t-cobalancing numbers and t-cobalancers

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Abstract: In this work, we determine the general terms of t-cobalancers, t-cobalancing numbers and Lucas t-cobalancing numbers by solving the Pell equation $2x^2 - y^2 = 2t^2 - 1$ for some fixed integer $t \geq 1$.

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1 Introduction

A positive integer $n$ is called a balancing number [2] if the Diophantine equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$$

(1)

holds for some positive integer $r$ which is called balancer corresponding to $n$. If $n$ is a balancing number with balancer $r$, then from (1)

$$n^2 = \frac{(n + r)(n + r + 1)}{2} \quad \text{and} \quad r = \frac{-2n - 1 + \sqrt{8n^2 + 1}}{2}. \quad (2)$$

Hence from (2) we get that $n$ is a balancing number if and only if $n^2$ is a triangular number (triangular numbers denoted by $T_n$ are the numbers of the form $T_n = \frac{n(n+1)}{2}$ for $n \geq 1$) and
$8n^2 + 1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (2), $8(0)^2 + 1 = 1$ and $8(1)^2 + 1 = 3^2$ are perfect squares. So we accept 0 and 1 to be balancing numbers. A balancing number is denoted by $B_n$ and hence $B_0 = 0, B_1 = 1, B_2 = 6$ and $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 2$.

Later Panda and Ray [16] defined that a positive integer $n$ is called a cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r)$$

holds for some positive integer $r$ which is called cobalancer corresponding to $n$. If $n$ is a cobalancing number with cobalancer $r$, then from (3)

$$n(n + 1) = \frac{(n + r)(n + r + 1)}{2} \quad \text{and} \quad r = -2n - 1 + \sqrt{8n^2 + 8n + 1}.$$  

Hence from (4) we get that $n$ is a cobalancing number if and only if $n(n + 1)$ is a triangular number and $8n^2 + 8n + 1$ is a perfect square. Since $8(0)^2 + 8(0) + 1 = 1$ is a perfect square, we accept 0 to be a cobalancing number, just like Behera and Panda accepted 0, 1 balancing numbers.

A cobalancing number is denoted by $b_n$ and $b_0 = b_1 = 0, b_2 = 2$ and $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$.

It is clear from (1) and (3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n = r_{n+1}$ and $R_n = b_n$ for $n \geq 1$, where $R_n$ is the $n$-th balancer and $r_n$ is the $n$-th cobalancer. Since $R_n = b_n$, we get from (1) that

$$b_n = -2B_n - 1 + \sqrt{8B_n^2 + 1} \quad \text{and} \quad B_n = \frac{2b_n + 1 + \sqrt{8b_n^2 + 8b_n + 1}}{2}.$$  

Thus from (5), we see that $B_n$ is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and $b_n$ is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2 + 1} \quad \text{and} \quad c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

are integers which are called the $n$-th Lucas-balancing number and $n$-th Lucas-cobalancing number, respectively.

Binet formulas for all balancing numbers are $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$ for $n \geq 1$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ which are the roots of the characteristic equation for Pell numbers $P_n$ (see also [6–8, 14, 15, 19, 21, 24]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [11], Liptai proved that there is no Fibonacci balancing number except 1 and in [12], he proved that there is no Lucas balancing number. In [23], Szalay considered the same problem and obtained some nice results by a different method. In [9], Kovács, Liptai and Olajos extended the concept of balancing numbers to the $(a, b)$-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b)$$

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for some positive integers \( n \) and \( r \), then \( an + b \) is an \((a, b)\)-balancing number. The sequence of \((a, b)\)-balancing numbers is denoted by \( B_m^{(a,b)} \) for \( m \geq 1 \). In [10], Liptai, Luca, Pint’er and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let \( y, k, l \in \mathbb{Z}^+ \) such that \( y \geq 4 \). Then a positive integer \( x \) with \( x \leq y - 2 \) is called a \((k, l)\)-power numerical center for \( y \) if
\[
1^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l.
\]
They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for \((k, l)\)-power numerical centers. For positive integers \( k, x \), let
\[
\Pi_k(x) = x(x + 1)\ldots(x + k - 1).
\]
Then it was proved in [9] that the equation \( B_m = \Pi_k(x) \) for a fixed integer \( k \geq 2 \) has only infinitely many solutions and for \( k \in \{2,3,4\} \) all solutions were determined. In [26], Tengely considered the case \( k = 5 \), that is, \( B_m = x(x + 1)(x + 2)(x + 3)(x + 4) \) and proved that this Diophantine equation has no solution for \( m \geq 0 \) and \( x \in \mathbb{Z} \). In [4], Frontczak considered the sums of balancing and Lucas-balancing numbers with binomial coefficients and in [5] he considered balancing polynomials. In [18], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [20], Patel, Irmak and Ray considered incomplete balancing and Lucas-balancing numbers and in [22], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [17], Panda and Panda defined almost balancing numbers. A natural number \( n \) is called an almost balancing number if the Diophantine equation
\[
||(n + 1) + (n + 2) + \cdots + (n + r)|| - [1 + 2 + \cdots + (n - 1)] = 1
\]
holds for some positive integer \( r \) which is called the almost balancer. In [25], the first author derived some new results on almost balancing numbers, triangular numbers and square triangular numbers.

