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Identities on the product of Jacobsthal-like and Jacobsthal–Lucas numbers

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Abstract: The purpose of this paper is to find the identities of Jacobsthal-like and Jacobsthal–Lucas numbers by using Binet's formula.
Keywords: Fibonacci number, Jacobsthal number, Fibonacci-like number, Jacobsthal–Lucas number, Jacobsthal-like number.
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1 Introduction

The Fibonacci numbers [6] F_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, \cdots\}$ wherein each term is the sum of the two previous terms beginning with the initial values $F_0 = 0$ and $F_1 = 1$.

The well-known Fibonacci sequence is defined as $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$ where $F_0 = 0, F_1 = 1$.

In a similar way, Lucas sequence is defined as $L_k = L_{k-1} + L_{k-2}$ for $k \geq 2$ where $L_0 = 2, L_1 = 1$.

Generalized Fibonacci sequence [4] is defined as $F_k = pF_{k-1} + qF_{k-2}, k \geq 2$ with initial condition $F_0 = a, F_1 = b$ where p, q are positive integers and a, b are nonnegative integers.

The Jacobsthal sequence [4] is defined by the recurrence relation

$$J_k = J_{k-1} + 2J_{k-2}, k \geq 2$$

with $J_0 = 0, J_1 = 1$.

Its Binet's formula is defined by

$$J_k = \frac{R_1^k - R_2^k}{R_1 - R_2}.$$

The Jacobsthal–Lucas sequence [5] is defined by the recurrence relation:

$$j_k = j_{k-1} + 2j_{k-2}, k \geq 2$$

with initial condition $j_0 = 2, j_1 = 1$.

Its Binet's formula is defined by

$$j_k = R_1^k + R_2^k.$$

Where $R_1 = 2$ and $R_2 = -1$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

In 2011–2012, Krassimir T. Atanassov studied and generalized about Jacobsthal number see more detail in [1, 2].

In 2011, Bijendra Singh, Omprakash Sikhwal, and Shikha Bhatnagar have studied some identities for even and odd Fibonacci-like and Lucas numbers [3].

In 2013, Yashwant K. Panwar, Bijendra Singh, and V. K. Gupta have presented some identities of common factors of generalized Fibonacci, Jacobsthal and Jacobsthal–Lucas numbers by using Binet's formula derivation [4].

In 2014, Bijendra Singh, Kiran Sisodiya, and Farooq Ahmad have studied the products of k-Fibonacci numbers and k-Lucas numbers see more details in [6].

In 2014, Deepika Jhala, G. P. S. Rathore, and Bijendra Singh have studied some identities involving common factors of k-Fibonacci and k-Lucas numbers see more details in [7].

In this paper, we give a new definition by using the recurrence relation of Jacobsthal sequence [4] and with the initial condition of Fibonacci-like sequence [4]. It is called Jacobsthal-like. We also provide generalized identities on the product of Jacobsthal-like and Jacobsthal–Lucas numbers by using Binet's formula for derivation.

2 Main results

Definition 2.1. *The Jacobsthal-like number is defined by the recurrence relation:*

$$V_k = V_{k-1} + 2V_{k-2}, k \geq 2$$

with initial condition $V_0 = 2, V_1 = 2$.

The Binet's formula is defined by

$$V_k = 2\frac{R_1^{k+1} - R_2^{k+1}}{R_1 - R_2},$$

where $R_1 = 2$ and $R_2 = -1$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

Definition 2.2. *The Jacobsthal–Lucas number [5] is defined by the recurrence relation:*

$$j_k = j_{k-1} + 2j_{k-2}, k \geq 2$$

with initial condition $j_0 = 2, j_1 = 1$.

The Binet's formula is defined by

$$j_k = R_1^k + R_2^k,$$

where $R_1 = 2$ and $R_2 = -1$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

Next, we present some identities on the product of Jacobsthal-like and Jacobsthal–Lucas numbers by using Binet's formula for derivation.

