Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 4, 150–157
DOI: 10.7546/nntdm.2019.25.4.150-157

On Beck's zero-divisor graph

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Received: 6 April 2019 **Revised:** 10 October 2019 **Accepted:** 12 October 2019

Abstract: For a commutative ring R with unity ($1 \neq 0$), the zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph with vertices as elements of R and two distinct vertices are adjacent whenever the product of the vertices is zero. This article aims at gaining a deeper insight into the basic structural properties of zero-divisor graphs given by Beck.
Keywords: Commutative ring, Zero-divisors, Diameter, Girth, Path graph, Complete graph, Complete bipartite graph, Star graph.
2010 Mathematics Subject Classification: 05C25, 05C75.

1 Introduction

For vocabulary and notations in graph theory, not explicitly mentioned in this article, we refer the reader to the standard textbooks [5, 9]; for ring theory, see [6, 10, 12]. Unless stated otherwise, the unity of a ring is different from its additive identity ($1 \neq 0$) and all rings examined in this article are commutative. All rings have at least two elements. As usual, the ring of integers modulo n is denoted by $\mathbb{Z}_n$ and finite field with p elements by $\mathbb{F}_p$.

The zero-divisor graph of a ring R, denoted by $\Gamma(R)$, is a simple graph with vertices as elements of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. This concept was introduced by Beck [4] in 1988, but he mainly worked on colorings of R. The study of colorings of a commutative ring was then continued by Anderson and Naseer [1]. Likewise, Anderson and Livingston [3] gave an altered definition. They associate a simple graph with vertices as elements of $Z(R)^* = Z(R) \setminus \{0\}$, where $Z(R)$ is the set of zero-divisors of R and the adjacency between two distinct vertices is defined in the same way as that of Beck's zero-divisor graph. Since then an ample research [2, 8, 11, 13, 14] has been done on zero-divisor graphs given by Anderson and Livingston. But to the best of our understanding, nobody has worked on the definition presented by Beck [4]. In this article, we limited our research to Beck's definition of zero-divisor graphs.

2 Main results

This section presents results analogous to the case for zero-divisor graphs found in Anderson and Livingston [3, Section 2]. We show that $\Gamma(R)$ is always connected and has small diameter and girth. We further determine which complete graphs and star graphs may be realized as $\Gamma(R)$. One can quickly notice the following basic facts about the zero-divisor graphs.

Proposition 2.1. *For any ring R, $\Gamma(R)$ always contains a universal vertex (i.e., the maximum degree $\Delta(\Gamma(R)) = |\Gamma(R)| - 1 = |R| - 1$). Further, if R is a commutative ring with unity, then minimum degree $\delta(\Gamma(R)) = 1$.*

Proof. It is clear, for $x \in R$ ($x \neq 0$), $0x$ is an edge in $\Gamma(R)$. Therefore $deg(0) = |R| - 1$ ($= |\Gamma(R)| - 1$). Also, for a commutative ring with unity R, since $1 \cdot x = x \cdot 1 = 0$ holds only when $x = 0$, $\Gamma(R)$ always contains at least one pendant vertex and hence $\delta(\Gamma(R)) = 1$. ☐

In fact, 0 is the only universal vertex. Further, for a commutative ring with unity R, it is the only cut vertex of $\Gamma(R)$. As for any $x \in R \setminus Z(R)$ and $y \in Z(R)^*$, 0 lies on every $x - y$ path.

Lemma 2.2. *For any ring R, there always exists at least one cycle of length 3 in $\Gamma(R)$, whenever there are more than one nonzero zero-divisors.*

Proof. Let us suppose $|Z(R)^*| > 1$. Due to this, there exist at least two nonzero zero-divisors x, $y \in R$ such that $xy = 0$ and consequently $0 - x - y - 0$ is a cycle of length 3 in $\Gamma(R)$. Hence, rest of the result follows. ☐

We now give a stronger condition for the existence of a cycle of length 3 in $\Gamma(R)$.

Theorem 2.3. *For any ring R, there exists a cycle of length 3 in $\Gamma(R)$ if and only if $|Z(R)^*| > 1$.*

Proof. For proving the necessity part, let us assume $|Z(R)^*| \leq 1$. Since $\Gamma(R)$ contains a cycle of length 3, it necessarily contains three distinct vertices x, y, and z such that $x - y - z - x$ is a cycle in $\Gamma(R)$. As a result, the elements x, y, and z are the zero-divisors of a ring R. Also, at the most one of these elements can be 0 of a ring. Hence, the set of nonzero zero-divisors must contain at least two elements, a contradiction to our assumption. Therefore $|Z(R)^*| > 1$.