Now let \( t \geq 1 \) be an integer. By considering (3), a positive integer \( n \) is called a \( t \)-cobalancing number if the Diophantine equation
\[
1 + 2 + \cdots + n = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)
\]
holds for some positive integer \( r \) which is called \( t \)-cobalancer corresponding to \( n \).

Let \( b_t^n \) denote the \( n \)-th \( t \)-cobalancing number and let \( r_t^n \) denote the \( n \)-th \( t \)-cobalancer. Then from (7), we get
\[
r_t^n = \frac{-2b_t^n - 2t - 1 + \sqrt{8(b_t^n)^2 + 8(t + 1)b_t^n + (2t + 1)^2}}{2}
\]
and
\[
b_t^n = \frac{2r_t^n - 1 + \sqrt{8(r_t^n)^2 + 8tr_t^n + 1}}{2}.
\]
Thus from (8), we notice that \( b_t^n \) is the \( n \)-th \( t \)-cobalancing number if and only if
\[
8(b_t^n)^2 + 8(t + 1)b_t^n + (2t + 1)^2
\]
is a perfect square. So
\[ c_n^t = \sqrt{8(b_n^t)^2 + 8(t + 1)b_n^t + (2t + 1)^2} \] (10)
is an integer which is called the \( n \)-th Lucas \( t \)-cobalancing number.

In order to determine the general terms of \( t \)-cobalancers, \( t \)-cobalancing numbers and Lucas \( t \)-cobalancing numbers, we have to determine the set of all (positive) integer solutions of the Pell equation ([1, 3, 13])
\[ 2x^2 - y^2 = 2t^2 - 1. \] (11)

From (9), we see that \( r_n^t \) is a \( t \)-cobalancer if and only if \( 8(r_n^t)^2 + 8r_n^t + 1 \) is a perfect square. So we set
\[ 8(r_n^t)^2 + 8r_n^t + 1 = y^2 \] (12)
for some integer \( y \geq 1 \). Then
\[ 2(2r_n^t + t)^2 - y^2 = 2t^2 - 1 \] and putting
\[ x = 2r_n^t + t, \] (13)
we get the Pell equation in (11). To get the set of all integer solutions of (11), we need some notations.

Let \( F(x, y) = ax^2 + bxy + cy^2 \) be an indefinite integral quadratic form [3] of discriminant \( \Delta = b^2 - 4ac \) and let \( m \) be any integer. Then the \( \Delta \)-order \( O_\Delta \) is defined for nonsquare discriminant \( \Delta \) to be the ring \( O_\Delta = \{ x + y\rho_\Delta : x, y \in \mathbb{Z} \} \), where \( \rho_\Delta = \sqrt{\frac{\Delta}{4}} \) if \( \Delta \equiv 0 \pmod{4} \) or \( 1 + \sqrt{\frac{\Delta}{2}} \) if \( \Delta \equiv 1 \pmod{4} \). So \( O_\Delta \) is a subring of \( \mathbb{Q}(\sqrt{\Delta}) = \{ x + y\sqrt{\Delta} : x, y \in \mathbb{Q} \} \). The unit group \( O_\Delta^* \) is defined to be the group of units of the ring \( O_\Delta \). We can rewrite \( F \) to be
\[ F(x, y) = \frac{(xa + y\frac{b+\sqrt{\Delta}}{2})(xa + y\frac{b-\sqrt{\Delta}}{2})}{a}. \]

So the module \( M_F \) of \( F \) is
\[ M_F = \{ xa + y\frac{b+\sqrt{\Delta}}{2} : x, y \in \mathbb{Z} \} \subset \mathbb{Q}(\sqrt{\Delta}). \]

Therefore, we get
\[ (u + v\rho_\Delta)(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'b + \frac{\sqrt{\Delta}}{2}, \]
where
\[
[x' \ y'] = \begin{cases} 
\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 \pmod{4}, \\
\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} u + \frac{b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 \pmod{4}.
\end{cases}
\] (14)

