A bound of sums with convolutions of Dirichlet characters

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Abstract: We use the exponent pair to bound sums \( \sum_{ab \leq x} \chi_1(a)\chi_2(b) \), where \( \chi_1 \) and \( \chi_2 \) are primitive Dirichlet characters with conductors \( q_1 \) and \( q_2 \), respectively.

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1 Introduction and statement of result

Let \( \chi_1 \) and \( \chi_2 \) be two primitive Dirichlet characters with conductors \( q_1 \) and \( q_2 \), respectively. For \( x > 1 \), define

\[
S_{\chi_1, \chi_2}(x) = \sum_{ab \leq x} \chi_1(a)\chi_2(b).
\]

The sum (1) is a generalized form of the character sum. A asymptotic formulas for (1) can found in [2] and [3]. In 2010 Banks and Shparlinski [1] bounded \( S_{\chi_1, \chi_2}(x) \) for small values of \( x \) and proved that, if \( x \geq q_2^{2/3} \geq q_1^{2/3} \) and \( \log x = q_2^o(1) \), then

\[
|S_{\chi_1, \chi_2}(x)| \leq x^{13/18} q_1^{2/27} q_2^{1/9+o(1)},
\]

and if \( x \geq q_2^{3/4} \geq q_1^{3/4} \) and \( \log x = q_2^o(1) \), then

\[
|S_{\chi_1, \chi_2}(x)| \leq x^{5/18} q_1^{3/32} q_2^{3/16+o(1)}.
\]
Banks and Shparlinski combined the Polya–Vinogradov bound with the Burgess bounds to prove (2) and (3).

In this paper we shall provide another bound for the sum in (1) by using exponent pairs. Our result is following:

**Theorem 1.1.** Let \( \chi_1 \) and \( \chi_2 \) be two primitive Dirichlet characters with conductors \( q_1 \) and \( q_2 \), respectively. For \( x \geq q_2^2 \), \( i = 1, 2 \), we have

\[
S_{\chi_1, \chi_2}(x) = O(x^{1/3}q_1^{5/9}q_2^{7/9}\log q_1).
\]

## 2 Prerequisites

**Notation.** Throughout this paper \( \epsilon \) denotes a fixed positive constant, not necessarily the same in all occurrences. Let \( \psi(x) = x - \lfloor x \rfloor - \frac{1}{2} \). For \( r = 1, 2, \ldots \) the exponent pair is \((k_r, l_r) = \left( \frac{1}{2} - \frac{r + 1}{2(2\Lambda - 1)}, \frac{1}{2} + \frac{1}{2(2\Lambda - 1)} \right), \Lambda = 2^r.\)

The following lemmas are needed in our proof.

**Lemma 2.1.** Let \( \chi \) be a primitive character modulo \( q \). For a real \( z > 1 \), we have

\[
\sum_{a \leq z} \chi(a) = \sum_{j \leq q} \chi(j) \left\lfloor \frac{z}{q} - \frac{j}{q} + 1 \right\rfloor.
\]

**Proof.** From the periodicity of the primitive character modulo \( q \), we have

\[
\sum_{a \leq z} \chi(a) = \sum_{j \leq q} \sum_{a \equiv j \pmod{q}} \chi(a) = \sum_{j \leq q} \sum_{a \equiv j \pmod{q}} \chi(j) = \sum_{j \leq q} \chi(j) \sum_{a \equiv j \pmod{q}} 1 = \sum_{j \leq q} \chi(j) \left\lfloor \frac{z}{q} - \frac{j}{q} + 1 \right\rfloor.
\]

\[\square\]

