+m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k+m+1} - R_2^{4k+m+1}}{R_1 - R_2}\right) + 2\left(R_1R_2\right)^{2k}\left(\frac{R_1^{m+1} - R_2^{m+1}}{R_1 - R_2}\right) \\
&= S_{p,q,4k+m} + q^{2k}S_{p,q,m}.
\end{aligned}$$

□

Using equation (1) and Theorem 2.3, we can get the following corollary.

Corollary 2.3.1. *For integers $k \geq 1$ and $m \geq 2$, we have*

1. $S_{1,q,2k}F_{1,q,2k+m} = S_{1,q,2k}S_{1,q,2k+m} - S_{1,q,4k+m} - q^{2k+1}S_{1,q,m-2}$;
2. $S_{1,q,2k+m}F_{1,q,2k} = S_{1,q,2k+m}S_{1,q,2k} - S_{1,q,4k+m} - q^{2k}S_{1,q,m}$.

Theorem 2.4. *For integers k, m such that $4k \geq m$, $k \geq 1$ and $m \geq 2$, we have*

1. $S_{p,q,2k}L_{p,q,2k-m} = S_{p,q,4k-m} + (-q)^{2k-m}S_{p,q,m}$;
2. $S_{p,q,2k-m}L_{p,q,2k} = S_{p,q,4k-m} - (-q)^{2k-m+1}S_{p,q,m-2}$.

Proof. Using Binet's formulas, we have

$$\begin{aligned}
S_{p,q,2k}L_{p,q,2k-m} &= 2\left(\frac{R_1^{2k+1} - R_2^{2k+1}}{R_1 - R_2}\right)\left(R_1^{2k-m} + R_2^{2k-m}\right) \\
&= 2\left(\frac{R_1^{4k-m+1} - R_2^{4k-m+1} + R_1^{2k+1}R_2^{2k-m} - R_1^{2k-m}R_2^{2k+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k-m+1} - R_2^{4k-m+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k+1}R_2^{2k-m} - R_1^{2k-m}R_2^{2k+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k-m+1} - R_2^{4k-m+1}}{R_1 - R_2}\right) + 2\left(R_1R_2\right)^{2k-m}\left(\frac{R_1^{m+1} - R_2^{m+1}}{R_1 - R_2}\right) \\
&= S_{p,q,4k-m} + (-q)^{2k-m}S_{p,q,m}
\end{aligned}$$

and

$$\begin{aligned}
S_{p,q,2k-m}L_{p,q,2k} &= 2\left(\frac{R_1^{2k-m+1} - R_2^{2k-m+1}}{R_1 - R_2}\right)\left(R_1^{2k} + R_2^{2k}\right) \\
&= 2\left(\frac{R_1^{4k-m+1} - R_2^{4k-m+1} + R_1^{2k-m+1}R_2^{2k} - R_1^{2k}R_2^{2k-m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k-m+1} - R_2^{4k-m+1}}{R_1 - R_2}\right) + 2\left(\frac{R_1^{2k-m+1}R_2^{2k} - R_1^{2k}R_2^{2k-m+1}}{R_1 - R_2}\right) \\
&= 2\left(\frac{R_1^{4k-m+1} - R_2^{4k-m+1}}{R_1 - R_2}\right) - 2\left(R_1R_2\right)^{2k-m+1}\left(\frac{R_1^{m-1} - R_2^{m-1}}{R_1 - R_2}\right) \\
&= S_{p,q,4k-m} - (-q)^{2k-m+1}S_{p,q,m-2}.
\end{aligned}$$

□

Using equation (1) and Theorem 2.4, we can get the corollary.

Corollary 2.4.1. *For integers k, m such that $4k \geq m$, $k \geq 1$ and $m \geq 2$, we have*

1. $S_{1,q,2k}F_{1,q,2k-m} = S_{1,q,2k}S_{1,q,2k-m} - S_{1,q,4k-m} - (-q)^{2k-m}S_{1,q,m}$;
2. $S_{1,q,2k-m}F_{1,q,2k} = S_{1,q,2k-m}S_{1,q,2k} - S_{1,q,4k-m} + (-q)^{2k-m+1}S_{1,q,m-2}$.

Theorem 2.5. *For integers $k \geq 1$ and $m \geq 0$, we have*

1. $S_{p,q,2k+m}L_{p,q,2k+m+1} = S_{p,q,4k+2m+1}$;
2. $S_{p,q,2k+m+1}L_{p,q,2k+m} = S_{p,q,4k+2m+1} + 2p(-q)^{2k+m}$.

Proof. Using Binet's formulas, we have

$$\begin{aligned}
S_{p,q,2k+m}L_{p,q,2k+m+1} &= 2\left(\frac{R_1^{2k+m+1}-R_2^{2k+m+1}}{R_1-R_2}\right)\left(R_1^{2k+m+1}+R_2^{2k+m+1}\right)\\
&= 2\left(\frac{R_1^{4k+2m+2}-R_2^{4k+2m+2}+R_1^{2k+m+1}R_2^{2k+m+1}-R_1^{2k+m+1}R_2^{2k+m+1}}{R_1-R_2}\right)\\
&= 2\left(\frac{R_1^{4k+2m+2}-R_2^{4k+2m+2}}{R_1-R_2}\right)+2\left(\frac{R_1^{2k+m+1}R_2^{2k+m+1}-R_1^{2k+m+1}R_2^{2k+m+1}}{R_1-R_2}\right)\\
&= S_{p,q,4k+2m+1}
\end{aligned}$$