So there is a bijection \( \Psi : \{ (x, y) : F(x, y) = m \} \to \{ \gamma \in M_F : N(\gamma) = am \} \) for solving \( F(x, y) = m \). The action of \( O_{\Delta,1}^* = \{ \alpha \in O_\Delta^* : N(\alpha) = 1 \} \) on \( \{ (x, y) : F(x, y) = m \} \) of integral solutions of the equation \( F(x, y) = m \) is most interesting when \( \Delta \) is a positive nonsquare since \( O_{\Delta,1}^* \) is infinite. Therefore, the orbit of each solution will be infinite and so the set \( \{ (x, y) : \} \).
\(F(x, y) = m\) is either empty or infinite. Since \(O_{\Delta, 1}\) can be explicitly determined, the set \(\{(x, y) : F(x, y) = m\}\) is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let \(\epsilon_\Delta > 1\) be the smallest unit of \(O_\Delta\) and let \(\tau_\Delta = \epsilon_\Delta\) if \(N(\epsilon_\Delta) = 1\) or \(\epsilon_\Delta^2\) if \(N(\epsilon_\Delta) = -1\). Then every \(O_{\Delta, 1}\) orbit of integral solutions of \(F(x, y) = m\) contains a solution \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) such that \(0 \leq y \leq U\), where

\[
U = \left\{ \left| \frac{am\tau_\Delta}{\Delta} \right|^\frac{1}{2} \left( 1 - \frac{1}{\tau_\Delta} \right) \text{ if } am > 0 \right. \\
\left. \left| \frac{am\tau_\Delta}{\Delta} \right|^\frac{1}{2} \left( 1 + \frac{1}{\tau_\Delta} \right) \text{ if } am < 0. \right.
\]

So for finding the a set of representatives of the \(O_{\Delta, 1}\) orbits of integral solutions of \(F(x, y) = m\), we must find for each integer \(y_0\) such that \(0 \leq y_0 \leq U\), all integers \(x_0\) that satisfy \(F(x_0, y_0) = m\). If \(F(x_0, y_0) = m\), then

\[
ax_0^2 + bx_0y_0 + cy_0^2 = m \iff \Delta y_0^2 + 4am = (2ax_0 + by_0)^2
\]

and hence

\[
x_0 = \frac{-by_0 \pm \sqrt{\Delta y_0^2 + 4am}}{2a}.
\]

Consequently, we get the set of representatives \(\text{Rep} = \{[x_0 \ y_0]\}\). Thus for the matrix \(M\) defined in (14), the set of all integer solutions of \(F(x, y) = m\) is

\[
\{\pm(x, y) : [x \ y] = [x_0 \ y_0]M^n, n \in \mathbb{Z}\}.
\]

2 Main results

For the set of all integer solutions of (11), the indefinite form \(F(x, y) = 2x^2 - y^2\) of discriminant \(\Delta = 8\). So \(\tau_8 = 3 + 2\sqrt{2}\) and

\[
M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}
\]

by (14). In order to determine the set of all integer solutions of (11), we have to consider our problem in three cases: \(t = 1, 2t^2 - 1\) is a perfect square or not a perfect square for \(t \geq 2\).

2.1 Case 1: \(t = 1\)

Theorem 2.1. If \(t = 1\), then

1. The set of all integer solutions of \(2x^2 - y^2 = 1\) is \(\{-2B_n + C_n, 4B_n - C_n\} : n \geq 1\}.

2. The general terms of 1-cobalancers, 1-cobalancing numbers and Lucas 1-cobalancing numbers are

\[
r_n^1 = b_{n+1}, \ b_n^1 = B_{n+1} - 1 \text{ and } c_n^1 = C_{n+1}
\]

for \(n \geq 1\).
Proof. (1) Let $t = 1$. Then for the Pell equation $2x^2 - y^2 = 1$, in the range
\[
0 \leq y_0 \leq U = \left| \frac{amr_t}{\Delta} \right| \left( 1 - \frac{1}{r_t} \right) = \left| \frac{2(1)(3 + 2\sqrt{2})}{8} \right| \left( \frac{2 + 2\sqrt{2}}{3 + 2\sqrt{2}} \right) = 1,
\]
$\Delta y_0^2 + 4am = 8y_0^2$ is a perfect square only for $y_0 = 1$ and hence
\[
x_0 = \frac{-by_0 \pm \sqrt{\Delta y_0^2 + 4am}}{2a} = \pm \sqrt{\frac{1}{4}8(1)^2 + 8} = \pm 1.
\]
So the set of representatives is $\text{Rep} = \{ [\pm 1 \ 1] \}$ and in this case $[1 - 1]M^n$ (where $M$ is defined in (15)) generates all integer solutions $(x_n, y_n)$ for $n \geq 1$. Since the $n$-th power of $M$ is
\[
M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}
\]
for $n \geq 1$, we conclude that the set of all integer solutions is $\{( -2B_n + C_n, 4B_n - C_n ) : n \geq 1 \}$.