The sufficient part follows from Lemma 2.2. ☐

Figure 1 gives the zero-divisor graphs for different rings. Parallel to the case in [3], nonisomorphic rings may have the same zero-divisor graph, and zero-divisor graph does not distinguish nilpotent elements.

Example 2.4. *Figure 1 shows possible graphs $\Gamma(R)$ with $|\Gamma(R)| \leq 4$, where up to isomorphism, each graph may be realized as $\Gamma(R)$ by precisely the following rings, respectively:*

(a) $\mathbb{Z}_2$
(b) $\mathbb{Z}_3$
(c) $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/\langle x^2\rangle$ *or* $\mathbb{F}_4$
(d) $\mathbb{Z}_2 \times \mathbb{Z}_2$

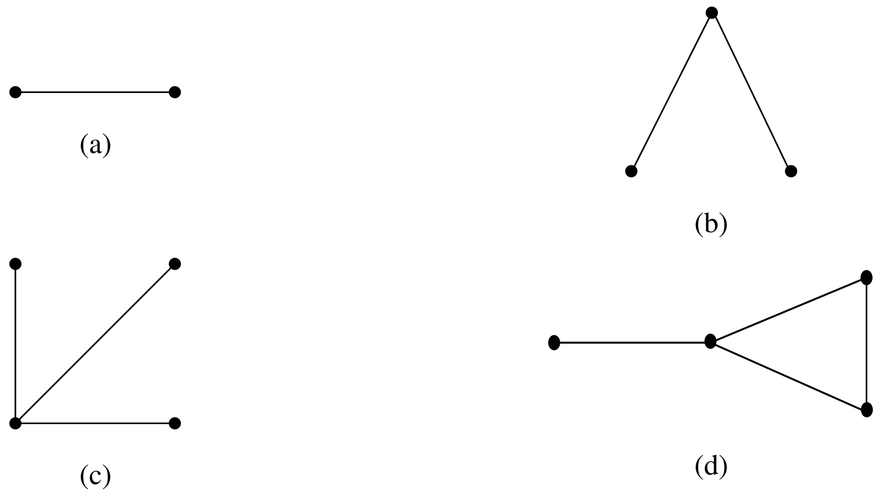

Figure 1. Examples showing all possible zero-divisor graphs with $|\Gamma(R)| \leq 4$

Up to isomorphism, there is a unique commutative ring with a unity of order p, where p is prime. As a result, Figure 1 shows the only possible zero-divisor graphs with $1 < |\Gamma(R)| \leq 3$. Note that $\Gamma(R)$ can never be a cycle of length 3. Our graphs are different from that of [3].

Also, it is well identified that out of eleven rings of order four, only four are commutative rings with unity [7], which are mentioned in above example. Again we get different graphs.

If $diam(G)$ and $g(G)$ represents diameter and girth of the graph G, respectively, then we have the following results.

Theorem 2.5. *For any ring R, the following conditions hold:*

1. *$\Gamma(R)$ is finite if and only if R is finite.*
2. *$\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 2$. Furthermore, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 5$.*

Proof.

1. The proof of this part is straightforward.

2. Let x and y be distinct vertices of $\Gamma(R)$. If either of $x = 0$ or $y = 0$, then xy is an edge in $\Gamma(R)$, thus $diam(x, y) = 1$. Assume that both x and y are nonzero.

Case 1. If $x, y \in R \setminus Z(R)$, then $x - 0 - y$ is a path of length 2, thus $d(x, y) = 2$.

Case 2. If $x, y \in Z(R)^*$. If $xy = 0$, then $d(x, y) = 1$. If xy is nonzero, then $x - 0 - y$ is a path of length 2, thus $d(x, y) = 2$.

Case 3. If $x \in R \setminus Z(R)$ and $y \in Z(R)^*$, then $x - 0 - y$ is a path of length 2, thus $d(x, y) = 2$.