Theorem 2.3.

$$V_{nk+p}j_{mk+q} = \begin{cases} V_{(n+m)k+p+q} + (-2)^{mk+q}V_{(n-m)k+p-q}, n \geq m > 0 \text{ and } p \geq q > 0, \\ V_{(n+m)k+p+q} - (-2)^{nk+p+1}V_{(m-n)k+q-p-2}, m \geq n > 0 \text{ and } q \geq p+2 > 0. \end{cases} \tag{1}$$

where m, n, p *and* q *are nonnegative integers.*

Proof. Case 1. If $n \geq m > 0$ and $p \geq q > 0$, then

$$\begin{aligned} V_{nk+p}j_{mk+q} &= 2\Big(\frac{R_1^{nk+p+1} - R_2^{nk+p+1}}{R_1 - R_2}\Big)\big(R_1^{mk+q} + R_2^{mk+q}\big) \\ &= 2\Big(\frac{R_1^{(n+m)k+p+q+1} + (R_1^{nk+p+1}R_2^{mk+q}) - (R_2^{nk+p+1}R_1^{mk+q}) - R_2^{(n+m)k+p+q+1}}{R_1 - R_2}\Big) \\ &= 2\Big(\frac{R_1^{(n+m)k+p+q+1} - R_2^{(n+m)k+p+q+1} + (R_1R_2)^{mk+q}(R_1^{(n-m)k+p-q+1} - R_2^{(n-m)k+p-q+1})}{R_1 - R_2}\Big) \\ &= 2\Big(\frac{R_1^{(n+m)k+p+q+1} - R_2^{(n+m)k+p+q+1}}{R_1 - R_2}\Big) + (-2)^{mk+q}2\Big(\frac{R_1^{(n-m)k+p-q+1} - R_2^{(n-m)k+p-q+1}}{R_1 - R_2}\Big) \\ &= V_{(n+m)k+p+q} + (-2)^{mk+q}V_{(n-m)k+p-q}. \end{aligned}$$

Case 2. If $m \geq n > 0$ and $q \geq p + 2 > 0$, then

$$\begin{aligned} V_{nk+p}j_{mk+q} &= 2\Big(\frac{R_1^{nk+p+1} - R_2^{nk+p+1}}{R_1 - R_2}\Big)\big(R_1^{mk+q} + R_2^{mk+q}\big) \\ &= 2\Big(\frac{R_1^{(n+m)k+p+q+1} + (R_1^{nk+p+1}R_2^{mk+q}) - (R_2^{nk+p+1}R_1^{mk+q}) - R_2^{(n+m)k+p+q+1}}{R_1 - R_2}\Big) \\ &= 2\Big(\frac{R_1^{(n+m)k+p+q+1} - R_2^{(n+m)k+p+q+1} - (R_1R_2)^{nk+p+1}(R_1^{(m-n)k+q-p-1} - R_2^{(m-n)k+q-p-1})}{R_1 - R_2}\Big) \\ &= 2\Big(\frac{R_1^{(n+m)k+p+q+1} - R_2^{(n+m)k+p+q+1}}{R_1 - R_2}\Big) - (-2)^{nk+p+1}2\Big(\frac{R_1^{(m-n)k+q-p-1} - R_2^{(m-n)k+q-p-1}}{R_1 - R_2}\Big) \\ &= V_{(n+m)k+p+q} - (-2)^{nk+p+1}V_{(m-n)k+q-p-2}. \end{aligned}$$

□

Corollary 2.3.1. *For different values of n, m, p and q (1), it can be expressed:*

- *If $n = 2, m = 1, p = 3$ and $q = 1$, then $V_{2k+3}j_{k+1} = V_{3k+4} + (-2)^{k+1}V_{k+2}$.*
- *If $n = 3, m = 3, p = 2$ and $q = 2$, then $V_{3k+2}j_{3k+2} = V_{6k+4} + 2(-2)^{3k+2}$.*
- *If $n = 2, m = 4, p = 0$ and $q = 2$, then $V_{2k}j_{4k+2} = V_{6k+2} - (-2)^{2k+1}V_{2k}$.*

Theorem 2.4. *Let m, n, p and q be nonnegative integers, then*

$$(\frac{R_1 - R_2}{2})^2 V_{nk+p}V_{mk+q} = j_{(n+m)k+p+q+2} - (-2)^{mk+q+1}j_{(n-m)k+p-q} \tag{2}$$

such that $n \geq m > 0$ and $p \geq q > 0$.