(2) Recall that $x = 2r_t^n + t$ by (13). So from (1), we easily deduce that
\[
r_t^n = \frac{-2B_{n+1} + C_{n+1} - 1}{2} = \frac{-2(\frac{\alpha^{2n+2} - \beta^{2n+2}}{2\sqrt{2}} + \frac{\alpha^{2n+2} + \beta^{2n+2}}{4\sqrt{2}} - 1}{2} = \alpha^{2n+2} \left( \frac{1}{4\sqrt{2}} + \frac{1}{4} \right) + \beta^{2n+2} \left( \frac{1}{4\sqrt{2}} + \frac{1}{4} \right) - \frac{1}{2}
\]
\[
= \frac{\alpha^{2n+2} + \beta^{2n+2} (\sqrt{2} + 1) - 1}{2} = \frac{\alpha^{2n+1} - \beta^{2n+1} - \frac{1}{2}}{2} = b_{n+1}.
\]
Note that $\sqrt{8(r_t^n)^2 + 8tr_n^n + 1} = y_n$ by (12). So from (9), we get
\[
b_t^n = \frac{2r_t^n - 1 + \sqrt{8(r_t^n)^2 + 8tr_t^n + 1}}{2} = \frac{2b_{n+1} + 4B_{n+1} - C_{n+1}}{2}
\]
\[
= \frac{2(\frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2} - 1 + 4(\frac{\alpha^{2n+2} - \beta^{2n+2}}{4\sqrt{2}} - \frac{\alpha^{2n+2} + \beta^{2n+2}}{4\sqrt{2}})}{2}
\]
\[
= \frac{\alpha^{2n+1} (\frac{1}{2\sqrt{2}} + \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}})}{4\sqrt{2}} + \beta^{2n+1} (\frac{1}{2\sqrt{2}} - \frac{\beta}{\sqrt{2}} - \frac{\beta}{2}) - 1
\]
\[
= \frac{\alpha^{2n+2} - \beta^{2n+2}}{4\sqrt{2}} - 1 = B_{n+1} - 1
\]
and from (10), we conclude that
\[
c_t^n = \sqrt{8(B_{n+1} - 1)^2 + 16(B_{n+1} - 1) + 9} = \sqrt{8B_{n+1}^2 + 1} = C_{n+1}
\]
as we wanted. □
2.2 Case 2: $2t^2 - 1$ is a perfect square

In this section, we assume that $2t^2 - 1$ is a perfect square for $t \geq 2$. Before considering our problem, we can give the following theorem.

**Theorem 2.2.** The quadratic equation $2t^2 - 1 = h^2$ is satisfied for $(t_n, h_n) = (P_{2n-1}, c_n)$ for $n \geq 1$.

**Proof.** Let $2t^2 - 1 = h^2$ for some positive integer $h$. Then we get the Pell equation $2t^2 - h^2 = 1$. In this case the set of representatives is $\text{Rep} = \{[\pm 1 \ 1] \}$. $M_n$ generates all integer solutions $(t_n, h_n)$ for $n \geq 1$. Thus $2t^2 - 1 = h^2$ is satisfied for $(t_n, h_n) = (-2B_n + C_n, 4B_n - C_n)$ for $n \geq 1$. On the other hand it can be easily seen that

$$-2B_n + C_n = P_{2n-1} \quad \text{and} \quad 4B_n - C_n = c_n,$$

so the quadratic equation $2t^2 - 1 = h^2$ is satisfied for $(t_n, h_n) = (P_{2n-1}, c_n)$ for $n \geq 1$. Indeed, since $P_n = \frac{a_n - \beta^n}{2\sqrt{2}}$ and $c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$, we easily get

$$2t^2 - 1 = \frac{\alpha^{4n-2} + \beta^{4n-2} + 2(\alpha \beta)^{2n-1}}{4} = \left(\frac{\alpha^{2n-1} + \beta^{2n-1}}{2}\right)^2 = c_n^2 = h^2$$

as we claimed. \qed

From Theorem 2.2, we see that $2t^2 - 1$ is a perfect square for $t = P_{2n-1}$. Consequently, we determine the general terms of all $P_{2n-1}$-cobalancers, $P_{2n-1}$-cobalancing numbers and Lucas $P_{2n-1}$-cobalancing numbers for $n \geq 2$ (For $n = 1$, we have $t = P_1 = 1$ and clearly we have already considered it in the previous section). In order to determine the set of all integer solutions of (11), we have two cases: $\#\text{Rep} = 4$ or $\#\text{Rep} > 4$.

**Theorem 2.3.** If $\#\text{Rep} = 4$, then

1. The set of all integer solutions of (11) is

$$\{(x_{3n+1}, y_{3n+1}), (x_{3n+2}, y_{3n+2}) : n \geq 0\} \cup \{(x_{3n}, y_{3n}) : n \geq 1\},$$

where

$$(x_{3n+1}, y_{3n+1}) = (2B_n + tC_n, 4tB_n + C_n)$$

$$(x_{3n+2}, y_{3n+2}) = (2hB_n + hC_n, 4hB_n + hC_n)$$

$$(x_{3n}, y_{3n}) = (-2B_n + tC_n, 4tB_n - C_n).$$

2. The general terms of $t$-cobalancers, $t$-cobalancing numbers and Lucas $t$-cobalancing numbers are

$$r_{3n}^t = \frac{2B_n + tC_n - t}{2}$$

$$r_{3n-1}^t = \frac{-2B_n + tC_n - t}{2}$$

$$b_{3n}^t = \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}$$

$$b_{3n-1}^t = \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2}$$

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Thus the set of all integer solutions is
\[
\begin{align*}
\{ & 8(b_{3n})^2 + 8(t + 1)b_{3n} + (2t + 1)^2 \\
\{ & 8(b_{3n-1})^2 + 8(t + 1)b_{3n-1} + (2t + 1)^2
\end{align*}
\]
for \( n \geq 1 \) and

\[
\begin{align*}
r_{3n+1}^t &= \frac{2hB_n + hC_n - t}{2} \\
b_{3n+1}^t &= \frac{2hB_{n+1} - t - 1}{2} \\
c_{3n+1}^t &= \sqrt{8(b_{3n+1})^2 + 8(t + 1)b_{3n+1} + (2t + 1)^2}
\end{align*}
\]
for \( n \geq 0 \).

