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A note on identities in two variables for a class of monoids

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Abstract: In this note we consider identities in the alphabet $X = \{x, y\}$. This note is self-contained and the aim is to describe gradually the identities partition (with three parameters) of the free semigroup X^+ for the class of monoids $B_n = \langle a, b \mid ba = b^n \rangle$ $(n > 0)$.
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1 Introduction and preliminaries

Section 4 of the recent paper by Geroldinger and Schwab [2] is devoted to the study of non-unique factorizations in a class of non-commutative monoids $\{B_n\}_{n>1}$. The monoids B_n, $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of nonnegative integers, are defined by the monoid presentation:

$$B_n = \langle a, b \mid ba = b^n \rangle.$$

The elements of B_n are words of the form $a^k b^m$ for $k, m \in \mathbb{N}$ with the understanding that $a^0 = b^0 = 1$. The monoid B_0 is the bicyclic monoid which plays a very important role in the structural theory of semigroups. The multiplication in B_0 is given by the rule

$$a^k b^m \cdot a^r b^s = \begin{cases} a^k b^{m-r+s} & \text{if} \quad m \geq r \\ a^{k-m+r} b^s & \text{if} \quad m < r. \end{cases}$$

If $n > 0$, then the multiplication in B_n is defined by:

$$a^k b^m \cdot a^r b^s = \begin{cases} a^{k+r} b^s & \text{if} \quad m = 0; \\ a^k b^{m+(n-1)r+s} & \text{if} \quad m > 0. \end{cases}$$

The identity (see [1])

$$(I) \qquad xyyxxyxyyx \approx xyyxyxxyyx, \qquad (xy^2x^2yxy^2x \approx xy^2xyx^2y^2x)$$

called Adjan's identity, is the first known and the shortest nontrivial identity satisfied in the bicyclic monoid B_0. It is known that (I) and the identity

$$(II) \qquad xyyxxyyxxy \approx xyyxyxyxxy \qquad (xy^2x^2y^2x^2y \approx xy^2xyxyx^2y)$$

are the only identities in the alphabet $\{x, y\}$ of length 10 satisfied in the bicyclic monoid B_0.

Now, we present some notations, definitions and remarks that are used throughout this paper. If X is a given finite set (called the set of alphabet), $x \in X$, and v is a word whose letters belong to X then $n_x(v)$ denotes the number of occurrences of x in v, and $\ell(v) = \sum_{x \in X} n_x(v)$ is called the *length* of v. A pair of words (v, w) is called *balanced* if $n_x(v) = n_x(w)$ for all $x \in X$. A mapping $\sigma : X \to S$, where S is a semigroup, can be extended in a unique way to a homomorphism of X^+ (the free semigroup generated by X) to S, called a *substitution by elements of* S, denoted again by σ. An *identity* $v \approx w$ $(v, w \in X^+)$ for (or, satisfied in) the semigroup S is a pair $(v, w) \in X^+ \times X^+$ such that $\sigma(v) = \sigma(w)$ for all substitutions by elements of S. An identity $v \approx w$ is called *balanced* if the pair of words (v, w) is balanced. In this case $\ell(v) = \ell(w)$ is called the *length of the identity* $v \approx w$. If $v \approx w$ is an identity for S then the identity $uvu' \approx uwu'$ satisfied in S, where $u, u' \in X^*$ (X^* being X^+ adjoined by the empty word) is called a *simple consequence* of $v \approx w$ (u and u' are called prefix and suffix, respectively, of the words uvu' and uwu'). The relation $\approx$ is an equivalence relation on the set X^+. The set of its equivalence classes is the *identities partition for S*, denoted by $\mathcal{P}_S$.

In this note we will consider the case $X = \{x, y\}$ and $S = B_n$ with $n > 0$. The problem is to give characterizations of identities in B_n $(n > 0)$. Shneerson [5] solved it (even in case of an arbitrary finite set X): *$v \approx w$ is an identity satisfied in B_n $(n > 0)$ if and only if $v = uv'$, $w = uw'$, where u is a prefix of v of the smallest length that contains both variables x and y, and the words v' and w' are equal in the free commutative monoid over the alphabet $X = \{x, y\}$.* Pastijn [3] solved this problem if $n = 0$ in all its generality (X being countable infinite set) in three distinct ways, and Shleifer [4] studied also (only if $X = \{x, y\}$) identities for B_0 and created such identities using computer assistance.

