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A new class of q-Hermite-based Apostol-type polynomials and its applications

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Abstract: The present article is to introduce a new class of q-Hermite based Apostol-type polynomials and to investigate their properties and characteristics. In particular, the generating functions, series expression and explicit and recurrence relations for these polynomials are established. We derive some relationships for q-Hermite based Apostol-type polynomials associated with q-Apostol-type Bernoulli polynomials, q-Apostol-type Euler and q-Apostol-type Genocchi polynomials.
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1 Introduction

Many mathematicians, physicists and engineers have been working for a long time in the field of q-calculus (see [5, 7–9, 15–18]). The q-calculus is a generalization of many subjects, like the hypergeometric series, complex analysis, and particle physics. By using q-analogs and umbral calculus, of many orthogonal polynomials and functions have been studied. The q-calculus is mostly being used by physicists at a high level. In short, q-calculus is a very much popular subject for researchers today.

Recently, due to fundamental importance in numerous areas such as applied mathematics, mechanics, mathematical physics, Lie theory and quantum algebra (see [1–3, 5]), a progressive instantaneous development has been found in the field of q-calculus. Throughout the article, $\mathbb{C}$ indicates the set of complex numbers, $\mathbb{N}$ designates set of natural numbers and $\mathbb{N}_0$ designates set of non-negative integers. Further, the variable $q \in \mathbb{C}$ such that $\mid q \mid < 1$.

We review certain definitions and concepts related to the q-calculus taken from [1], which will be used throughout this work.

The q-analogue of $a \in \mathbb{C}$ is defined by:

$$[a]_q = \frac{1-q^a}{1-q};\ q \in \mathbb{C}\backslash\{1\}. \tag{1}$$

The q-factorial function is defined by:

$$[n]_q! = \prod_{m=1}^{n} [m]_q = \frac{(q;q)}{(1-q)^n}, q \neq 1; n \in \mathbb{N}, [0]_q! = 1; 0 < q < 1. \tag{2}$$

The q-binomial coefficient $\binom{n}{k}_q$ is defined by:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, k = 0, 1, 2..., n; n \in \mathbb{N}_0. \tag{3}$$

The q-exponential function is defined by:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, \mid x \mid < \mid 1-q \mid^{-1}. \tag{4}$$

The q-Hermite polynomials are special or limited cased of the orthogonal polynomials as they contain no parameter other than q and appear to be at the bottom of a hierarchy of the classical q-orthogonal polynomials (see [2]).

We recall that the q-Hermite polynomials $H_{n,q}(x)$ are defined by means of the following generating function, (see [16]):

$$F_q(x,t) = F_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{[n]_q!}, F_q(t) = \sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} \frac{t^{2n}}{[2n]_q!}. \tag{5}$$

Recently, many mathematicians studied the unification of the Bernoulli and Euler polynomials. Luo and Srivastava (see [10, 11]) introduced the generalized Apostol–Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α. Further, the generalized Apostol–Euler polynomials $E_n^{(\alpha)}(x)$ of order α and the generalized Apostol–Genochhi polynomials $G_n^{(\alpha)}(x)$ of order α are investigated by Luo (see [12, 13]).

Thereafter, in 2014 Ernst [4] defined the q-analogues of the generalized Apostol type polynomials.

The generalized q-Apostol–Bernoulli polynomials of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function, (see [4]):

$$\left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, (\mid t \mid < \mid \log(-\lambda) \mid). \tag{6}$$

The generalized q-Apostol–Euler polynomials of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function, (see [4]):

$$\left(\frac{2}{\lambda e_q(t)+1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, (\mid t \mid < \mid \log(-\lambda) \mid). \tag{7}$$

The generalized q-Apostol–Genocchi polynomials of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function, (see [4]):

$$\left(\frac{2t}{\lambda e_q(t)+1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, (\mid t \mid < \mid \log(-\lambda) \mid). \tag{8}$$