Proof. (1) If \( \#\text{Rep} = 4 \), then the set of representatives is \( \text{Rep} = \{ [\pm t, 1], [\pm h, h] \} \) and in this case

1. \([t, 1]M^n \) generates all integer solutions \((x_{3n+1}, y_{3n+1}) \) for \( n \geq 0 \),
2. \([t, -1]M^n \) generates all integer solutions \((x_{3n}, y_{3n}) \) for \( n \geq 1 \),
3. \([h, h]M^n \) generates all integer solutions \((x_{3n+2}, y_{3n+2}) \) for \( n \geq 0 \).

Thus the set of all integer solutions is \( \{(2B_n + tC_n, 4tB_n + C_n), (2hB_n + hC_n, 4hB_n + hC_n): n \geq 0 \} \cup \{(-2B_n + tC_n, 4tB_n - C_n): n \geq 1 \} \).

(2) From (1), we easily get

\[
r_{3n}^t = \frac{2B_n + tC_n - t}{2}
\]
for \( n \geq 1 \). Thus from (9), we get

\[
b_{3n}^t = \frac{2B_n + tC_n - t - 1 + 4tB_n + C_n}{2}
\]
\[
= \frac{t(4B_n + C_n - 1) + 2B_n + C_n - 1}{2}
\]
\[
= \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}
\]
for \( n \geq 1 \) since \( 4B_n + C_n - 1 = 2(B_n + b_{n+1}) \). From (10), we observe that

\[
c_{3n}^t = \sqrt{8(b_{3n})^2 + 8(t + 1)b_{3n} + (2t + 1)^2}
\]
for \( n \geq 1 \). The other cases can be proved similarly. \( \square \)

**Theorem 2.4.** If \( \#\text{Rep} = 2k > 4 \), then

1. The set of all integer solutions of (11) is

\[
\{(x(2k-1)n+1, y(2k-1)n+1), (x(2k-1)n+i+1, y(2k-1)n+i+1), (x(2k-1)n+k, y(2k-1)n+k): n \geq 0 \} \cup \{(x(2k-1)n, y(2k-1)n), (x(2k-1)n-i, y(2k-1)n-i): n \geq 1 \},
\]

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where
\[
(x_{(2k-1)n+1}, y_{(2k-1)n+1}) = (2B_n + tC_n, 4tB_n + C_n)
\]
\[
(x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}) = (2t_2iB_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_2iC_n)
\]
\[
(x_{(2k-1)n+k}, y_{(2k-1)n+k}) = (2hB_n + hC_n, 4hB_n + hC_n)
\]
\[
(x_{(2k-1)n}, y_{(2k-1)n}) = (-2B_n + tC_n, 4tB_n - C_n)
\]
\[
(x_{(2k-1)n-i}, y_{(2k-1)n-i}) = (-2t_2iB_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_2iC_n).
\]

2. The general terms of \(t\)-cobalancers, \(t\)-cobalancing numbers and Lucas \(t\)-cobalancing numbers are

\[
r^t_{(2k-1)n} = \frac{2B_n + tC_n - t}{2}
\]
\[
r^t_{(2k-1)n-1} = \frac{-2B_n + tC_n - t}{2}
\]
\[
r^t_{(2k-1)n-i-1} = \frac{-2t_2iB_n + t_{2i-1}C_n - t}{2}
\]
\[
b^t_{(2k-1)n} = \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}
\]
\[
b^t_{(2k-1)n-1} = \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2}
\]
\[
b^t_{(2k-1)n-i-1} = \frac{(-2t_2i + 4t_{2i-1})B_n + (t_{2i-1} - t_2i)C_n - t - 1}{2}
\]
\[
c^t_{(2k-1)n} = \sqrt{8(b^t_{(2k-1)n})^2 + 8(t + 1)b^t_{(2k-1)n} + (2t + 1)^2}
\]
\[
c^t_{(2k-1)n-1} = \sqrt{8(b^t_{(2k-1)n-1})^2 + 8(t + 1)b^t_{(2k-1)n-1} + (2t + 1)^2}
\]
\[
c^t_{(2k-1)n-i-1} = \sqrt{8(b^t_{(2k-1)n-i-1})^2 + 8(t + 1)b^t_{(2k-1)n-i-1} + (2t + 1)^2}
\]

