On the Generalized Fibonacci-circulant-Hurwitz numbers

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Abstract: The theory of Fibonacci-circulant numbers was introduced by Deveci et al. (see [5]). In this paper, we define the Fibonacci-circulant-Hurwitz sequence of the second kind by Hurwitz matrix of the generating function of the Fibonacci-circulant sequence of the second kind and give a fair generalization of the sequence defined, which we call the generalized Fibonacci-circulant-Hurwitz sequence. First, we derive relationships between the generalized Fibonacci-circulant-Hurwitz numbers and the generating matrices for these numbers. Also, we give miscellaneous properties of the generalized Fibonacci-circulant-Hurwitz numbers such as the Binet formula, the combinatorial, permanental, determinantal representations, the generating function, the exponential representation and the sums.

Keywords: Fibonacci-circulant-Hurwitz Sequence, Circulant matrix, Hurwitz matrix, Representation.

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1 Introduction

The $k$-step Fibonacci sequence $\{F_n^k\}$ is defined by initial values $F_0^k = F_1^k = F_{k-2}^k = 0$, $F_{k-1}^k = 1$ and recurrence relation

$$F_{n+k}^k = F_{n+k-1}^k + F_{n+k-2}^k + \cdots + F_n^k$$

for $n \geq 0$.

For detailed information about the $k$-step Fibonacci sequence, see [9, 21].

In [5], Deveci et al. defined the Fibonacci-circulant sequence of the second kind as shown:

$$x_2^1 = \cdots = x_4^2 = 0, \quad x_5^2 = 1 \quad \text{and} \quad x_n^2 = -x_{n-3}^2 + x_{n-4}^2 - x_{n-5}^2$$

for $n \geq 6$.

Note that the characteristic polynomial of the Fibonacci-circulant sequence of the second kind is as follows:

$$f(x) = -x^5 + x^2 + x - 1.$$  

Let an $n$-th degree real polynomial $f$ be given by

$$f^2(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n.$$  

In [8], the Hurwitz matrix $H_n = [h_{i,j}]_{n \times n}$ associated to the polynomial $f$ was defined as shown:

$$H_n = \begin{bmatrix}
  c_1 & c_3 & c_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
  c_0 & c_2 & c_4 & \cdots & \cdots & \cdots & & & \\
  0 & c_1 & c_3 & \cdots & \cdots & \cdots & & & \\
  : & c_0 & c_2 & \ddots & \ddots & \ddots & 0 & \ddots & \\
  : & 0 & c_1 & \ddots & \ddots & \ddots & c_n & \ddots & \\
  : & : & c_0 & \ddots & \ddots & \ddots & c_{n-1} & 0 & \ddots \\
  : & : & 0 & \cdots & \cdots & \cdots & c_{n-2} & c_n & \ddots \\
  : & : & : & \cdots & \cdots & \cdots & c_{n-3} & c_{n-1} & 0 \\
  0 & 0 & 0 & \cdots & \cdots & \cdots & c_{n-4} & c_{n-2} & c_n
\end{bmatrix}$$

Consider the $k$-step homogeneous linear recurrence sequence $\{a_n\}$,

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [9], Kalman derived a number of closed-form formulas for the sequence $\{a_n\}$ by matrix method as follows:

$$A^n \begin{bmatrix}
  a_0 \\
  a_1 \\
  \vdots \\
  a_{k-1}
\end{bmatrix} = \begin{bmatrix}
  a_n \\
  a_{n+1} \\
  \vdots \\
  a_{n+k-1}
\end{bmatrix}.$$
where

\[
A = [a_{ij}]_{k \times k} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
c_0 & c_1 & c_2 & c_{k-2} & c_{k-1}
\end{bmatrix}.
\]

Number theoretic properties such as these obtained from Fibonacci numbers relevant to this paper have been studied by many authors [1, 4, 7, 11, 12, 20, 23, 27, 28]. Now we define the generalized Fibonacci-circulant-Hurwitz numbers and then, we obtain their miscellaneous properties using the generating matrix and the generating function of these numbers.