Proof.

$$\begin{aligned}
&(\frac{R_1 - R_2}{2})^2 V_{nk+p}V_{mk+q} \\
&\quad =(\frac{R_1 - R_2}{2})^2 2(\frac{R_1^{nk+p+1} - R_2^{nk+p+1}}{R_1 - R_2})2(\frac{R_1^{mk+q+1} - R_2^{mk+q+1}}{R_1 - R_2}) \\
&\quad =R_1^{(n+m)k+p+q+2} - R_1^{nk+p+1}R_2^{mk+q+1} - R_2^{nk+p+1}R_1^{mk+q+1} + R_2^{(n+m)k+p+q+2} \\
&\quad =\big(R_1^{(n+m)k+p+q+2} + R_2^{(n+m)k+p+q+2}\big) - (R_1R_2)^{mk+q+1}\big(R_1^{(n-m)k+p-q} + R_2^{(n-m)k+p-q}\big) \\
&\quad =j_{(n+m)k+p+q+2} - (-2)^{mk+q+1}j_{(n-m)k+p-q}.
\end{aligned}$$

□

Corollary 2.4.1. *For different values of n, m, p and q (2), it can be expressed:*

- *If $n = 2, m = 1, p = 3$ and $q = 1$, then $\big(\frac{R_1-R_2}{2}\big)^2 V_{2k+3}V_{k+1} = j_{3k+6} - (-2)^{k+2}j_{k+2}$.*
- *If $n = 3, m = 3, p = 2$ and $q = 2$, then $\big(\frac{R_1-R_2}{2}\big)^2 V_{3k+2}V_{3k+2} = j_{6k+6} - 2(-2)^{3k+3}$.*
- *If $n = 2, m = 4, p = 0$ and $q = 2$, then $\big(\frac{R_1-R_2}{2}\big)^2 V_{2k}V_{4k+2} = j_{6k+4} - (-2)^{2k+1}j_{2k+2}$.*

Theorem 2.5. *Let m, n, p and q are nonnegative integers, then*

$$j_{nk+p}j_{mk+q} = j_{(n+m)k+p+q} + (-2)^{mk+q}j_{(n-m)k+p-q} \tag{3}$$

such that $n \geq m > 0$ and $p \geq q > 0$.

Proof.

$$\begin{aligned}
j_{nk+p}j_{mk+q} &= \big(R_1^{nk+p} + R_2^{nk+p}\big)\big(R_1^{mk+q} + R_2^{mk+q}\big) \\
&= \big(R_1^{(n+m)k+p+q} + R_1^{nk+p}R_2^{mk+q} + R_2^{nk+p}R_1^{mk+q} + R_2^{(n+m)k+p+q}\big) \\
&= \big(R_1^{(n+m)k+p+q} + R_2^{(n+m)k+p+q}\big) + (R_1R_2)^{mk+q}\big(R_1^{(n-m)k+p-q} + R_2^{(n-m)k+p-q}\big) \\
&= j_{(n+m)k+p+q} + (-2)^{mk+q}j_{(n-m)k+p-q}.
\end{aligned}$$

□

Corollary 2.5.1. *For different values of n, m, p and q (3), it can be expressed*

- *If $n = 2, m = 1, p = 3$ and $q = 1$, then $j_{2k+3}j_{k+1} = j_{3k+4} + (-2)^{k+1}j_{k+2}$.*
- *If $n = 3, m = 3, p = 2$ and $q = 2$, then $j_{3k+2}j_{3k+2} = j_{6k+4} + 2(-2)^{3k+2}$.*
- *If $n = 2, m = 4, p = 0$ and $q = 2$, then $j_{2k}j_{4k+2} = j_{6k+2} + (-2)^{2k}j_{2k+2}$.*

Theorem 2.6. *Let m, n, p, q, r and s are nonnegative integers, then*

$$\begin{aligned} V_{nk+p}j_{mk+q}j_{sk+r} = V_{(n+m+s)k+p+q+r} + (-2)^{sk+r}V_{(n+m-s)k+p+q-r} \\ + (-2)^{mk+q}V_{(n+s-m)k+p+r-q} + (-2)^{(m+s)k+q+r}V_{(n-m-s)k+p-q-r} \end{aligned} \tag{4}$$

such that $n \geq m + s > 0$ and $p \geq q + r > 0$.