The purpose of this note is to reach the above Shneerson's characterization gradually, if $X = \{x, y\}$. It is straightforward to see that Theorems 3.1 and 3.2 of Section 3 express this fact. In preparation we used only a few identities (Section 2).

2 The identities $(A_{i,j})$

Since B_n (for all $n \in \mathbb{N}$) contains a copy of the infinite cyclic semigroup, any identity $v \approx w$ for B_n is balanced. From the multiplication defined in B_n it follows that if $v \approx w$ is an identity satisfied in B_n then the first letter of v and w coincide. We will consider identities with the first letter x; changing the two letters x and y between them in each of the words of the identity $v \approx w$ does not lead to a new identity in our convention. If $n > 0$ then the right cancellation law holds in the set of all identities for B_n because the monoid B_n $(n > 0)$ is right cancellative

Proposition 2.1. *For any positive integers $i \geq j$ and $n > 0$,*

$$(A_{i,j}) \quad xy^{i+1}x \approx xy^j xy^{i-j+1} \qquad ((A'_{i,j}) \quad yx^{i+1}y \approx yx^j yx^{i-j+1})$$

is an identity satisfied in B_n.

Proof. To prove that $(A_{i,j})$ is an identity for B_n $(n > 0)$ we consider the substitution $y = a^k b^m$, $x = a^r b^s$. Then

$$(a) \quad y^2x = \begin{cases} a^{2k+r}b^s & \text{if} \quad m = 0 \\ a^k b^{2m+(n-1)k+(n-1)r+s} & \text{if} \quad m > 0, \end{cases}$$

and

$$(b) \quad yxy = \begin{cases} a^{2k+r} & \text{if} \quad m = 0 \text{ and } s = 0 \\ a^{k+r}b^{s+(n-1)k} & \text{if} \quad m = 0 \text{ and } s > 0 \\ a^k b^{2m+(n-1)k+(n-1)r+s} & \text{if} \quad m > 0. \end{cases}$$

Since $y^2x = yxy$ in the cases $m > 0$ and $m = 0 = s$, we will consider hereinafter $m = 0$, $s > 0$. Then

$$xy^{i+1}x = xy^{i-1}(y^2x) = a^r b^s a^{(i-1)k} a^{2k+r} b^s = a^r b^s a^{(i+1)k+r} b^s = a^r b^{2s+(n-1)[(i+1)k+r]},$$

and

$$xy^i xy = xy^{i-1}(yxy) = a^r b^s a^{(i-1)k} a^{k+r} b^{s+(n-1)k} = a^r b^s a^{ik+r} b^{s+(n-1)k} = a^r b^{2s+(n-1)[(i+1)k+r]}.$$

So, $(A_{i,j})$ is an identity satisfied in B_n $(n > 0)$ if $i = j$.

Now, the following sequence of identities satisfied in B_n, $n > 0$,

$$xy^{i+1}x \approx xy^i xy \approx xy^{i-1}xy^2 \approx xy^{i-2}xy^3 \approx \cdots \approx xy^j xy^{i-j+1}$$

finishes the proof of the proposition. □

It is easy to check that nontrivial identities for B_n of length 2 and 3 do not exist. It follows that:

Corollary 2.1. *For any $n > 0$, the identity*

$$(A_{1,1}) \quad xy^2x \approx xyxy \qquad ((A'_{1,1}) \quad yx^2y \approx yxyx)$$

is the shortest nontrivial identity satisfied in the monoid B_n.