for \(n \geq 1\) and

\[
r^t_{(2k-1)n+i} = \frac{2t_2iB_n + t_{2i-1}C_n - t}{2}
\]
\[
r^t_{(2k-1)n+k-1} = \frac{2hB_n + hC_n - t}{2}
\]
\[
b^t_{(2k-1)n+i} = \frac{(2t_2i + 4t_{2i-1})B_n + (t_{2i-1} + t_2i)C_n - t - 1}{2}
\]
\[
b^t_{(2k-1)n+k-1} = \frac{6hB_n + 2hC_n - t - 1}{2}
\]
\[
c^t_{(2k-1)n+i} = \sqrt{8(b^t_{(2k-1)n+i})^2 + 8(t + 1)b^t_{(2k-1)n+i} + (2t + 1)^2}
\]
\[
c^t_{(2k-1)n+k-1} = \sqrt{8(b^t_{(2k-1)n+k-1})^2 + 8(t + 1)b^t_{(2k-1)n+k-1} + (2t + 1)^2}
\]

for \(n \geq 0\),

where \(t_{2i-1}\) and \(t_{2i}\) are positive integers such that \(2t_{2i-1}^2 - t_{2i}^2 = 2t^2 - 1\) for \(1 \leq i \leq k-2\), \(t < t_1 < t_3 < \cdots < t_{2k-5} < h\) and \(1 < t_2 < t_4 < \cdots < t_{2k-4} < h\).
Proof. (1) Let \( \#\text{Rep} = 2k > 4 \), then the set of representatives is

\[
\text{Rep} = \{[\pm t \ 1], [\pm t_{2i-1} \ t_{2i}], [\pm h \ h] \},
\]

where \( t_{2i-1} \) and \( t_{2i} \) are positive integers such that \( 2t^2_{2i-1} - t^2_{2i} = 2t^2 - 1 \) for \( 1 \leq i \leq k-2, t < t_1 < t_3 < \cdots < t_{2k-5} < h \) and \( 1 < t_2 < t_4 < \cdots < t_{2k-4} < h \). In this case

1. \( [t \ 1]^M \) generates all integer solutions \( (x_{(2k-1)n+1}, y_{(2k-1)n+1}) \) for \( n \geq 0 \),
2. \( [t \ -1]^M \) generates all integer solutions \( (x_{(2k-1)n}, y_{(2k-1)n}) \) for \( n \geq 1 \),
3. \( [h \ h]^M \) generates all integer solutions \( (x_{(2k-1)n+k}, y_{(2k-1)n+k}) \) for \( n \geq 0 \),
4. \( [t_{2i-1} \ t_{2i}]^M \) generates all integer solutions \( (x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}) \) for \( n \geq 0 \),
5. \( [t_{2i-1} - t_{2i}]^M \) generates all integer solutions \( (x_{(2k-1)n-i}, y_{(2k-1)n-i}) \) for \( n \geq 1 \).

So the set of all integer solutions is \( \{(2B_n + tC_n, 4tB_n + C_n), (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n), (2hB_n + hC_n, 4hB_n + hC_n) : n \geq 0\} \cup \{(-2B_n + tC_n, 4tB_n - C_n), (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n) : n \geq 1\} \).

(2) It can be proved in the same way that Theorem 2.3 was proved. \( \square \)

In Table 1, the set of representatives is given for some values of \( t \). As we can see in Table 1, when \( \#\text{Rep} = 2k > 4 \), it is impossible to determine the set of representatives and \#\text{Rep} in terms of \( t \). That is why we assume that \( \text{Rep} = \{[\pm t \ 1], [\pm t_{2i-1} \ t_{2i}], [\pm h \ h] \} \), where \( t_{2i-1} \) and \( t_{2i} \) are positive integers such that \( 2t^2_{2i-1} - t^2_{2i} = 2t^2 - 1 \) for \( 1 \leq i \leq k-2, t < t_1 < t_3 < \cdots < t_{2k-5} < h \) and \( 1 < t_2 < t_4 < \cdots < t_{2k-4} < h \).

<table>
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<th>( t )</th>
<th>Set of representatives</th>
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<td>985</td>
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<td>33461</td>
<td>{[\pm 33461 \ 1], [\pm 35155 \ 15247], [\pm 38935 \ 28153], [\pm 40409 \ 32039], [\pm 47321 \ 47321]}</td>
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<td>195025</td>
<td>{[\pm 195025 \ 1], [\pm 195083 \ 6767], [\pm 195257 \ 13457], [\pm 197005 \ 39401], [\pm 197743 \ 46207], [\pm 199547 \ 59737], [\pm 202985 \ 79601], [\pm 205933 \ 93527], [\pm 205973 \ 93703], [\pm 207607 \ 100657], [\pm 209405 \ 107849], [\pm 211327 \ 115103], [\pm 219883 \ 143623], [\pm 222425 \ 151249], [\pm 227837 \ 166583], [\pm 236623 \ 189503], [\pm 243355 \ 205849], [\pm 243443 \ 206057], [\pm 246977 \ 214303], [\pm 250747 \ 222887], [\pm 254665 \ 231601], [\pm 271133 \ 266377], [\pm 275807 \ 275807]}</td>
</tr>
</tbody>
</table>

Table 1. \( 2t^2 - 1 \) is a perfect square

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2.3 Case 3: $2t^2 - 1$ is not a perfect square

When $2t^2 - 1$ is not a perfect square for $t \geq 2$, we have two cases: $\#\text{Rep} = 2$ or $\#\text{Rep} > 2$.