2 Significance

As it is well-known that recurrence sequences, circulant matrix and Hurwitz matrix appear in modern research in many fields from mathematics, physics, computer science, architecture to nature and art (see, for example, [6, 10, 13, 14, 17, 18, 22, 24, 25, 26]). This paper is expanded the concept to the generalized Fibonacci-circulant-Hurwitz sequence which is defined by using circulant and Hurwitz matrices.

3 The main results

By the polynomial \( f^2(x) \), we can write the following Hurwitz matrix:

\[
M^2 = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1
\end{bmatrix}.
\]

Using the matrix \( M^2 \), we define the Fibonacci-circulant-Hurwitz sequence of the second kind as shown:

\[
a_1^2 = \cdots = a_4^2 = 0, \ a_5^2 = 1 \text{ and } a_{n+1}^2 = -a_n^2 + a_{n-1}^2 + a_{n-2}^2 + a_{n-4}^2 \text{ for } n \geq 5.
\]

Now we consider a new sequence which is a generalized form of the the Fibonacci-circulant-Hurwitz sequence of the second kind and is called the generalized Fibonacci-circulant-Hurwitz sequence. The sequence is defined by integer constants \( a_1^k = \cdots = a_{k-1}^k = 0, \ a_k^k = 1 \) and the recurrence relation

\[
a_{n+1}^k = -a_n^k + a_{n-1}^k + \cdots + a_{n-k+1}^k + a_{n-k+3}^k + a_{n-k+1}^k \tag{1}
\]

for \( n \geq k \), where \( k \) is a positive integer such that \( k \geq 4 \).
From (1), we may write the following matrix:

\[
M_k = [m_{i,j}]_{k \times k} = \begin{bmatrix}
-1 & 1 & 1 & \cdots & 1 & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{bmatrix}.
\]

(2)

The matrix \( M_k \) is called the generalized Fibonacci-circulant-Hurwitz matrix.

Note that \( \det(M_k) = (-1)^{k+1} \) for \( k \geq 4 \).

By induction on \( n \), we get

\[
(M_4)^n = \begin{bmatrix}
a_{n+4}^4 & a_{n+3}^4 + a_{n+1}^4 & a_{n+2}^4 & a_{n+3}^4 \\
a_{n+3}^4 & a_{n+2}^4 + a_n^4 & a_{n+1}^4 & a_{n+2}^4 \\
a_{n+2}^4 & a_{n+1}^4 + a_n^4 & a_n^4 & a_{n+1}^4 \\
a_{n+1}^4 & a_n^4 + a_{n-1}^4 & a_{n-1}^4 & a_n^4 \\
\end{bmatrix},
\]

\[
(M_5)^n = \begin{bmatrix}
a_{n+5}^5 & a_{n+4}^5 + a_{n+3}^5 & a_{n+2}^5 & a_{n+3}^5 & a_{n+4}^5 \\
a_{n+4}^5 & a_{n+3}^5 + a_{n+2}^5 & a_{n+1}^5 & a_{n+2}^5 & a_{n+3}^5 \\
a_{n+3}^5 & a_{n+2}^5 + a_{n+1}^5 & a_{n+1}^5 & a_{n+2}^5 & a_{n+3}^5 \\
a_{n+2}^5 & a_{n+1}^5 + a_n^5 & a_n^5 & a_{n+1}^5 & a_{n+2}^5 \\
a_{n+1}^5 & a_n^5 + a_{n-1}^5 & a_{n-1}^5 & a_n^5 & a_{n+1}^5 \\
\end{bmatrix},
\]

and

\[
(M_k)^n = \begin{bmatrix}
a_{n+k}^k & a_{n+k+1}^k + a_{n+k}^k & a_{n+k-1}^k + a_{n+k-3}^k & a_{n+k-2}^k & a_{n+k-1}^k \\
a_{n+k-1}^k & a_{n+k}^k + a_{n+k-1}^k & a_{n+k-2}^k + a_{n+k-4}^k & a_{n+k-3}^k & a_{n+k-2}^k \\
a_{n+k-2}^k & a_{n+k-1}^k + a_{n+k-2}^k & a_{n+k-3}^k + a_{n+k-5}^k & a_{n+k-4}^k & a_{n+k-3}^k \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n+1}^k & a_{n+2}^k + a_{n+1}^k & a_n^k + a_{n-2}^k & a_{n-1}^k & a_n^k \\
\end{bmatrix}.
\]