Remark 2.1. The Adjan identity (I) and the identity (II) are both satisfied in B_n since they are simple consequences of $(A'_{1,1})$ if $n > 0$:

$$(A'_{1,1}) \Rightarrow \underline{xy}\,\overbrace{yx^2y}\,\underline{\underline{xy^2x}} \approx \underline{xy}\,\overbrace{yxyx}\,\underline{\underline{xy^2x}} \text{ that is (I);}$$

$$(A'_{1,1}) \Rightarrow \underline{xy}\,\overbrace{yx^2y}\,\underline{\underline{yx^2y}} \approx \underline{xy}\,\overbrace{yxyx}\,\underline{\underline{yx^2y}} \text{ that is (II).}$$

Remark 2.2. Example 4.4 of [3] sets that

$$xyxx^iyx^{\ell-i}y^kx \approx xyxx^jyx^{\ell-j}y^kx \quad (0 \le i < j < \ell \text{ and } k \ge 1)$$

is an identity for B_0 if and only if $(k+1)(i+1) \ge \ell+1 \ge 2(j+1)$. The problem gets a new look in the case $n > 0$. Using $(A'_{i,1})$ and $(A'_{j,1})$ we obtain the following two identities satisfied in B_n, $n > 0$:

$$xyxx^iyx^{\ell-i}y^kx \approx xyxyx^\ell y^kx \quad \text{and} \quad xyxx^jyx^{\ell-j}y^kx \approx xyxyx^\ell y^kx.$$

So, $xyxx^iyx^{\ell-i}y^kx \approx xyxx^jyx^{\ell-j}y^kx$ is an identity for B_n (if $n > 0$) for any $i, j, k, \ell \in \mathbb{N}$ with $i, j \le \ell$.

3 Main results

Unless otherwise indicated, we consider words v (and identities) with x the first letter and with $n_y(v) > 0$ (that is, words v of the form $v = x^ku$, where u is non-empty and y is the first letter of u). We say that a word of the form

$$(*) \qquad x^{\ell_1}(yx)^{\ell_2}z^{\ell_3} \qquad (\text{where } z \in \{x, y\},\ \ell_1 > 0 \text{ and } \ell_2, \ell_3 \ge 0)$$

is a canonical form of the word v (the words $(yx)^{\ell_2}$ and z^{ℓ_3} are the empty word if $\ell_2 = 0$ and $\ell_3 = 0$, respectively) if

$$v \approx x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}$$

is an identity satisfied in B_n, $n > 0$ (ℓ_2 can be 0 only if $\ell_3 > 0$ and $z = y$ since $n_y(v) > 0$).

Lemma 3.1. *A canonical form of the word $v = x^ku$ (y being the first letter of u), is given by*

$$v \approx \begin{cases} x^k(yx)^{n_x(u)}y^{n_y(u)-n_x(u)} & \text{if} \quad n_y(u) \ge n_x(u) \\ x^k(yx)^{n_y(u)}x^{n_x(u)-n_y(u)} & \text{if} \quad n_y(u) < n_x(u) \end{cases}.$$

Proof. A sequence of identities obtained by using (from left to right) only the identities $(A_{i,1})$ and $(A'_{i,1})$ (i.e., $xy^{i+1}x \approx xyxy^i$ and $yx^{i+1}y \approx yxyx^i$) leads us in the end to an identity for B_n of the form

$$v \approx x^kyxyx\cdots yxz^m \quad (m \ge 0),$$

where $z \in \{x, y\}$. It is clear that if $n_y(u) > n_x(u)$ then $z = y$ and the number of occurrences of (yx) is $n_x(u)$. If $n_y(u) = n_x(u)$ then the number of occurrences of (yx) is also $n_x(u)$. Since any identity for B_n is balanced, it follows that $m = n_y(u) - n_x(u)$. Now, if $n_y(u) < n_x(u)$ then $z = x$ and the number of occurrences of (yx) is $n_y(u)$. Obviously in this case $m = n_x(u) - n_y(u)$. $\square$

Theorem 3.1. *Let v and w be two words in the alphabet $\{x, y\}$, $v = x^k u$ and $w = x^{k'} u'$ (y being the first letter of both words u and u'). Then the following statements are equivalent:*

(i) *$v \approx w$ is an identity satisfied in B_n, $n > 0$;*

(ii) *v and w have the same canonical form;*

(iii) *$n_x(u) = n_x(u')$, $n_y(u) = n_y(u')$ and $k = k'$;*

(iv) *(v, w) is balanced and $k = k'$.*

Proof. $(i) \Leftrightarrow (ii)$ If v and w have the same canonical form then obviously $v \approx w$ is an identity satisfied in B_n if $n > 0$.