and

$$\begin{aligned}
S_{p,q,2k+m+1}L_{p,q,2k+m} &= 2\left(\frac{R_1^{2k+m+2}-R_2^{2k+m+2}}{R_1-R_2}\right)\left(R_1^{2k+m}+R_2^{2k+m}\right)\\
&= 2\left(\frac{R_1^{4k+2m+2}-R_2^{4k+2m+2}+R_1^{2k+m+2}R_2^{2k+m}-R_1^{2k+m}R_2^{2k+m+2}}{R_1-R_2}\right)\\
&= 2\left(\frac{R_1^{4k+2m+2}-R_2^{4k+2m+2}}{R_1-R_2}\right)+2\left(\frac{R_1^{2k+m+2}R_2^{2k+m}-R_1^{2k+m}R_2^{2k+2m+2}}{R_1-R_2}\right)\\
&= 2\left(\frac{R_1^{4k+2m+2}-R_2^{4k+2m+2}}{R_1-R_2}\right)+2\left(R_1R_2\right)^{2k+m}\left(\frac{R_1^2-R_2^2}{R_1-R_2}\right)\\
&= S_{p,q,4k+2m+1}+(-q)^{2k+m}S_{p,q,1}\\
&= S_{p,q,4k+2m+1}+2p(-q)^{2k+m}.
\end{aligned}$$

□

Using equation (1) and Theorem 2.5, we can get the following corollary.

Corollary 2.5.1. *For integers $k \geq 1$ and $m \geq 0$, we have*

1. $S_{1,q,2k+m}F_{1,q,2k+m+1} = S_{1,q,2k+m}S_{1,q,2k+m+1} - S_{1,q,4k+2m+1}$;
2. $S_{1,q,2k+m+1}F_{1,q,2k+m} = S_{1,q,2k+m+1}S_{1,q,2k+m} - S_{1,q,4k+2m+1} - 2p(-q)^{2k+m}$.

Theorem 2.6. *For integers $k \geq 1$ and $m \geq 0$, we have*

1. $S_{p,q,2k+m}L_{p,q,2k+m-1} = S_{p,q,4k+2m-1} + 2p(-q)^{2k+m-1}$;
2. $S_{p,q,2k+m-1}L_{p,q,2k+m} = S_{p,q,4k+2m-1}$.

Proof. Using Binet's formulas, we have

$$\begin{aligned}
S_{p,q,2k+m}L_{p,q,2k+m-1} &= 2\left(\frac{R_1^{2k+m+1}-R_2^{2k+m+1}}{R_1-R_2}\right)\left(R_1^{2k+m-1}+R_2^{2k+m-1}\right)\\
&= 2\left(\frac{R_1^{4k+2m}-R_2^{4k+2m}+R_1^{2k+m+1}R_2^{2k+m-1}-R_1^{2k+m-1}R_2^{2k+m+1}}{R_1-R_2}\right)\\
&= 2\left(\frac{R_1^{4k+2m}-R_2^{4k+2m}}{R_1-R_2}\right)+2\left(\frac{R_1^{2k+m+1}R_2^{2k+m-1}-R_1^{2k+m-1}R_2^{2k+m+1}}{R_1-R_2}\right)\\
&= 2\left(\frac{R_1^{4k+2m}-R_2^{4k+2m}}{R_1-R_2}\right)+2\left(R_1R_2\right)^{2k+m-1}\left(\frac{R_1^2-R_2^2}{R_1-R_2}\right)\\
&= S_{p,q,4k+2m-1}+(-q)^{2k+m-1}S_{p,q,1}\\
&= S_{p,q,4k+2m-1}+2p(-q)^{2k+m-1}
\end{aligned}$$

and

$$\begin{aligned}
S_{p,q,2k+m-1}L_{p,q,2k+m} &= 2\left(\frac{R_1^{2k+m}-R_2^{2k+m}}{R_1-R_2}\right)\left(R_1^{2k+m}+R_2^{2k+m}\right)\\
&= 2\left(\frac{R_1^{4k+2m}-R_2^{4k+2m}+R_1^{2k+m}R_2^{2k+m}-R_1^{2k+m}R_2^{2k+m}}{R_1-R_2}\right)\\
&= 2\left(\frac{R_1^{4k+2m}-R_2^{4k+2m}}{R_1-R_2}\right)+2\left(\frac{R_1^{2k+m}R_2^{2k+m}-R_1^{2k+m}R_2^{2k+m}}{R_1-R_2}\right)\\
&= S_{p,q,4k+2m-1}. \qquad \square
\end{aligned}$$

Using equation (1) and Theorem 2.6, we can get the following corollary.

Corollary 2.6.1. *For integers $k \geq 1$ and $m \geq 0$, we have*

1. $S_{1,q,2k+m}F_{1,q,2k+m-1} = S_{1,q,2k+m}S_{1,q,2k+m-1} - S_{1,q,4k+2m-1} - 2(-q)^{2k+m-1}$;
2. $S_{1,q,2k+m-1}F_{1,q,2k+m} = S_{1,q,2k+m-1}S_{1,q,2k+m} - S_{1,q,4k+2m-1}$.

Acknowledgements

The authors would like to thank the referees for their valuable comments. The author would like to thank the Ubon Ratchathani Rajabhat University for the full support in this research.

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