(3)

for \( k \geq 6 \), where \((M_k)^*\) is a matrix with \( k \) row and \( k-5 \) column given below:

\[
\begin{bmatrix}
a_{n+k-1}^k + \cdots + a_{n+4}^k & a_{n+k-2}^k + \cdots + a_{n+3}^k & a_{n+k-1}^k & a_{n+k-2}^k \\
a_{n+k-2}^k + \cdots + a_{n+3}^k & a_{n+k-2}^k + \cdots + a_{n+3}^k & a_{n+k-1}^k & a_{n+k-2}^k \\
\vdots & \vdots & \ddots & \vdots \\
a_n^k + \cdots + a_{n-k+4}^k & a_n^k + \cdots + a_{n-k+4}^k & a_n^k + \cdots + a_{n-k+4}^k & a_n^k + a_{n-1}^k + a_{n-3}^k \\
\end{bmatrix}.
\]

Lemma 3.1. The characteristic equation of all the generalized Fibonacci-circulant-Hurwitz numbers \( x^k + x^{k-1} - x^{k-2} - \cdots - x^2 - 1 = 0 \) does not have multiple roots for \( k \geq 4 \).
Proof. Let \( f(x) = x^k + x^{k-1} - x^{k-2} - \cdots - x^2 - 1 \). We easily see that \( f(1) \neq 1 \). Consider \( h(x) = (x - 1) f(x) \). Since \( f(1) \neq 1, 1 \) is root but not a multiple root of \( h(x) \). Assume that \( u \) a multiple root of \( h(x) \). Then \( h(u) = 0 \) and \( h'(u) = 0 \). So we get

\[(1 - k) u^4 + ku^3 + (k - 7) u^2 + (4 - 2k) u + 2 (k - 1) = 0.\]

Using appropriate softwares such as Wolfram Mathematica 10.0 [29], one can see that this last equation does not have a solution which is a contradiction. This contradiction proves that the equation \( f(x) \) does not have multiple roots. \( \square \)

If \( x_1, x_2, \ldots, x_k \) are the eigenvalues of the generalized Fibonacci-circulant-Hurwitz matrix \( M_k \), then by Lemma 3.1, it is known that \( x_1, x_2, \ldots, x_k \) are distinct. Let a \( k \times k \) Vandermonde matrix \( V^k \) be given by

\[
V^k = \begin{bmatrix}
(x_1)^{k-1} & (x_2)^{k-1} & \cdots & (x_k)^{k-1} \\
(x_1)^{k-2} & (x_2)^{k-2} & \cdots & (x_k)^{k-2} \\
\vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & \cdots & x_k \\
1 & 1 & \cdots & 1
\end{bmatrix}.
\]

Now assume that \( W^k (i) \) is a \( (p + 2) \times 1 \) matrix as shown:

\[
W^k (i) = \begin{bmatrix}
(x_1)^{n+k-i} \\
(x_2)^{n+k-i} \\
\vdots \\
(x_{p+2})^{n+k-i}
\end{bmatrix}
\]

and \( V^k (i, j) \) is a \( k \times k \) matrix derived from the Vandermonde matrix \( V^k \) by replacing the \( j \)-th column of \( V^k \) by matrix \( W^k (i) \).

Now we give the Binet formulas for the generalized Fibonacci-circulant-Hurwitz numbers by the following Theorem.

