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A bound of sums with convolutions of Dirichlet characters

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Abstract: We use the exponent pair to bound sums $\sum_{ab\leq x}\chi_1(a)\chi_2(b)$, where χ_1 and χ_2 are primitive Dirichlet characters with conductors q_1 and q_2, respectively.
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1 Introduction and statement of result

Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2, respectively. For $x > 1$, define

$$S_{\chi_1,\chi_2}(x) = \sum_{ab\leq x}\chi_1(a)\chi_2(b). \tag{1}$$

The sum (1) is a generalized form of the character sum. A symptotic formulas for (1) can found in [2] and [3]. In 2010 Banks and Shparlinski [1] bounded $S_{\chi_1,\chi_2}(x)$ for small values of x and proved that, if $x \geq q_2^{2/3} \geq q_1^{2/3}$ and $\log x = q_2^{o(1)}$, then

$$|S_{\chi_1,\chi_2}(x)| \leq x^{13/18}q_1^{2/27}q_2^{1/9+o(1)}, \tag{2}$$

and if $x \geq q_2^{3/4} \geq q_1^{3/4}$ and $\log x = q_2^{o(1)}$, then

$$|S_{\chi_1,\chi_2}(x)| \leq x^{5/18}q_1^{3/32}q_2^{3/16+o(1)}. \tag{3}$$

Banks and Shparlinski combined the *Polya–Vinogranov bound* with the *Burgess bounds* to prove (2) and (3).

In this paper we shall provide another bound for the sum in (1) by using exponent pairs. Our result is following;

Theorem 1.1. *Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2, respectively. For $x \geq q_i^2$, $i = 1, 2$, we have*

$$S_{\chi_1,\chi_2}(x) = O(x^{1/3} q_1^{5/9} q_2^{7/9} \log q_1).$$

2 Prerequisites

Notation. Throughout this paper ϵ denotes a fixed positive constant, not necessarily the same in all occurrenes. Let $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$. For $r = 1, 2, ...$ the exponent pair is

$$(k_r, l_r) = \left(\frac{1}{2} - \frac{r+1}{2(2\Lambda - 1)}, \frac{1}{2} + \frac{1}{2(2\Lambda - 1)}\right), \Lambda = 2^r.$$

The following lemmas are needed in our proof.

Lemma 2.1. *Let χ be a primitive character modulo q. For a real $z > 1$, we have*

$$\sum_{a \leq z} \chi(a) = \sum_{j \leq q} \chi(j) \left\lfloor \frac{z}{q} - \frac{j}{q} + 1 \right\rfloor.$$

Proof. From the periodicity of the primitive character modulo q, we have

$$\sum_{a \leq z} \chi(a) = \sum_{j \leq q} \sum_{\substack{a \leq z \\ a \equiv j \pmod q}} \chi(a) = \sum_{j \leq q} \sum_{\substack{a \leq z \\ a \equiv j \pmod q}} \chi(j) = \sum_{j \leq q} \chi(j) \sum_{\substack{a \leq z \\ a \equiv j \pmod q}} 1 = \sum_{j \leq q} \chi(j) \left\lfloor \frac{z}{q} - \frac{j}{q} + 1 \right\rfloor.$$

□

Lemma 2.2 (see [4, Lemma 17]). *Let x, η, α, ω be real numbers, j and q be positive numbers, where $x \geq 1, \alpha > 0, \eta \geq 1, 1 \leq j \leq q$, and (k, l) is an exponent pair with $k > 0$ and*

$$R(x, \eta, \alpha; q, j; \omega) = \sum_{\substack{n \leq \eta \\ n \equiv j \pmod q}} \psi\left(\frac{x}{n^\alpha} + \omega\right),$$

if ω is independent on n. Then

$$R(x, \eta, \alpha; q, j; \omega) = O(1) + O(x^{\frac{-1}{2}} \eta^{1+\frac{\alpha}{2}} q^{-1}) + \begin{cases} O\left(x^{\frac{k}{k+1}} \eta^{\frac{l-\alpha k}{k+1}} q^{\frac{-l}{k+1}}\right) & \text{for } l > \alpha k, \\ O\left(x^{\frac{k}{k+1}} \log \eta \, q^{\frac{-\alpha k}{k+1}}\right) & \text{for } l = \alpha k, \\ O\left((x q^{-\alpha})^{\frac{k}{1+(1+\alpha)k-l}}\right) & \text{for } l < \alpha k, \end{cases}$$

where the O-Constants is dependent on only α.