In view of equations (6)–(8), we introduce the generalized q-Apostol type polynomials $F_{n,q}^{(\alpha)}(x;a,b;\lambda)$ of order α by means of the following generating function, (see [4]):

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda e_q(t)+a^b}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, \tag{9}$$

$$(\alpha \in \mathbb{N}_0, \lambda, a, b \in \mathbb{C}, \mid t \mid < \mid \log(-\lambda) \mid).$$

Where $F_{n,q}^{(\alpha)}(a,b;\lambda) = F_{n,q}^{(\alpha)}(0;a,b;\lambda)$ are known as q-Apostol-type numbers of order α.

If we take the $\lim_{q\to 1}$; the generalized q-Apostol type polynomials defined by equation (9) reduces to the unified Apostol type polynomials (see [14]). In fact, the following special case holds:

$$\lim_{q\to 1} \mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda) = \mathcal{F}_{n}^{(\alpha)}(x;a,b;\mu,\nu;\lambda).$$

The following special cases hold true:

$$\lim_{q\to 1} \mathcal{F}_{n,q}^{(\alpha)}(x;1,1,1;\lambda) = B_{n}^{(\alpha)}(x;\lambda),$$

$$\lim_{q\to 1} \mathcal{F}_{n,q}^{(\alpha)}(x;0,-1,0;\lambda) = E_{n}^{(\alpha)}(x;\lambda),$$

$$\lim_{q\to 1} \mathcal{F}_{n,q}^{(\alpha)}(x;1,-1/2,1;\lambda) = G_{n}^{(\alpha)}(x;\lambda),$$

where $B_{n}^{(\alpha)}(x;\lambda), E_{n}^{(\alpha)}(x;\lambda)$ and $G_{n}^{(\alpha)}(x;\lambda)$ are the generalized forms of the Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials, (see [14]). The Stirling numbers of the second kind are defined as, (see [19]):

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{10}$$

This paper is organized as follows. In Section 2, we consider q-Hermite-based Apostol-type polynomials and we derive some properties of these polynomials. In Section 3, we derive some relationships in between q-Apostol-type Bernoulli polynomials, q-Apostol-type Euler polynomials and q-Apostol-type Genocchi polynomials.

2 q-Hermite-based Apostol-type polynomials

This section is designed with certain properties of q-Hermite-based Apostol-type polynomials and some properties. We begin with the following definition as follows.

Definition 2.1. For $q \in \mathbb{C}, 0 < | q | < 1$, the generalized q-Hermite-based Apostol-type polynomials are defined by means of the following generating function:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda e_q(t)+a^b}\right)^{\alpha} F_q(t)e_q(xt)=\sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!}, \tag{11}$$

where, $\lambda,\ \mu,\ \nu,\ a,\ b \in \mathbb{C},\ n \geq 0$ and $| \ t \ |<| \ \ln(-\lambda) \ |$.

If we take $x = 0$ in (11), we have

$${}_H\mathcal{F}_{n,q}^{(\alpha)}(a,b;\mu,\nu;\lambda) = {}_H\mathcal{F}_{n,q}^{(\alpha)}(0;a,b;\mu,\nu;\lambda),$$

where ${}_H\mathcal{F}_{n,q}^{(\alpha)}(a,b;\mu,\nu;\lambda)$ are known as q-Hermite-based-Apostol-type numbers of order α.

For $\lambda = 1$ in (11), we get

$${}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\mu,\nu) = {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\mu,\nu;1),$$

where ${}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\mu,\nu)$ are known as q-Hermite-based unified polynomials of order α.

On setting $\lambda = \alpha = 1$ in (11), we have

$${}_H\mathcal{F}_{n,q}(x;a,b;\mu,\nu) = {}_H\mathcal{F}_{n,q}^{(1)}(x;a,b;\mu,\nu;1),$$

where ${}_H\mathcal{F}_{n,q}(x;a,b;\mu,\nu)$ are known as q-Hermite-based unified polynomials.