Thus $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 2$. By [5, Proposition 1.3.2], if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 5$. □

Remark 2.6. *For each integer $n \geq 1$, let P_n be a path graph on n vertices. By Example 2.4 and Theorem 2.5, the path graph P_n can be realized as $\Gamma(R)$ if and only if $n = 2$ or 3.*

Theorem 2.7. *For a finite commutative ring with unity R, $\Gamma(R)$ is a path graph on n vertices if and only if R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. The proof of this theorem is pretty much apparent from the above remark. □

We now discuss girth in detail.

Theorem 2.8. *For a finite commutative ring with unity R, $g(\Gamma(R)) = \infty$ if and only if R is isomorphic to a finite field or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2 \rangle$.*

Proof. For the sake of necessity, suppose $g(\Gamma(R)) = \infty$. To prove the result, we work with the cardinality of the set of nonzero zero-divisors. In this view, we claim that $|Z(R)^*| \leq 1$. Suppose to the contrary that $|Z(R)^*| > 1$. By Lemma 2.2, $\Gamma(R)$ contains a cycle of length 3, and therefore $g(\Gamma(R)) = 3$, which is a contradiction to our hypothesis. Hence $|Z(R)^*| \leq 1$. Furthermore, up to isomorphism, the only finite commutative rings with unity having $|Z(R)^*| \leq 1$ are finite fields, $\mathbb{Z}_4$, and $\mathbb{Z}_2[x]/\langle x^2 \rangle$. Hence, R must be one of them.

The sufficient part is easy to see as if we take any of the above-listed rings, then $\Gamma(R)$ contains no cycle, and hence result follows. □

Corollary 2.9. *For an infinite field $\mathbb{F}$, $g(\Gamma(\mathbb{F})) = \infty$.*

Proof. Since for an infinite field $\mathbb{F}$, $|Z(\mathbb{F})^*| = 0$ (i.e., $|Z(\mathbb{F})^*| \leq 1$). Hence, result follows. □

We now aim at improving the condition regarding girth given in Theorem 2.5.

Theorem 2.10. *Let R be a ring. If $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) = 3$.*

Proof. Let us suppose $\Gamma(R)$ contains a cycle. We claim that the length of shortest cycle present in $\Gamma(R)$ is 3. In this view, if there is a cycle of length 3, then result follows itself. Now, let us assume that there is no cycle of length 3 in $\Gamma(R)$. In this case, it contains a cycle $x_1 - x_2 - \cdots - x_n - x_1$ for $n \geq 4$. Next, if for all i such that $1 \leq i \leq n$, x_i is a nonzero element of a ring R, then $0x_i$ is an edge in $\Gamma(R)$. Therefore, for $1 \leq i \leq n - 1$, $0 - x_i - x_{i+1} - 0$ is a cycle of length 3 in $\Gamma(R)$. Further, at the most one of the x_i can be 0 of a ring. Without loss of generality assume $x_1 = 0$. Consequently, $0 - x_2 - x_3 - 0$ is a cycle of length 3 in $\Gamma(R)$. Hence, in both the cases $g(\Gamma(R)) = 3$. □

We have seen above that if $\Gamma(R)$ contains a cycle, then it always contains a cycle of length 3 and further, there exists no ring R with $\Gamma(R)$ isomorphic to a cycle C_3. We can now state that $\Gamma(R)$ cannot be an n-gon for any $n \geq 3$. However, for each $n \geq 3$, there is a zero-divisor graph with an n-cycle. For $n \geq 2$, let $R_n = \mathbb{Z}_2[x_1, x_2, ..., x_n]/\langle x_1^2, x_2^2, \ldots, x_n^2, x_1x_2, x_2x_3, \ldots, x_nx_1\rangle$. In this case, $\Gamma(R_n)$ is finite and has a cycle of length $n+1$, i.e., $0 - x_1 - x_2 - \cdots - x_n - 0$. According to the definition given by Anderson and Livingston [3], $\Gamma(R)$ can be a cycle C_3 or C_4. But, for Beck's definition, C_n can never be realized as $\Gamma(R)$ for any $n \geq 3$.

With this let us now shift our focus to complete graphs and star graphs. Why we are putting stress on the study of star graphs as a zero-divisor graph is because of the subgraph of $\Gamma(R)$ induced by the set $H_R = R \setminus Z(R)^*$. The subgraph induced by the set H_R is a star graph on $|R| - |Z(R)| + 1$ vertices with 0 as a center.