**Theorem 3.1.** Let \( k \) be a positive integer such that \( k \geq 4 \) and let \( (M_k)^\alpha = \left[ m_{i,j}^{(\alpha)} \right] \) for \( \alpha \geq 1, \) then

\[
m_{i,j}^{(\alpha)} = \frac{\det V^k (i, j)}{V^k}.
\]

**Proof.** Since the eigenvalues of the generalized Fibonacci-circulant-Hurwitz matrix \( M_k \) are distinct, \( M_k \) is diagonalizable. Then, we may write \( M_k V^k = V^k D_k, \) where \( D_k = \text{diag}(x_1, x_2, \ldots, x_k) \). Since \( \det V^k \neq 0 \), we get

\[
(V^k)^{-1} M_k V^k = D_k.
\]

It will thus be seen that the matrices \( M_k \) and \( D_k \) are similar. Then we can write the matrix equation \((M_k)^\alpha V^k = V^k (D_k)^\alpha\) for \( \alpha \geq 1. \) Since \( (M_k)^\alpha = \left[ m_{i,j}^{(\alpha)} \right], \) we get

\[
\begin{align*}
m_{i,1}^{(\alpha)} (x_1)^{k-1} &+ m_{i,2}^{(\alpha)} (x_1)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_1)^{\alpha+k-i} \\
m_{i,1}^{(\alpha)} (x_2)^{k-1} &+ m_{i,2}^{(\alpha)} (x_2)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_2)^{\alpha+k-i} \\
\vdots & \\
m_{i,1}^{(\alpha)} (x_k)^{k-1} &+ m_{i,2}^{(\alpha)} (x_k)^{k-2} + \cdots + m_{i,k}^{(\alpha)} = (x_k)^{\alpha+k-i}
\end{align*}
\]

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So we conclude that
\[ m_{i,j}^{(\alpha)} = \frac{\det V^k(i,j)}{V_k^k} \]
for each \( i, j = 1, 2, \ldots, k \).

Thus by Theorem 3.1 and the matrix \((M_k)^n\), we have the following useful results.

**Corollary 3.1.** Let \( a_n^k \) be the \( n \)-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then
\[ a_n^k = \frac{\det V^k(k,k)}{V_k^k} = \frac{\det V^k(k-1,k-1)}{V_k^k} \]
for \( k \geq 4 \).

Now we consider the combinatorial representations for all the generalized Fibonacci-circulant-Hurwitz numbers.

Let a \( k \times k \) companion matrix \( C(c_1, c_2, \ldots, c_k) \) be given by
\[
C(c_1, c_2, \ldots, c_k) = \begin{bmatrix}
  c_1 & c_2 & \cdots & c_k \\
  1 & 0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 1 & 0
\end{bmatrix}.
\]

For more details on the companion type matrices, see [15, 16].

**Theorem 3.2** (Chen and Louck [3]). The \((i,j)\) entry \( c_{i,j}^{(\alpha)}(c_1, c_2, \ldots, c_k) \) in the matrix \( C^{\alpha}(c_1, c_2, \ldots, c_k) \) is given by the following formula:
\[
c_{i,j}^{(\alpha)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1,t_2,\ldots,t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} c_1^{t_1} \cdots c_k^{t_k} \quad (4)
\]
where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + kt_k = \alpha - i + j \), \( \binom{t_1 + \cdots + t_k}{t_1,\ldots,t_k} = \frac{(t_1 + \cdots + t_k)!}{t_1! \cdots t_k!} \) is a multinomial coefficient, and the coefficients in (4) are defined to be 1 if \( \alpha = i - j \).

**Corollary 3.2.** Let \( k \) be a positive integer such that \( k \geq 4 \) and let \( a_n^k \) be the \( n \)-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then
\[
a_n^k = \sum_{(t_1,t_2,\ldots,t_k)} \frac{t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} = \sum_{(t_1,t_2,\ldots,t_{p+2})} \frac{t_{k-1} + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k}
\]
where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + kt_k = n \).

**Proof.** In Theorem 3.2, if we choose \( i = j = k \) and \( i = j = k - 1 \), then the proof is immediately seen from (3).
Definition 3.1. An $u \times v$ real matrix $A = [a_{i,j}]$ is called a contractible matrix in the $n$-th column (resp. row) if the $n$-th column (resp. row) contains exactly two non-zero entries.