3 Proof of Theorem 1.1

Proof. For $x > 1$, we have

$$S_{\chi_1,\chi_2}(x) = \sum_{a\le x^{1/2}} \chi_1(a) \sum_{b\le x/a} \chi_2(b) + \sum_{b\le x^{1/2}} \chi_2(b) \sum_{a\le x/b} \chi_1(a) - \sum_{a\le x^{1/2}} \chi_1(a) \sum_{b\le x^{1/2}} \chi_2(b).$$

In view of Lemma 2.1 we have

$$S_{\chi_1,\chi_2}(x) = E_1 + E_2 - E_3,$$

where

$$E_1 = \sum_{j\le q_2} \chi_2(j) \sum_{a\le x^{1/2}} \chi_1(a) \left\lfloor \frac{x}{aq_2} - \frac{j}{q_2} + 1 \right\rfloor,$$

$$E_2 = \sum_{h\le q_1} \chi_1(h) \sum_{b\le x^{1/2}} \chi_2(b) \left\lfloor \frac{x}{bq_1} - \frac{h}{q_1} + 1 \right\rfloor,$$

$$E_3 = \sum_{h\le q_1} \chi_1(h) \sum_{j\le q_2} \chi_2(j) \left\lfloor \frac{x^{1/2}}{q_1} - \frac{h}{q_1} + 1 \right\rfloor \left\lfloor \frac{x^{1/2}}{bq_2} - \frac{j}{q_2} + 1 \right\rfloor.$$

From $\lfloor x \rfloor = x - \psi(x) - \frac{1}{2}$, $\psi(x) = \psi(x+1)$ and the identity $\sum_{j\le q_i} \chi_i(j) = 0$, for $i = 1, 2$, we have

$$\begin{aligned}
E_1 &= \sum_{j\le q_2} \chi_2(j) \sum_{a\le x^{1/2}} \chi_1(a) \left(\frac{x}{aq_2} - \frac{j}{q_2} + \frac{1}{2} - \psi\left(\frac{x}{aq_2} - \frac{j}{q_2} \right) \right) \\
&= -\sum_{j\le q_2} \chi_2(j) \sum_{a\le x^{1/2}} \chi_1(a) \left(\frac{j}{q_2} + \psi\left(\frac{x}{aq_2} - \frac{j}{q_2} \right) \right) \\
&= -\frac{1}{q_2} \sum_{j\le q_2} j\chi_2(j) \sum_{a\le x^{1/2}} \chi_1(a) - \sum_{j\le q_2} \chi_2(j) \sum_{a\le x^{1/2}} \chi_1(a)\psi\left(\frac{x}{aq_2} - \frac{j}{q_2} \right).
\end{aligned}$$

In view of Lemma 2.1 and the periodicity of the character modulo q, we have

$$\begin{aligned}
E_1 = &-\frac{1}{q_2} \sum_{\substack{h\le q_1 \\ j\le q_2}} j\chi_1(h)\chi_2(j) \left\lfloor \frac{x^{1/2}}{q_1} - \frac{h}{q_1} + 1 \right\rfloor \\
&- \sum_{\substack{h\le q_1 \\ j\le q_2}} \chi_1(h)\chi_2(j) \sum_{\substack{a\le x^{1/2} \\ a\equiv h \pmod{q_1}}} \psi\left(\frac{x}{aq_2} - \frac{j}{q_2} \right).
\end{aligned}$$