**Theorem 2.5.** If $\#\text{Rep} = 2$, then

1. The set of all integer solutions of (11) is \( \{(x_{2n+1}, y_{2n+1}) : n \geq 0\} \cup \{(x_{2n}, y_{2n}) : n \geq 1\} \), where
   \[
   (x_{2n+1}, y_{2n+1}) = (2B_n + tC_n, 4tB_n + C_n) \\
   (x_{2n}, y_{2n}) = (-2B_n + tC_n, 4tB_n - C_n).
   \]

2. The general terms of $t$-cobalancers, $t$-cobalancing numbers and Lucas $t$-cobalancing numbers are
   \[
   r_{2n}^t = \frac{2B_n + tC_n - t}{2} \\
   r_{2n-1}^t = \frac{-2B_n + tC_n - t}{2} \\
   b_{2n}^t = \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2} \\
   b_{2n-1}^t = \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2} \\
   c_{2n}^t = \sqrt{8(b_{2n}^t)^2 + 8(t + 1)b_{2n}^t + (2t + 1)^2} \\
   c_{2n-1}^t = \sqrt{8(b_{2n-1}^t)^2 + 8(t + 1)b_{2n-1}^t + (2t + 1)^2}
   \]
   for $n \geq 1$.

**Proof.** (1) If $\#\text{Rep} = 2$, then the set of representatives is $\text{Rep} = \{[\pm t \ 1]\}$ and in this case, $[t \ 1]M^n$ generates all integer solutions $(x_{2n+1}, y_{2n+1})$ for $n \geq 0$ and $[t \ -1]M^n$ generates all integer solutions $(x_{2n}, y_{2n})$ for $n \geq 1$. Thus the set of all integer solutions is \( \{(2B_n + tC_n, 4tB_n + C_n) : n \geq 0\} \cup \{(-2B_n + tC_n, 4tB_n - C_n) : n \geq 1\} \).

(2) From (1), we get
   \[
   r_{2n}^t = \frac{2B_n + tC_n - t}{2}.
   \]
   Hence we get from (9)
   \[
   b_{2n}^t = \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}
   \]
   and from (10)
   \[
   c_{2n}^t = \sqrt{8(b_{2n}^t)^2 + 8(t + 1)b_{2n}^t + (2t + 1)^2}
   \]
   for $n \geq 1$. \hfill \Box

**Theorem 2.6.** If $\#\text{Rep} = 2k > 2$, then

1. The set of all integer solutions of (11) is
   \[
   \{(x_{2kn+1}, y_{2kn+1}), (x_{2kn+i+1}, y_{2kn+i+1}) : n \geq 0\} \cup \{(x_{2kn}, y_{2kn}), (x_{2kn-i}, y_{2kn-i}) : n \geq 1\},
   \]

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where
\[
(x_{2kn+1}, y_{2kn+1}) = (2B_n + tC_n, 4tB_n + C_n)
\]
\[
(x_{2kn+i+1}, y_{2kn+i+1}) = (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n)
\]
\[
(x_{2kn}, y_{2kn}) = (-2B_n + tC_n, 4tB_n - C_n)
\]
\[
(x_{2kn-i}, y_{2kn-i}) = (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n).
\]

2. The general terms of \( t \)-cobalancers, \( t \)-cobalancing numbers and Lucas \( t \)-cobalancing numbers are
\[
r_{2kn}^t = \frac{2B_n + tC_n - t}{2}
\]
\[
r_{2kn-1}^t = \frac{-2B_n + tC_n - t}{2}
\]
\[
r_{2kn-i-1}^t = \frac{-2t_{2i}B_n + t_{2i-1}C_n - t}{2}
\]
\[
b_{2kn}^t = \frac{2t(B_n + b_{n+1}) + 2B_n + C_n - 1}{2}
\]
\[
b_{2kn-1}^t = \frac{2t(B_n + b_{n+1}) - 2B_n - C_n - 1}{2}
\]
\[
b_{2kn-i-1}^t = \frac{(-2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} - t_{2i})C_n - t - 1}{2}
\]
\[
c_{2kn}^t = \sqrt{8(b_{2kn}^t)^2 + 8(t + 1)b_{2kn}^t + (2t + 1)^2}
\]
\[
c_{2kn-1}^t = \sqrt{8(b_{2kn-1}^t)^2 + 8(t + 1)b_{2kn-1}^t + (2t + 1)^2}
\]
\[
c_{2kn-i-1}^t = \sqrt{8(b_{2kn-i-1}^t)^2 + 8(t + 1)b_{2kn-i-1}^t + (2t + 1)^2}
\]