Now, we give some special cases for q-Hermite-based unified Apostol-type polynomials with the help of the following table:

S.No.	Case	Name of the polynomial	Generating function
i	$\mu=0,\nu=1,a=-1,b=1$	q-HATBP of order α	$\left(\frac{t}{\lambda e_q(t)-1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(\alpha)}(x;\lambda;0,1)\frac{t^n}{[n]_q!}$.
	$\mu=0,\nu=1,a=-1,b=1,\lambda=1$	q-HBP order α	$\left(\frac{t}{e_q(t)-1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(\alpha)}(x;1;0,1)\frac{t^n}{[n]_q!}$.
	$\mu=0,\nu=1,a=-1,b=1,\lambda=\alpha=1$	q-HBP	$\left(\frac{t}{e_q(t)-1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}(x;1;0,1)\frac{t^n}{[n]_q!}$.
ii	$\mu=1,\nu=0,a=b=1$	q-HATEP of order α	$\left(\frac{t}{\lambda e_q(t)+1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{E}_{n,q}^{(\alpha)}(x;\lambda;1,0)\frac{t^n}{[n]_q!}$.
	$\mu=1,\nu=0,a=b=1,\lambda=1$	q-HEP order α	$\left(\frac{t}{e_q(t)+1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{E}_{n,q}^{(\alpha)}(x;1;1,0)\frac{t^n}{[n]_q!}$.
	$\mu=1,\nu=0,a=b=1,\lambda=\alpha=1$	q-HEP	$\left(\frac{t}{e_q(t)+1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{E}_{n,q}(x;1;1,0)\frac{t^n}{[n]_q!}$.
iii	$\mu=1,\nu=1,a=b=1$	q-HATGP of order α	$\left(\frac{2t}{\lambda e_q(t)+1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{G}_{n,q}^{(\alpha)}(x;\lambda;1,1)\frac{t^n}{[n]_q!}$.
	$\mu=1,\nu=1,a=b=1,\lambda=1$	q-HGP order α	$\left(\frac{2t}{e_q(t)+1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{G}_{n,q}^{(\alpha)}(x;1;1,1)\frac{t^n}{[n]_q!}$.
	$\mu=1,\nu=1,a=b=1,\lambda=\alpha=1$	q-HGP	$\left(\frac{t}{e_q(t)+1}\right)F_q(t)e_q(xt) = \sum_{n=0}^{\infty} {}_H\mathcal{G}_{n,q}(x;1;1,1)\frac{t^n}{[n]_q!}$.

Table 1. Special cases for q-Hermite-based unified Apostol-type polynomials

Theorem 2.1. *The following relations hold true:*

$$ {}_H\mathcal{F}_{m,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu) = \sum_{n=0}^{m} \binom{m}{n}_q \mathcal{F}_{n,q}^{(\alpha)}(a,b;\lambda;\mu,\nu) H_{m-n,q}(x), \tag{12} $$

$$ {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu) = \sum_{n=0}^{m} \binom{m}{n}_q {}_H\mathcal{F}_{n,q}^{(\alpha)}(a,b;\lambda;\mu,\nu) x^{m-n}. \tag{13} $$

Proof. From (11), we have

$$ \sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} = \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right) F_q(t)e_q(xt). $$

$$ \sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{F}_{n,q}^{(\alpha)}(a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} \sum_{m=0}^{\infty} H_{m,q}(x)\frac{t^m}{[m]_q!}. $$

Now using the Cauchy product and comparing the coefficients of t^n, we obtain the desired result (12). Again, by using (11), we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} = \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right) F_q(t)e_q(xt).$$

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty}\frac{(xt)^m}{[m]_q!}.$$

Using the Cauchy product and comparing the coefficients of t^n, we get the result (13). □