Conversely, if $v \approx w$ is an identity for B_n, $n > 0$, and $v \approx x^{\ell_1}(yx)^{\ell_2} z^{\ell_3}$, $w \approx x^{\ell'_1}(yx)^{\ell'_2} z'^{\ell'_3}$, are two canonical forms of v and w respectively, then we will prove that the two canonical forms are the same, that is: (1) $\ell_1 = \ell'_1$, $\ell_2 = \ell'_2$, $\ell_3 = \ell'_3$, and (2) $z = z'$ if $\ell_3 = \ell'_3 \neq 0$.

Using the substitution $\sigma_{1,1}$ by elements of B_n $(n > 0)$ defined by $x = a$, $y = b$,

$$\sigma_{1,1}(x^{\ell_1}(yx)^{\ell_2} z^{\ell_3}) = a^{\ell_1} b^{n\ell_2} b^{\ell_3} = a^{\ell_1} b^{n\ell_2 + \ell_3} \quad \text{if } z = y$$

and

$$\sigma_{1,1}(x^{\ell_1}(yx)^{\ell_2} z^{\ell_3}) = a^{\ell_1} b^{n\ell_2} a^{\ell_3} = a^{\ell_1} b^{n\ell_2 + (n-1)\ell_3} \quad \text{if } z = x.$$

Analogously,

$$\sigma_{1,1}(x^{\ell'_1}(yx)^{\ell'_2} z'^{\ell'_3}) = a^{\ell'_1} b^{n\ell'_2 + \ell'_3} \quad \text{if } z' = y$$

and

$$\sigma_{1,1}(x^{\ell'_1}(yx)^{\ell'_2} z'^{\ell'_3}) = a^{\ell'_1} b^{n\ell'_2 + (n-1)\ell'_3} \quad \text{if } z' = x.$$

It is clear that $\sigma_{1,1}(x^{\ell_1}(yx)^{\ell_2} z^{\ell_3}) = \sigma_{1,1}(x^{\ell'_1}(yx)^{\ell'_2} z^{\ell'_3})$ implies

$$\ell_1 = \ell'_1.$$

Since any identity for B_n is balanced, it follows that

$$2\ell_2 + \ell_3 = 2\ell'_2 + \ell'_3.$$

The equality $\sigma_{1,1}(x^{\ell_1}(yx)^{\ell_2} z^{\ell_3}) = \sigma_{1,1}(x^{\ell'_1}(yx)^{\ell'_2} z'^{\ell'_3})$ implies also:

Case 1. $(z = z' = y)$: $n\ell_2 + \ell_3 = n\ell'_2 + \ell'_3$, that is $(n-2)(\ell'_2 - \ell_2) = 0$.

Case 2. $(z = z' = x)$: $n\ell_2 + (n-1)\ell_3 = n\ell'_2 + (n-1)\ell'_3$, that is $(n-2)(\ell'_2 - \ell_2) = 0$.

Case 3. $(z \neq z')$: if $z = y$ and $z' = x$ then $n\ell_2 + \ell_3 = n\ell'_2 + (n-1)\ell'_3$ implies $2n(\ell'_2 - \ell_2) = 2\ell_3 - 2(n-1)\ell'_3$ and so, $(n-2)(\ell_3 + \ell'_3) = 0$; analogously if $z = x$ and $z' = y$. Thus the hypothesis $z \neq z'$ implies $\ell_3 = \ell'_3 = 0$ if $n \neq 2$, and therefore z^{ℓ_3} and $z'^{\ell'_3}$ are the empty word.

Since $\ell_2' = \ell_2$ if and only if $\ell_3' = \ell_3$ (any identity satisfied in B_n is balanced), the conclusion is that $\ell_2' = \ell_2$, $\ell_3' = \ell_3$ if $n \neq 2$, and $z = z'$ if $\ell_3 = \ell_3' \neq 0$ and $n \neq 2$ ($n > 0$). The case $n = 2$ will be discussed below.