for \( n \geq 1 \) and
\[
r_{2kn+i}^t = \frac{2t_{2i}B_n + t_{2i-1}C_n - t}{2}
\]
\[
b_{2kn+i}^t = \frac{(2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} + t_{2i})C_n - t - 1}{2}
\]
\[
c_{2kn+i}^t = \sqrt{8(b_{2kn+i}^t)^2 + 8(t + 1)b_{2kn+i}^t + (2t + 1)^2}
\]

for \( n \geq 0 \),

where \( t_{2i-1} \) and \( t_{2i} \) are positive integers such that \( 2t_{2i-1}^2 - t_{2i}^2 = 2t^2 - 1 \) for \( 1 \leq i \leq k-1, t < t_1 < t_3 < \cdots < t_{2k-3} \) and \( 1 < t_2 < t_4 < \cdots < t_{2k-2} \).

Proof. (1) If \( \# \text{Rep} = 2k > 2 \), then the set of representatives is \( \text{Rep} = \{[\pm t \ 1], [\pm t_{2i-1} \ t_{2i}]\} \), where \( t_{2i-1} \) and \( t_{2i} \) are positive integers such that \( 2t_{2i-1}^2 - t_{2i}^2 = 2t^2 - 1 \) for \( 1 \leq i \leq k-1, t < t_1 < t_3 < \cdots < t_{2k-3} \) and \( 1 < t_2 < t_4 < \cdots < t_{2k-2} \). Here,

1. \([t \ 1]M^n\) generates all integer solutions \((x_{2kn+1}, y_{2kn+1})\) for \( n \geq 0 \),
2. \([t \ -1]M^n\) generates all integer solutions \((x_{2kn}, y_{2kn})\) for \( n \geq 1 \),
3. \([t_{2i-1} \ t_{2i}]M^n\) generates all integer solutions \((x_{2kn+i+1}, y_{2kn+i+1})\) for \( n \geq 0 \),
4. \([t_{2i-1} \ -t_{2i}]M^n\) generates all integer solutions \((x_{2kn-i}, y_{2kn-i})\) for \( n \geq 1 \).
So the set of all integer solutions is \( \{(2B_n + tC_n, 4tB_n + C_n), (2t_2B_n + t_{2i-1}C_n, At_{2i-1}B_n + t_{2i}C_n) : n \geq 0\} \cup \{(-2B_n + tC_n, 4tB_n - C_n), (-2t_2B_n + t_{2i-1}C_n, At_{2i-1}B_n - t_{2i}C_n) : n \geq 1\} \).

(2) It can be proved in the same way that Theorem 2.5 was proved. □

Again when \#Rep = 2k > 2, it is impossible to determine the set of representatives and \#Rep in terms of \( t \). For example in Table 2, the set of representatives is given for some values of \( t \). That is why we assume that Rep = \{[±1], [±t_{2i-1} t_{2i}]\}, where \( t_{2i-1} \) and \( t_{2i} \) are positive integers such that \( 2t_{2i-1}^2 - t_{2i}^2 = 2t^2 - 1 \) for \( 1 \leq i \leq k - 1, t < t_1 < t_2 < \cdots < t_{2k-3} \) and \( 1 < t_2 < t_4 < \cdots < t_{2k-2} \).

<table>
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<tr>
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<tbody>
<tr>
<td>58</td>
<td>{[±58 1], [±62 31], [±74 65]}</td>
</tr>
<tr>
<td>142</td>
<td>{[±142 1], [±148 59], [±182 161]}</td>
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<td>{[±54 1], [±56 21], [±60 37], [±70 63]}</td>
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<td>135</td>
<td>{[±135 1], [±137 33], [±173 153], [±187 183]}</td>
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<td>152</td>
<td>{[±152 1], [±154 35], [±158 61], [±178 131], [±196 175], [±212 209]}</td>
</tr>
<tr>
<td>299</td>
<td>{[±299 1], [±301 49], [±311 121], [±359 281], [±385 343], [±415 407]}</td>
</tr>
<tr>
<td>275</td>
<td>{[±275 1], [±277 47], [±293 143], [±295 151], [±307 193], [±317 223], [±353 313], [±383 377]}</td>
</tr>
</tbody>
</table>

Table 2. \( 2t^2 - 1 \) is not a perfect square

3 Concluding remark

In this paper, we determine the general terms of all \( t \)-cobalancers, \( t \)-cobalancing numbers and Lucas \( t \)-cobalancing numbers by solving the Pell equation \( 2x^2 - y^2 = 2t^2 - 1 \) for some fixed integer \( t \geq 1 \) in three cases: \( t = 1, 2t^2 - 1 \) is a perfect square or not a perfect square for \( t \geq 2 \). But in all cases, we are able to determine the general terms of them.

References