Theorem 2.2. *The following relations hold true:*

$$ {}_H\mathcal{F}_{n,q}^{(\alpha+\beta)}(x;a,b;\lambda;\mu,\nu) = \sum_{r=0}^{n}\binom{n}{r}_q {}_H\mathcal{F}_{r,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu){}_H\mathcal{F}_{n-r,q}^{(\beta)}(a,b;\lambda;\mu,\nu), \tag{14}$$

$$ {}_H\mathcal{F}_{n,q}^{(\alpha+\beta)}(x+u;a,b;\lambda;\mu,\nu) = \sum_{r=0}^{n}\binom{n}{r}_q {}_H\mathcal{F}_{r,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu){}_H\mathcal{F}_{n-r,q}^{(\beta)}(u;a,b;\lambda;\mu,\nu), \tag{15}$$

$$\lambda {}_H\mathcal{F}_{n,q}^{(\alpha)}(x+1;a,b;\lambda;\mu,\nu)+a^b {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu) = \frac{2^\mu[n]_q!}{[n-k]_q!}{}_H\mathcal{F}_{n-k,q}^{(\alpha-1)}(x;a,b;\lambda;\mu,\nu). \tag{16}$$

Proof. Utilizing (11) and making use of lemma (see [20, p.100, eq.2]), we can easily obtain results (14) and (15).

For obtaining the result (16), we take

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x+1;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} + a^b\sum_{n=0}^{\infty} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!}$$
$$= \lambda\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha F_q(t)e_q((x+1)t) + a^b\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha F_q(t)e_q(xt)$$
$$= \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha F_q(t)e_q(xt)[\lambda e_q(t)+a^b]$$
$$= \sum_{n=0}^{\infty} 2^\mu {}_H\mathcal{F}_{n,q}^{(\alpha-1)}(x;a,b;\lambda;\mu,\nu)\frac{t^{n+k}}{[n]_q!}.$$

Using the Cauchy product and comparing of the coefficients of t^n, we arrive at the required result (16). □

Theorem 2.3. *The following recurrence relations hold true:*

$$ {}_H\mathcal{B}_{k,q}^{(\alpha)}\left(\frac{1}{m}\right) - \sum_{j=0}^{k}\binom{k}{j}_q\left(\frac{1}{m}-1\right)^{k-j}\mathcal{B}_{j,q}^{(\alpha)} = [k]_q\sum_{j=0}^{k-1}\binom{k-1}{j}_q\times\left(\frac{1}{m}-1\right)_q^{(k-j-1)}{}_H\mathcal{B}_{j,q}^{(\alpha-1)}, \tag{17}$$

$$ {}_H\mathcal{E}_{k,q}^{(\alpha)}\left(\frac{1}{m}\right) + \sum_{j=0}^{k}\binom{k}{j}_q\left(\frac{1}{m}-1\right)^{k-j}\mathcal{E}_{j,q}^{(\alpha)} = 2\sum_{j=0}^{k}\binom{k}{j}_q\times\left(\frac{1}{m}-1\right)_q^{(k-j)}{}_H\mathcal{E}_{j,q}^{(\alpha-1)}. \tag{18}$$

Proof. These relations can be obtained by making use of (11) with replacement of x with $\frac{1}{m}$. □

Proposition. The following differential relation holds true:

$$D_{q,x\,H}\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) = [n]_{q\,H}\mathcal{F}^{(\alpha)}_{n-1,q}(x;a,b;\lambda;\mu,\nu). \tag{19}$$

Proof. Using (11), we get

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} = \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^{\alpha} e_q(xt)F_q(t).$$

Differentiating the above equation with respect to x and using the result $D_{q,x}e_q(xt) = xe_q(xt)$, we have

$$D_{q,x}\sum_{n=0}^{\infty} {}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} = \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^{\alpha} e(xt)F_q(t)t.$$

Now using the Cauchy product and comparing the coefficients of t, we lead to the required result. □