Let $x_1, x_2, \ldots, x_u$ be row vectors of the matrix $A$. If $A$ is contractible in the $n$-th column such that $a_{\tau,n} \neq 0, a_{\sigma,n} \neq 0$ and $\tau \neq \sigma$, then the $(u - 1) \times (v - 1)$ matrix $A_{\tau,\sigma;n}$ obtained from $A$ by replacing the $\tau$-th row with $a_{\tau,n}x_\sigma + a_{\sigma,n}x_\tau$ and deleting the $\sigma$-th row. We call the $n$-th column the contraction in the $n$-th column relative to the $\tau$-th row and the $\sigma$-th row.

In [2], it was shown that $\text{per}(A) = \text{per}(B)$ if $A$ is a real matrix of order $u > 1$ and the matrix $B$ is a contraction of $A$.

Let $u \geq k$ and let a $u \times u$ super-diagonal matrix $N^k_u = [n^p_{i,j}]$ be given by

$$n^p_{i,j} = \begin{cases} 1 & \text{if } i = s \text{ and } j = s + 1 \text{ for } 1 \leq s \leq u - 1, \\ 1 & \text{if } i = s \text{ and } j = s + 2 \text{ for } 1 \leq s \leq u - 2, \\ \vdots & \vdots \\ 1 & \text{if } i = s \text{ and } j = s + k - 3 \text{ for } 1 \leq s \leq u - k + 3, \\ i = s \text{ and } j = s + k - 1 \text{ for } 1 \leq s \leq u - k + 1 & \text{and} \\ i = s + 1 \text{ and } j = s \text{ for } 1 \leq s \leq u - 1, \\ -1 & \text{if } i = s \text{ and } j = s \text{ for } 1 \leq s \leq u, \\ 0 & \text{otherwise}, \end{cases}$$

where $k \geq 4$.

Now we give the permanental representations for the generalized Fibonacci-circulant-Hurwitz numbers by the following Theorems.

Theorem 3.3. Let $a_n$ be the $n$-th element of the generalized Fibonacci-circulant-Hurwitz sequence, then

$$\text{per}(N^k_u) = a^{k}_{u+k}$$

for $u \geq k$.

Proof. The assertion may be proved by induction on $u$. Assume that the result hold for any integer grater than or equal to $k$. Then we show the equation holds for $u + 1$. Expanding the $\text{per}(N^k_u)$ by the Laplace expansion of permanent according to the first row gives us

$$\text{per}(N^k_{u+1}) = -\text{per}(N^k_u) + \text{per}(N^k_{u-1}) + \cdots + \text{per}(N^k_{u-k+3}) + \text{per}(N^k_{u-k+1}).$$

Since

$$\text{per}(N^k_u) = a^{k}_{u+k}, \text{ per}(N^k_{u-1}) = a^{k}_{u+k-1}, \ldots, \text{ per}(N^k_{u-k+3}) = a^{k}_{u+3}, \text{ per}(N^k_{u-k+1}) = a^{k}_{u+1},$$

by using the recurrence relation of the generalized Fibonacci circulant-Hurwitz numbers, we obtain $\text{per}(N^k_{u+1}) = a^{k}_{u+k+1}$. \hfill \square
Suppose that \( u > k \) and the \( u \times u \) matrices \( H^k_u = [h^k_{i,j}] \) and \( T^k_u = [t^k_{i,j}] \) are defined by

\[
h^k_{i,j} = \begin{cases} 
1 & \text{if } i = s \text{ and } j = s + \rho \text{ for } 1 \leq s \leq u - k + 2, \\
& \text{and } 1 \leq \rho \leq k - 3, \\
1 & \text{if } i = s \text{ and } j = s + k - 1 \text{ for } 1 \leq s \leq u - k + 1 \\
& \text{and } 1 \leq s \leq u - 1, \\
-1 & \text{if } i = s + 1 \text{ and } j = s \text{ for } 1 \leq s \leq u - k + 1, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
T^k_u = \begin{bmatrix} 
(u - k) - \text{th} \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & 0 & H^k_{u-1} \\
\vdots \\
0
\end{bmatrix},
\]

\( k \geq 4 \).