Proof.

$$\begin{aligned} V_{nk+p}j_{mk+q}j_{sk+r} &= 2\Big(\frac{R_1^{nk+p+1} - R_2^{nk+p+1}}{R_1 - R_2}\Big)\big(R_1^{mk+q} + R_2^{mk+q}\big)\big(R_1^{sk+r} + R_2^{sk+r}\big) \\ &= 2\Big(\frac{R_1^{nk+p+1} - R_2^{nk+p+1}}{R_1 - R_2}\Big)\big(R_1^{(m+s)k+q+r} + R_1^{mk+q}R_2^{sk+r} + R_2^{mk+q}R_1^{sk+r} + R_2^{(m+s)k+q+r}\big) \\ &= 2\Big[\Big(\frac{R_1^{(m+s+n)k+p+q+r+1} - R_2^{(m+s+n)k+p+q+r+1}}{R_1 - R_2}\Big) \\ &\quad + \Big(\frac{R_1^{(n+m)k+p+q+1}R_2^{sk+r} - R_2^{(n+m)k+p+q+1}R_1^{sk+r}}{R_1 - R_2}\Big) \\ &\quad + \Big(\frac{R_1^{(n+s)k+p+r+1}R_2^{mk+q} - R_1^{mk+q}R_2^{(n+s)k+p+r+1}}{R_1 - R_2}\Big) \\ &\quad + \Big(\frac{R_1^{nk+p+1}R_2^{(m+s)k+q+r} - R_1^{(m+s)k+q+r}R_2^{nk+p+1}}{R_1 - R_2}\Big)\Big] \\ &= 2\Big[\Big(\frac{R_1^{(m+s+n)k+p+q+r+1} - R_2^{(m+s+n)k+p+q+r+1}}{R_1 - R_2}\Big) \\ &\quad + (R_1R_2)^{sk+r}\Big(\frac{R_1^{(m+n-s)k+p+q-r+1} - R_2^{(m+n-s)k+p+q-r+1}}{R_1 - R_2}\Big) \\ &\quad + (R_1R_2)^{mk+q}\Big(\frac{R_1^{(n+s-m)k+p+r-q+1} - R_2^{(n+s-m)k+p+r-q+1}}{R_1 - R_2}\Big) \\ &\quad + (R_1R_2)^{(m+s)k+q+r}\Big(\frac{R_1^{(n-m-s)kp-q-r+1} - R_2^{(n-m-s)k+p-q-r+1}}{R_1 - R_2}\Big)\Big] \\ &= 2\Big(\frac{R_1^{(m+s+n)k+p+q+r+1} - R_2^{(m+s+n)k+p+q+r+1}}{R_1 - R_2}\Big) \\ &\quad + (-2)^{sk+r}2\Big(\frac{R_1^{(m+n-s)k+p+q-r+1} - R_2^{(m+n-s)k+p+q-r+1}}{R_1 - R_2}\Big) \\ &\quad + (-2)^{mk+q}2\Big(\frac{R_1^{(n+s-m)k+p+r-q+1} - R_2^{(n+s-m)k+p+r-q+1}}{R_1 - R_2}\Big) \\ &\quad + (-2)^{(m+s)k+q+r}2\Big(\frac{R_1^{(n-m-s)k+p-q-r+1} - R_2^{(n-m-s)k+p-q-r+1}}{R_1 - R_2}\Big) \end{aligned}$$

$$=V_{(n+m+s)k+p+q+r}+(-2)^{sk+r}V_{(n+m-s)k+p+q-r}+(-2)^{mk+q}V_{(n+s-m)k+p+r-q}$$
$$+(-2)^{(m+s)k+q+r}V_{(n-m-s)k+p-q-r}. \qquad \square$$

Corollary 2.6.1. *For different values of n, m, s, p, q and r (4) can be expressed*

- *If $n=3, m=1, s=1, p=3, q=2$ and $r=1$, then*
$$V_{3k+3}j_{k+2}j_{k+1}=V_{5k+6}+(-2)^{k+1}V_{3k+4}+(-2)^{k+2}V_{3k+2}+(-2)^{2k+3}V_k.$$
- *If $n=3, m=1, s=2, p=2, q=1$ and $r=1$, then*
$$V_{3k+2}j_{k+1}j_{2k+1}=V_{6k+4}+(-2)^{2k+1}V_{2k+2}+(-2)^{k+1}V_{4k+2}+2(-2)^{3k+2}.$$

Theorem 2.7. *Let m, n, p, q, r and s are nonnegative integers,*

$$\begin{aligned}(\frac{R_1-R_2}{2})^2V_{nk+p}V_{mk+q}j_{sk+r}=&j_{(n+m+s)k+p+q+r+2}-(-2)^{mk+q+1}j_{(n+s-m)k+p+r-q}\\&-(-2)^{nk+p+1}j_{(m+s-n)k+q+r-p}\\&+(-2)^{(n+m)k+p+q+2}j_{(s-n-m)k+r-p-q-2}\end{aligned} \tag{5}$$

such that $s\geq n+m>0$ and $r\geq p+q+2>0$.