Theorem 2.4. *The following relations hold true:*

$${}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) = \sum_{m=0}^{\left[\frac{n}{2}\right]}(-1)^m q^{\binom{m}{2}}\binom{n}{2m}_q \mathcal{F}^{(\alpha)}_{n-2m,q}(x;a,b;\lambda;\mu,\nu), \tag{20}$$

$${}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) = \sum_{m=0}^{n}\binom{n}{m}_q {}_H\mathcal{F}^{(\alpha)}_{n-m,q}(a,b;\lambda;\mu,\nu)x^m. \tag{21}$$

Proof. From (11), we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!} = \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right) F_q(t)e_q(xt)$$

$$= \sum_{n=0}^{\infty} \mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty}(-1)^m q^{\binom{m}{2}}\frac{t^{2m}}{[2m]_q!}.$$

Using the Cauchy product and comparing the coefficients of t, we arrive at the desired result (20). The proof of (21) is similar. □

Corollary 2.4.1. *On setting* $x = 1$ *in* (20) *and* (21)*, we have*

$${}_H\mathcal{F}^{(\alpha)}_{n,q}(1;a,b;\lambda;\mu,\nu) = \sum_{m=0}^{\left[\frac{n}{2}\right]}(-1)^m q^{\binom{m}{2}}\binom{n}{2m}_q \mathcal{F}^{(\alpha)}_{n-2m,q}(1;a,b;\lambda;\mu,\nu), \tag{22}$$

$${}_H\mathcal{F}^{(\alpha)}_{n,q}(1;a,b;\lambda;\mu,\nu) = \sum_{m=0}^{n}\binom{n}{m}_q {}_H\mathcal{F}^{(\alpha)}_{n-m,q}(a,b;\lambda;\mu,\nu). \tag{23}$$

3 Relationships between Bernoulli, Euler and Genocchi polynomials

In this section, we establish some relationships for q-Hermite-based Apostol-type polynomials related to q-Apostol–Bernoulli polynomials, q-Apostol–Euler polynomials and q-Apostol-type Genocchi polynomials.

Theorem 3.1. *The following relation holds true:*

$$
{}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) =
$$

$$
\frac{1}{[n+1]_q!}\left[\lambda\sum_{r=0}^{n+1}\binom{n+1}{r}_q\sum_{m=0}^{n+1}\binom{n+1}{m}_q\mathcal{B}_{n+1-m-r,q}(x;\lambda)\right]{}_H\mathcal{F}^{(\alpha)}_{m,q}(a,b;\lambda;\mu,\nu)
$$

$$
-\left[\frac{1}{[n+1]_q!}\sum_{m=0}^{n+1}\binom{n+1}{m}_q\mathcal{B}_{n+1-m,q}\right]{}_H\mathcal{F}^{(\alpha)}_{m,q}(a,b;\lambda;\mu,\nu). \tag{24}
$$

Proof. From (11), we have

$$
\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha e_q(xt)F_q(t) = \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha \frac{t}{\lambda e_q(t)-1}F_q(t)\frac{\lambda e_q(t)-1}{t}e_q(xt)
$$

$$
= \left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha F_q(t)\left[\frac{t}{\lambda e_q(t)-1}e_q(xt)\right]\frac{\lambda}{t}e_q(t)
$$

$$
-\frac{1}{t}\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha F_q(t)\left[\frac{t}{\lambda e_q(t)-1}e_q(xt)\right]
$$

$$
= \frac{1}{t}\left(\lambda\sum_{m=0}^{\infty}{}_H\mathcal{F}^{(\alpha)}_{m,q}(a,b;\lambda;\mu,\nu)\frac{t^m}{[m]_q!}\sum_{r=0}^{\infty}\mathcal{B}_{r,q}(x;\lambda)\frac{t^r}{[r]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\right.
$$

$$
\left.-\sum_{m=0}^{\infty}{}_H\mathcal{F}^{(\alpha)}_{m,q}(a,b;\lambda;\mu,\nu)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty}\mathcal{B}_{n,q}(x;\lambda)\frac{t^n}{[n]_q!}\right).
$$