**Lemma 2.2** (see [4, Lemma 17]). Let \( x, \eta, \alpha, \omega \) be real numbers, \( j \) and \( q \) be positive numbers, where \( x \geq 1, \alpha > 0, \eta \geq 1, 1 \leq j \leq q \), and \((k, l)\) is an exponent pair with \( k > 0 \) and

\[
R(x, \eta, \alpha; q, j; \omega) = \sum_{n \equiv j \pmod{q}} \psi \left( \frac{x}{n^\alpha} + \omega \right),
\]

if \( \omega \) is independent on \( n \). Then

\[
R(x, \eta, \alpha; q, j; \omega) = O(1) + O(x^{\frac{1}{2}}\eta^{1+\frac{l}{q-1}}) \quad \text{for } l > \alpha k,
\]

\[
O(x^{k+\frac{1}{q+1}}) \quad \text{for } l = \alpha k,
\]

\[
O\left((xq^{-\alpha})^{\frac{l}{l+\alpha}\log(1+\omega)}\right) \quad \text{for } l < \alpha k,
\]

where the \( O \)-Constants is dependent on only \( \alpha \).
3 Proof of Theorem 1.1

Proof. For $x > 1$, we have
\[ S_{χ_1,χ_2}(x) = \sum_{a \leq x^{1/2}} χ_1(a) \sum_{b \leq x/a} χ_2(b) + \sum_{b \leq x^{1/2}} χ_2(b) \sum_{a \leq x/b} χ_1(a) - \sum_{a \leq x^{1/2}} χ_1(a) \sum_{b \leq x^{1/2}} χ_2(b). \]

In view of Lemma 2.1 we have
\[ S_{χ_1,χ_2}(x) = E_1 + E_2 - E_3, \]
where
\[
\begin{align*}
E_1 &= \sum_{j \leq q_2} χ_2(j) \sum_{a \leq x^{1/2}} χ_1(a) \left\lfloor \frac{x}{aq_2} - \frac{j}{q_2} + 1 \right\rfloor, \\
E_2 &= \sum_{h \leq q_1} χ_1(h) \sum_{b \leq x^{1/2}} χ_2(b) \left\lfloor \frac{x}{bq_1} - \frac{h}{q_1} + 1 \right\rfloor, \\
E_3 &= \sum_{h \leq q_1} χ_1(h) \sum_{j \leq q_2} χ_2(j) \left\lfloor \frac{x^{1/2}}{q_1} - \frac{h}{q_1} + 1 \right\rfloor \left\lfloor \frac{x^{1/2}}{aq_2} - \frac{j}{q_2} + 1 \right\rfloor.
\end{align*}
\]

From $\lfloor x \rfloor = x - \psi(x) - \frac{1}{2}$, $\psi(x) = \psi(x + 1)$ and the identity $\sum_{j \leq q_1} χ_1(j) = 0$, for $i = 1, 2$, we have
\[
E_1 = \sum_{j \leq q_2} χ_2(j) \sum_{a \leq x^{1/2}} χ_1(a) \left( \frac{x}{aq_2} - \frac{j}{q_2} + 1 - \frac{1}{2} - \psi \left( \frac{x}{aq_2} - \frac{j}{q_2} \right) \right) \\
= -\sum_{j \leq q_2} χ_2(j) \sum_{a \leq x^{1/2}} χ_1(a) \left( \frac{j}{q_2} + \psi \left( \frac{x}{aq_2} - \frac{j}{q_2} \right) \right) \\
= -\frac{1}{q_2} \sum_{j \leq q_2} j χ_2(j) \sum_{a \leq x^{1/2}} χ_1(a) - \sum_{j \leq q_2} χ_2(j) \sum_{a \leq x^{1/2}} χ_1(a) \psi \left( \frac{x}{aq_2} - \frac{j}{q_2} \right).
\]

In view of Lemma 2.1 and the periodicity of the character modulo $q$, we have
\[
E_1 = -\frac{1}{q_2} \sum_{h \leq q_1 \atop j \leq q_2} j χ_1(h) χ_2(j) \left\lfloor \frac{x^{1/2}}{q_1} - \frac{h}{q_1} + 1 \right\rfloor \\
- \sum_{h \leq q_1 \atop j \leq q_2} χ_1(h) χ_2(j) \sum_{a \leq x^{1/2} \atop a \equiv h \pmod{q_1}} ψ \left( \frac{x}{aq_2} - \frac{j}{q_2} \right).
\]