Proof.

$$\begin{aligned}&(\frac{R_1-R_2}{2})^2V_{nk+p}V_{mk+q}j_{sk+r}\\
&=(\frac{R_1-R_2}{2})^2 2(\frac{R_1^{nk+p+1}-R_2^{nk+p+1}}{R_1-R_2})2(\frac{R_1^{mk+q+1}-R_2^{mk+q+1}}{R_1-R_2})(R_1^{sk+r}+R_2^{sk+r})\\
&=(R_1^{nk+p+1}-R_2^{nk+p+1})(R_1^{mk+q+1}-R_2^{mk+q+1})(R_1^{sk+r}+R_2^{sk+r})\\
&=R_1^{(n+m+s)k+p+q+r+2}-R_1^{(n+s)k+p+r+1}R_2^{mk+q+1}-R_2^{nk+p+1}R_1^{(m+s)k+q+r+1}\\
&\quad+R_2^{(n+m)k+p+q+2}R_1^{sk+r}+R_1^{(n+m)k+p+q+2}R_2^{sk+r}-R_1^{nk+p+1}R_2^{(m+s)k+q+r+1}\\
&\quad-R_2^{(n+s)k+p+r+1}R_1^{mk+q+1}+R_2^{(n+m+s)k+p+q+r+2}\\
&=(R_1^{(n+m+s)k+p+q+r+2}+R_2^{(n+m+s)k+p+q+r+2})\\
&\quad-(R_1^{(n+s)k+p+r+1}R_2^{mk+q+1}+R_2^{(n+s)k+p+r+1}R_1^{mk+q+1})\\
&\quad-(R_2^{nk+p+1}R_1^{(m+s)k+q+r+1}+R_1^{nk+p+1}R_2^{(m+s)k+q+r+1})\\
&\quad+(R_2^{(n+m)k+p+q+2}R_1^{sk+r}+R_1^{(n+m)k+p+q+2}R_2^{sk+r})\\
&=(R_1^{(n+m+s)k+p+q+r+2}+R_2^{(n+m+s)k+p+q+r+2})\\
&\quad-(R_1R_2)^{mk+q+1}(R_1^{(n+s-m)k+p+r-q}+R_2^{(n+s-m)k+p+r-q})\\
&\quad-(R_1R_2)^{nk+p+1}(R_1^{(m+s-n)k+q+r-p}+R_2^{(m+s-n)k+q+r-p})\\
&\quad+(R_1R_2)^{(n+m)k+p+q+2}(R_1^{(s-n-m)k+r-p-q-2}+R_2^{(s-n-m)k+r-p-q-2})\\
&=j_{(n+m+s)k+p+q+r+2}-(-2)^{mk+q+1}j_{(n+s-m)k+p+r-q}-(-2)^{nk+p+1}j_{(m+s-n)k+q+r-p}\\
&\quad+(-2)^{(n+m)k+p+q+2}j_{(s-n-m)k+r-p-q-2}.\end{aligned} \qquad \square$$

Corollary 2.7.1. *For different values of n, m, s, p, q and r. (5), it can be expressed:*

- *If $n = 1, m = 2, s = 3, p = 1, q = 1$ and $r = 4$, then*

$$(\frac{R_1 - R_2}{2})^2 V_{k+1} V_{2k+1} j_{3k+4} = j_{6k+8} - (-2)^{2k+2} j_{2k+4} - (-2)^{k+2} j_{4k+4} + 2(-2)^{3k+4}.$$

- *If $n = 1, m = 1, s = 3, p = 2, q = 1$ and $r = 5$, then*

$$(\frac{R_1 - R_2}{2})^2 V_{k+2} V_{k+1} j_{3k+5} = j_{5k+10} - (-2)^{k+2} j_{3k+6} - (-2)^{k+3} j_{3k+4} + (-2)^{2k+5} j_k.$$

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