Moreover, the subgraph of $\Gamma(R)$ induced by the set $H_R^C = Z(R)^*$ is zero-divisor graph reported in the article [3] by Anderson and Livingston. This work is a generalization of their zero-divisor graphs.

Remark 2.11. *For $n \geq 1$, let K_n be the complete graph on n vertices. In Figure 1, only graph (a) is a complete graph K_2.*

Next, we show K_2 (i.e., the complete graph on two vertices) is the only complete graph which can be realized as $\Gamma(R)$.

Theorem 2.12. *For a finite commutative ring with unity R, $\Gamma(R)$ is complete graph if and only if R is isomorphic to $\mathbb{Z}_2$.*

Proof. Before proving the result, note that for a finite commutative ring with unity, a nonzero element is either a unit or a zero-divisor. Towards proving the necessity part, suppose that $\Gamma(R)$ is complete graph K_n for any $n \geq 1$. Now, the following cases may occur:

1. If $n = 1$.
   In this case, R must be a trivial ring, and since we are concerned with a commutative ring having unity $(1 \neq 0)$, it does not serve our hypothesis. Therefore, this case is not possible.
2. If $n = 2$.
   For this regard, R must contain precisely two elements. Moreover, the only commutative ring with unity having two elements is $\mathbb{Z}_2$. Hence, R is isomorphic to $\mathbb{Z}_2$.
3. If $n \geq 3$.
   For $x, y \in R$, since $\Gamma(R)$ is complete graph, $xy = 0$. As a consequence, for $x \in R$, x is a zero-divisor. Due to this, every nonzero element of a ring R is a zero-divisor, and none of the element is a unit, which contradicts our hypothesis. Therefore, $\Gamma(R)$ cannot be a complete graph K_n for any $n \geq 3$.

So, if $\Gamma(R)$ is a complete graph, then $\Gamma(R) \cong K_2$ and $R \cong \mathbb{Z}_2$. The sufficient part follows directly from the remark above. □

However, if we relax any of the conditions in the hypothesis, then the result may differ. For $n \geq 1$, let $R_n = \mathbb{Z}_n$, the ring of integer modulo n in which product of any two elements is zero.

For any $n \geq 1$, R_n is then a finite commutative ring "without unity" such that $\Gamma(\mathbb{Z}_n) \cong K_n$. Therefore, for $n \geq 1$, K_n may be realized as $\Gamma(R)$ if and only if R is a finite commutative ring without unity.

Additionally, an infinite complete graph may be realized as $\Gamma(R)$. Let $R = \mathbb{Z}$ with usual addition and product of any two elements is zero. In this case, $\Gamma(\mathbb{Z})$ is an infinite complete graph. Therefore, one can find infinite rings and rings without unity for K_n to be realized as $\Gamma(R)$.

If $K_{m,n}$ denotes a complete bipartite graph with vertex set having m and n elements, then we have the following results. Further, $K_{1,n}$ denotes a star graph. The center of a graph is the set of all vertices of minimum eccentricity; the center of a star graph $K_{1,n}$ is the vertex with degree $n - 1$.

Theorem 2.13. *For a finite commutative ring with unity R, $\Gamma(R)$ is a complete bipartite graph if and only if R is isomorphic to a finite field or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\langle x^2\rangle$. In particular, a star graph $K_{1,r}$ may be realized as $\Gamma(R)$ if and only if $r = p^n - 1$ for some prime p and an integer $n \geq 1$.*

Proof. For necessity, let $\Gamma(R)$ be a complete bipartite graph. We prove the result on the same lines as that of Theorem 2.8. Of course, if $|Z(R)^*| > 1$, then $\Gamma(R)$ contains C_3 as a subgraph, which is a cycle of an odd length. It contradicts our hypothesis, as a bipartite graph can never have a cycle of an odd length as a subgraph. So we have $|Z(R)^*| \leq 1$. Again, the only possible rings are finite fields, $\mathbb{Z}_4$, and $\mathbb{Z}_2[x]/\langle x^2\rangle$. Moreover, $\Gamma(\mathbb{F}_{p^n})$ is a complete bipartite graph K_{1,p^n-1}, and $\Gamma(\mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2[x]/\langle x^2\rangle)$ is a complete bipartite graph $K_{1,3}$.