Using the matrices \( H^k_u = [h^k_{i,j}] \) and \( T^k_u = [t^k_{i,j}] \) and the above results we can give more general permanental representations.

**Theorem 3.4.** For \( u > k \),

\[
\text{per} \left( H^k_u \right) = a^k_u,
\]

and

\[
\text{per} \left( T^k_u \right) = \sum_{\tau=0}^{u-1} a^k_{\tau}.
\]

**Proof.** Consider the first part of the theorem. We prove this by the induction method. Suppose that the equation holds for \( u > k \), then we show that the equation holds for \( u + 1 \). If we expand the \( \text{per} \left( H^k_u \right) \) by the Laplace expansion of permanent according to the first row, then we get

\[
\text{per} \left( H^k_{u+1} \right) = -\text{per} \left( H^k_u \right) + \text{per} \left( H^k_{u-1} \right) + \cdots + \text{per} \left( H^k_{u-k+3} \right) + \text{per} \left( H^k_{u-k+1} \right)
\]

\[
= -a^k_u + a^k_{u-1} + \cdots + a^k_{u-k+3} + a^k_{u-k+1}
\]

\[
= a^k_{u+1}.
\]

Prove the second part of the theorem: Expanding the \( \text{per} \left( T^k_u \right) \) with respect to the first row, we can write

\[
\text{per} \left( T^k_u \right) = \text{per} \left( T^k_{u-1} \right) + \text{per} \left( H^k_{u-1} \right).
\]

Thus, by the results and an inductive argument, the proof is easily seen. \( \square \)
Using the definition of the generalized Fibonacci-circulant-Hurwitz numbers we find the generating function $g(x)$ as shown

$$g(x) = \frac{x^k}{1 + x - x^2 - \cdots - x^{k-2} - x^k}$$

where $k \geq 4$.

Now we investigate an exponential representation for the generalized Fibonacci-circulant-Hurwitz numbers.

**Theorem 3.5.** For $k \geq 4$, the generalized Fibonacci-circulant-Hurwitz numbers have the following exponential representation:

$$g(x) = x^k \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} \left( -1 + x + \cdots + x^{k-3} + x^{k-1} \right)^n \right).$$

*Proof.* We consider the generating function $g(x) = \frac{x^k}{1 + x - x^2 - \cdots - x^{k-2} - x^k}$. Since

$$\ln g(x) = \ln \left( \frac{x^k}{1 + x - x^2 - \cdots - x^{k-2} - x^k} \right),$$

$$\ln g(x) = \ln x^k - \ln (1 + x - x^2 - \cdots - x^{k-2} - x^k)$$

and

$$\ln \left( 1 + x - x^2 - \cdots - x^{k-2} - x^k \right) = -[x \left( -1 + x + x^2 + \cdots + x^{k-3} + x^{k-1} \right)$$

$$+ \frac{1}{2} x^{2 \left( -1 + x + x^2 + \cdots + x^{k-3} + x^{k-1} \right)} + \cdots$$

$$+ \frac{1}{i} x^i \left( -1 + x + x^2 + \cdots + x^{k-3} + x^{k-1} \right)^i + \cdots \big],$$

it is clear that

$$\ln \frac{g(x)}{x^k} = \sum_{n=1}^{\infty} \frac{x^n}{n} \left( -1 + x + \cdots + x^{k-3} + x^{k-1} \right)^n. \quad \Box$$

Now we consider the sums of all the generalized Fibonacci-circulant-Hurwitz numbers. Let the $k \times k$ matrix $M_k$ be as in (2) and let the sums of the generalized Fibonacci-circulant-Hurwitz numbers from 1 to $n$, ($n > 1$) be denoted by $S_n$, that is,

$$S_n = \sum_{i=1}^{n} a_i^k.$$

If we define the $(k + 1) \times (k + 1)$ matrix $Z_k$ as in the following form:

$$Z_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & & M_k \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix},$$

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then by using induction on \( n \), we may write

\[
(Z_k)^n = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
S_{n+k-1} & & & \\
& S_{n+k-1} & & (M_k)^n \\
& & \ddots & \\
& & & S_n
\end{bmatrix}.
\]

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References