On making use of the Cauchy product and comparing the coefficients of t^n, we arrive at the desired result. □

Corollary 3.1.1. *The following relations hold true for Euler and Genocchi polynomials with q-Hermite-based Apostol-type polynomials.*

$$
\begin{aligned}
{}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) = &\frac{1}{2}\left[\lambda\sum_{r=0}^{n}\binom{n}{r}_q\sum_{m=0}^{n}\binom{n}{m}_q\mathcal{E}_{n-m-r,q}(x;\lambda)\right]{}_H\mathcal{F}^{(\alpha)}_{m,q}(a,b;\lambda;\mu,\nu)\\
&-\left[\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}_q\mathcal{E}_{n-m,q}\right]{}_H\mathcal{F}^{(\alpha)}_{m,q}(a,b;\lambda;\mu,\nu),
\end{aligned} \tag{25}
$$

and

$$
{}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) =
$$

$$\frac{1}{2[n+1]_q!}\left[\lambda\sum_{r=0}^{n+1}\binom{n+1}{r}_q\sum_{m=0}^{n+1}\binom{n+1}{m}_q\mathcal{G}_{n+1-m-r,q}(x;\lambda)\right]{}_H\mathcal{F}_{m,q}^{(\alpha)}(a,b;\lambda;\mu,\nu)$$
$$-\left[\frac{1}{2[n+1]_q!}\sum_{m=0}^{n+1}\binom{n+1}{m}_q\mathcal{G}_{n+1-m,q}\right]{}_H\mathcal{F}_{m,q}^{(\alpha)}(a,b;\lambda;\mu,\nu). \tag{26}$$

Theorem 3.2. *The following explicit relationship between q-Hermite-Apostol Bernoulli polynomials, q-Hermite-Apostol Euler polynomials and q-Hermite-Apostol Genocchi polynomials holds true:*

$${}_H\mathcal{G}_{n,q}^{(\alpha)}(x;a,b;\lambda)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}_q\left[\sum_{j=0}^{k}\binom{n}{j}_q{}_H\mathcal{E}_{n-j-k,q}(x;\lambda)+{}_H\mathcal{E}_{n-k,q}(x;\lambda)\right]{}_H\mathcal{G}_{k,q}^{(\alpha)}(x;a,b;\lambda). \tag{27}$$

Proof. From (11), we get

$$\left(\frac{2t}{\lambda e_q(t)+a^b}\right)^{\alpha}e_q(xt)F_q(t)=\frac{2}{\lambda e_q(t)+1}F_q(t)\left(\frac{e_q(t)+1}{2}\right)\left(\frac{2t}{\lambda e_q(t)+a^b}\right)^{\alpha}e_q(xt)$$

$$\sum_{n=0}^{\infty}{}_H\mathcal{G}_{n,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,q}(0;\lambda)\frac{t^n}{[n]_q!}\sum_{j=0}^{\infty}\frac{t^j}{[j]_q!}\sum_{k=0}^{\infty}{}_H\mathcal{G}_{k,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^k}{[k]_q!}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,q}(0;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}{}_H\mathcal{G}_{k,q}^{(\alpha)}(x;a,b;)\frac{t^k}{[k]_q!}.$$
$$\sum_{n=0}^{\infty}{}_H\mathcal{G}_{n,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^n}{[n]_q!}=I_1+I_2. \tag{28}$$