For E_2 and E_3, computation similar to E_1 yields,

$$\begin{aligned}
E_2 = &-\frac{1}{q_1} \sum_{\substack{h\le q_1 \\ j\le q_2}} h\chi_1(h)\chi_2(j) \left\lfloor \frac{x^{1/2}}{q_2} - \frac{j}{q_2} + 1 \right\rfloor \\
&- \sum_{\substack{h\le q_1 \\ j\le q_2}} \chi_1(h)\chi_2(j) \sum_{\substack{a\le x^{1/2} \\ a\equiv j \pmod{q_2}}} \psi\left(\frac{x}{aq_1} - \frac{h}{q_1} \right),
\end{aligned}$$

and

$$E_3 = -\frac{1}{q_2}\sum_{\substack{h\le q_1\\ j\le q_2}} j\chi_1(h)\chi_2(j)\left\lfloor \frac{x^{1/2}}{q_1} - \frac{h}{q_1} + 1\right\rfloor$$
$$-\frac{1}{q_1}\sum_{\substack{h\le q_1\\ j\le q_2}} h\chi_1(h)\chi_2(j)\left\lfloor \frac{x^{1/2}}{q_2} - \frac{j}{q_2} + 1\right\rfloor + O(q_1q_2).$$

Thus, we have

$$S_{\chi_1,\chi_2}(x) = -\sum_{\substack{h\le q_1\\ j\le q_2}} \chi_1(h)\chi_2(j) \sum_{\substack{a\le x^{1/2}\\ a\equiv h \pmod{q_1}}} \psi\left(\frac{x}{aq_2} - \frac{j}{q_2}\right)$$
$$-\sum_{\substack{h\le q_1\\ j\le q_2}} \chi_1(h)\chi_2(j) \sum_{\substack{a\le x^{1/2}\\ a\equiv j \pmod{q_2}}} \psi\left(\frac{x}{aq_1} - \frac{h}{q_1}\right) + O(q_1q_2).$$

In view of Lemma 2.2, for the exponent pair $(2/7, 4/7)$, we have

$$\sum_{\substack{h\le q_1\\ j\le q_2}} \chi_1(h)\chi_2(j) \sum_{\substack{a\le x^{1/2}\\ a\equiv h \pmod{q_1}}} \psi\left(\frac{x}{aq_2} - \frac{j}{q_2}\right)$$
$$= \sum_{\substack{h\le q_1\\ j\le q_2}} \chi_1(h)\chi_2(j) R\Big(\frac{x}{q_2}, x^{1/2}, 1, q_1, h, \frac{-j}{q_2}\Big)$$
$$\ll \sum_{\substack{h\le q_1\\ j\le q_2}} \Big(1 + \frac{x^{1/4}q_2^{1/2}}{q_1} + \frac{x^{1/3}}{q_2^{2/9}q_1^{4/9}}\Big)$$
$$= O\Big(q_1q_2 + x^{1/4}q_2^{3/2}\log q_1 + x^{1/3}q_1^{5/9}q_2^{7/9}\Big).$$

In the same way, we have

$$\sum_{\substack{h\le q_1\\ j\le q_2}} \chi_1(h)\chi_2(j) \sum_{\substack{a\le x^{1/2}\\ a\equiv j \pmod{q_2}}} \psi\left(\frac{x}{aq_1} - \frac{h}{q_1}\right) = O\Big(q_1q_2 + x^{1/4}q_1^{3/2}\log q_2 + x^{1/3}q_2^{5/9}q_1^{7/9}\Big).$$

Thus, we have

$$S_{\chi_1,\chi_2}(x) = O\Big(x^{1/4}q_2^{3/2}\log q_1 + x^{1/3}q_1^{5/9}q_2^{7/9}\Big) + O\Big(x^{1/4}q_1^{3/2}\log q_2 + x^{1/3}q_2^{5/9}q_1^{7/9}\Big) + O(q_1q_2).$$

We note that, for $x \ge q_1^2, q_2^2$, the error term $x^{1/3}q_1^{5/9}q_2^{7/9}\log q_1$ dominates the remaining terms. Thus, for $x \ge q_1^2, q_2^2$, we obtain the result in Theorem 1.1. □

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