For $E_2$ and $E_3$, computation similar to $E_1$ yields,
\[
E_2 = -\frac{1}{q_1} \sum_{h \leq q_1 \atop j \leq q_2} h χ_1(h) χ_2(j) \left\lfloor \frac{x^{1/2}}{q_2} - \frac{j}{q_2} + 1 \right\rfloor \\
- \sum_{h \leq q_1 \atop j \leq q_2} χ_1(h) χ_2(j) \sum_{a \leq x^{1/2} \atop a \equiv j \pmod{q_2}} ψ \left( \frac{x}{aq_1} - \frac{h}{q_1} \right),
\]
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and

\[ E_3 = -\frac{1}{q_2} \sum_{h \leq q_1 \atop j \leq q_2} j \chi_1(h) \chi_2(j) \left( \frac{x^{1/2}}{q_1} - \frac{h}{q_1} + 1 \right) \]

\[ -\frac{1}{q_1} \sum_{h \leq q_1 \atop j \leq q_2} h \chi_1(h) \chi_2(j) \left( \frac{x^{1/2}}{q_2} - \frac{j}{q_2} + 1 \right) + O(q_1 q_2). \]

Thus, we have

\[ S_{\chi_1 \chi_2}(x) = -\sum_{h \leq q_1 \atop j \leq q_2} \chi_1(h) \chi_2(j) \sum_{a \leq x^{1/2}} \psi \left( \frac{x}{aq_2} - \frac{j}{q_2} \right) \]

\[ -\sum_{h \leq q_1 \atop j \leq q_2} \chi_1(h) \chi_2(j) \sum_{a \leq x^{1/2}} \psi \left( \frac{x}{aq_1} - \frac{h}{q_1} \right) + O(q_1 q_2). \]

In view of Lemma 2.2, for the exponent pair \((2/7, 4/7)\), we have

\[ \sum_{h \leq q_1 \atop j \leq q_2} \chi_1(h) \chi_2(j) \sum_{a \equiv h \pmod{q_1}} \psi \left( \frac{x}{aq_2} - \frac{j}{q_2} \right) \]

\[ = \sum_{h \leq q_1 \atop j \leq q_2} \chi_1(h) \chi_2(j) R \left( \frac{x}{q_2}, x^{1/2}, 1, q_1, h, -\frac{j}{q_2} \right) \]

\[ \ll \sum_{h \leq q_1 \atop j \leq q_2} \left( 1 + \frac{x^{1/4} q_1^{1/2}}{q_2} + \frac{x^{1/3}}{q_2^{9/4} q_1^{7/9}} \right) \]

\[ = O \left( q_1 q_2 + x^{1/4} q_2^{3/2} \log q_1 + x^{1/3} q_1^{5/9} q_2^{7/9} \right). \]

In the same way, we have

\[ \sum_{h \leq q_1 \atop j \leq q_2} \chi_1(h) \chi_2(j) \sum_{a \equiv j \pmod{q_2}} \psi \left( \frac{x}{aq_1} - \frac{h}{q_1} \right) = O \left( q_1 q_2 + x^{1/4} q_1^{3/2} \log q_2 + x^{1/3} q_1^{5/9} q_2^{7/9} \right). \]

Thus, we have

\[ S_{\chi_1 \chi_2}(x) = O \left( x^{1/4} q_2^{3/2} \log q_1 + x^{1/3} q_1^{5/9} q_2^{7/9} \right) + O \left( x^{1/4} q_1^{3/2} \log q_2 + x^{1/3} q_2^{5/9} q_1^{7/9} \right) + O(q_1 q_2). \]

We note that, for \( x \geq q_1^2, q_2^2 \), the error term \( x^{1/3} q_1^{5/9} q_2^{7/9} \log q_1 \) dominates the remaining terms. Thus, for \( x \geq q_1^2, q_2^2 \), we obtain the result in Theorem 1.1.

\[ \square \]

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