For I_1,

$$I_1=\frac{1}{2}\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,q}(0;\lambda)\frac{t^n}{[n]_q!}\sum_{j=0}^{\infty}\frac{t^j}{[j]_q!}\sum_{k=0}^{\infty}{}_H\mathcal{G}_{k,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^k}{[k]_q!}$$
$$I_1=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{k}_q{}_H\mathcal{G}_{k,q}^{(\alpha)}(x;a,b;\lambda)\binom{n}{j}_q{}_H\mathcal{E}_{n-j-k,q}(0;\lambda)\frac{t^n}{[n]_q!}. \tag{29}$$

For I_2,

$$I_2=\frac{1}{2}\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,q}(0;\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}{}_H\mathcal{G}_{k,q}^{(\alpha)}(x;a,b;)\frac{t^k}{[k]_q!}$$
$$I_2=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_q{}_H\mathcal{E}_{n-k,q}(0;\lambda){}_H\mathcal{G}_{k,q}^{(\alpha)}(x;a,b;)\frac{t^n}{[n]_q!}. \tag{30}$$

From (28), we get

$$\sum_{n=0}^{\infty}{}_H\mathcal{G}_{n,q}^{(\alpha)}(x;a,b;\lambda)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_q\left[\sum_{j=0}^{n}\binom{n}{j}_q{}_H\mathcal{E}_{n-j-k,q}(x;\lambda)\right.$$
$$\left.+{}_H\mathcal{E}_{n-k,q}(x;\lambda)\right]{}_H\mathcal{G}_{k,q}^{\alpha}(x;a,b;\lambda)\frac{t^n}{[n]_q!}.$$

On comparing the coefficients of t^n, we obtain the required result. □

Theorem 3.3. *The following relation holds true:*

$$\mathcal{E}_{n,q}^{(\alpha)}(x;a,b;\lambda) = \sum_{k=0}^{n} \binom{n}{k}_q \frac{1}{m^{n-1}[k+1]_q!} \left[2\sum_{j=0}^{k+1} \binom{k+1}{j}_q \left(\frac{1}{m}-1\right)^{k+1-j} \mathcal{E}_{j,q}^{(\alpha-1)}(0;a,b;\lambda) \right.$$
$$\left. - \sum_{j=0}^{k+1} \binom{k+1}{j}_q \left(\frac{1}{m}-1\right)^{k+1-j} \mathcal{E}_{j,q}^{(\alpha)}(0;a,b;\lambda) - \mathcal{E}_{k+1,q}^{(\alpha)}(0;a,b;\lambda) \right]$$
$$\times {}_H\mathcal{B}_{n-k,q}(mx;a,b;\lambda). \tag{31}$$

Proof. Using (11), we have

$$\left(\frac{2}{\lambda e_q(t)+a^b}\right)^{\alpha} e_q(xt)F_q(t) = \left(\frac{2}{\lambda e_q(t)+a^b}\right)^{\alpha} \frac{\lambda e_q(t/m)-a^b}{t} \frac{t}{\lambda e_q(t/m-a^b)}$$
$$\times e_q\left(\frac{t}{m}mx\right) F_q(t).$$

By using equations (7) and (11), we arrive at the desired result. □

Theorem 3.4. *The following relations hold true:*

$${}_H\mathcal{B}_{n,q}^{(\alpha)}(x;a,b;\lambda) = \sum_{j=0}^{n} \binom{mx}{j} j! \sum_{k=0}^{n-j} \binom{n}{k}_q m^{j-n} {}_H\mathcal{B}_{k,q}^{(\alpha)}(0;a,b;\lambda) S_2(n-k,j), \tag{32}$$

$${}_H\mathcal{E}_{n,q}^{(\alpha)}(x;a,b;\lambda) = \sum_{j=0}^{n} \binom{mx}{j} j! \sum_{k=0}^{n-j} \binom{n}{k}_q m^{j-n} {}_H\mathcal{E}_{k,q}^{(\alpha)}(0;a,b;\lambda) S_2(n-k,j). \tag{33}$$

Proof. By using eq. (10) and (11), we obtain the results (32) and (33). We omit the proof. □

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