Let $\sigma_{1,2}$ be the substitution by elements of B_2 defined by $x = a$, $y = b^2$. Then,

$$\sigma_{1,2}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = a^{\ell_1}b^{3\ell_2}a^{\ell_3} = a^{\ell_1}b^{3\ell_2+\ell_3} \quad \text{if } z = x$$

and

$$\sigma_{1,2}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = a^{\ell_1}b^{3\ell_2}b^{2\ell_3} = a^{\ell_1}b^{3\ell_2+2\ell_3} \quad \text{if } z = y.$$

Analogously,

$$\sigma_{1,2}(x^{\ell_1'}(yx)^{\ell_2'}z'^{\ell_3'}) = a^{\ell_1'}b^{3\ell_2'+\ell_3'} \quad \text{if } z' = x$$

and

$$\sigma_{1,2}(x^{\ell_1'}(yx)^{\ell_2'}z'^{\ell_3'}) = a^{\ell_1'}b^{3\ell_2'+2\ell_3'} \quad \text{if } z' = y.$$

Taking into account that $\ell_1 = \ell_1'$ and $2\ell_2 + \ell_3 = 2\ell_2' + \ell_3'$, the equality $\sigma_{1,1}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = \sigma_{1,1}(x^{\ell_1'}(yx)^{\ell_2'}z^{\ell_3'})$ implies:

<u>Case 1.</u> ($z = z' = y$): $3\ell_2 + \ell_3 = 3\ell_2' + \ell_3' \Rightarrow \ell_2 = \ell_2'$ (and therefore $\ell_3 = \ell_3'$).

<u>Case 2.</u> ($z = z' = x$): $3\ell_2 + 2\ell_3 = 3\ell_2' + 2\ell_3' \Rightarrow \ell_2 = \ell_2'$ (and therefore $\ell_3 = \ell_3'$).

<u>Case 3.</u> ($z \neq z'$): if $z = y$ and $z' = x$ then $3\ell_2 + \ell_3 = 3\ell_2' + 2\ell_3' \Rightarrow \ell_2 = \ell_2' + \ell_3'$ and so $2\ell_2' + \ell_3' = 2(\ell_2' + \ell_3') + \ell_3$, that is $\ell_3' + \ell_3 = 0$ and therefore $\ell_3' = \ell_3 = 0$ (analogously if $z = x$ and $z' = y$).

Thus, if $v \approx w$ is an identity for B_n, $n > 0$, and $v \approx x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}$, $w \approx x^{\ell_1'}(yx)^{\ell_2'}z'^{\ell_3'}$, are two canonical forms of v and w respectively, then the two canonical forms coincide.

$(ii) \Leftrightarrow (iii)$ follows from Lemma 3.1.

$(iii) \Leftrightarrow (iv)$ holds obviously. □

Remark 3.1. Given two different words v and w, if x^k ($k > 0$) is the leftmost subword of the maximal length of both words v and w consisting of repetitions of x, $n_y(v) = n_y(w) = \ell > 1$ and $n_x(v) - k = n_x(w) - k = m > 0$ then, and only then, $v \approx w$ is a nontrivial identity for B_n ($n > 0$). So, a triple of positive integers (k, l, m), $l > 1$, determine a set of words and thus a set of nontrivial identities. For example, the triple of positive integers $(4, 2, 2)$ determine the set of words

$$\{x^4y^2x^2,\ x^4yx^2y,\ x^4yxyx\}$$

and the set of nontrivial identities

$$\{x^4y^2x^2 \approx x^4yx^2y,\ \ x^4yx^2y \approx x^4yxyx,\ \ x^4y^2x^2 \approx x^4yxyx\}.$$

Taking into account all possible cases, we conclude that

Theorem 3.2. *The identities partition $\mathcal{P}_{B_n}$ $(n > 0)$ is given by*

$$\mathcal{P}_{B_n} = \{P_{k,l,m}\}_{k,l>0,m\geq 0} \cup \{P_{k,0,0}\}_{k>0},$$

where

$$P_{k,l,m} = \{x^k u \mid \text{the first letter of } u \text{ is } y,\ n_y(u) = l \text{ and } n_x(u) = m\}$$

if $k, l > 0, m \geq 0$, and $P_{k,0,0}$ $(k > 0)$ are the singletons $\{x^k\}$. The elements of this partition are finite sets and if $k, l > 0, m \geq 0$, then

$$|P_{k,l,m}| = \binom{l+m-1}{l-1}.$$

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