For sufficiency, it is easy to analyze that zero-divisor graphs of above-mentioned finite commutative rings with unity are complete bipartite graphs. Hence, the result follows. □

Corollary 2.14. *For an infinite field $\mathbb{F}$, $\Gamma(\mathbb{F})$ is an infinite complete bipartite graph.*

Proof. The proof of the above corollary is easy to see. □

Although, infinite fields are not only the case where a zero-divisor graph is an infinite complete bipartite graph. Let $R = \mathbb{Z}$ with usual addition and multiplication. Since $\mathbb{Z}$ is an infinite integral domain (i.e., it is not a field), $\Gamma(\mathbb{Z})$ is an infinite star graph.

Besides, at this point, we take into account that in the case of fields, since the set of zero-divisors contains only one element, i.e., the additive identity 0, the set H_R as discussed earlier is equal to the whole ring R. The subgraph induced by this set H_R is itself the zero-divisor graph of that ring. Out of the three possibilities discussed in Theorem 2.13, only fields carry this behaviour. The subgraph induced by the set H_R for the rings $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/\langle x^2\rangle$ is isomorphic to a star graph $K_{1,2}$, and hence does not contain a cycle.

However, on the other hand, if $\Gamma(R)$ contains a cycle, then also the subgraph induced by the set H_R is a star graph on $|R| - |Z(R)| + 1$ vertices with 0 as a center. It is easy to verify that 0 is the only vertex adjacent to every other vertex and no additional adjacency can be found in this induced subgraph.

We end with the following theorems which are an outcome of the preceding discussion.

Theorem 2.15. *Let R be a commutative ring with unity. Then exactly one of the following holds:*

1. *$\Gamma(R)$ is a star graph.*
2. *$\Gamma(R)$ has a cycle of length 3.*

Proof. By using Theorem 2.3, Theorem 2.13 and from the fact that either $|Z(R)^*| \leq 1$ or otherwise, the proof is evident. □

Theorem 2.16. *Let R be a finite commutative ring with unity. The following statements are equivalent:*

1. $|Z(R)^*| \leq 1$ *(i.e., the set of nonzero zero-divisors contains at the most one element); or*
2. $g(\Gamma(R)) = \infty$ *(i.e., $\Gamma(R)$ contains no cycle); or*
3. $\Gamma(R)$ *is a complete bipartite graph, precisely a star graph; or*
4. R *is isomorphic to one of the following rings: $\mathbb{F}_{p^n}$, where p is prime and $n \in \mathbb{N}$, $\mathbb{Z}_4$, and $\mathbb{Z}_2[x]/\langle x^2 \rangle$.*

Proof. The proof follows from Theorem 2.3, Theorem 2.8 and Theorem 2.13. □

Theorem 2.16 is still valid if we rest the "finite" condition, except for the statement 4 where infinite field must accompany other rings to the list.

3 Conclusion

In this article, we present the readers, characteristics of zero-divisor graphs originated by Beck in the year 1988. We further compare the results with a well-known work done by Anderson and Livingston in the year 1999. If $\Gamma(R)$ (respectively, $\Gamma'(R)$) denotes the zero-divisor graph given by Beck (respectively, Anderson and Livingston), then Table 1 explains the need for doing this analysis. The concept of Beck's zero-divisor graph has been beautifully extended to Beck's signed zero-divisor graph in some other article for application purposes.

Property	Definition given by Anderson and Livingston	Definition given by Beck
Number of possible graphs	There are seven $\Gamma'(R)$ with $1 \leq \lvert\Gamma'(R)\rvert \leq 4$.	There are only four $\Gamma(R)$ with $1 \leq \lvert\Gamma(R)\rvert \leq 4$; one of each order 2, 3 and two of order 4.
Cycle C_n, $n \geq 1$	$\Gamma'(R)$ can be a cycle C_3 or C_4, but not C_n for any $n \geq 5$.	$\Gamma(R)$ can never be a cycle C_n for any n.
Complete graph K_r, $r \geq 1$	K_r may be realized as $\Gamma'(R)$ if and only if $r = p^n - 1$, for some prime p and an integer $n \geq 1$.	K_r may be realized as $\Gamma(R)$ if and only if $r = 2$.
Complete Bipartite graph $K_{m,n}$, $m, n \geq 1$	$K_{m,n}$ may be realized as $\Gamma'(R)$ if and only if $m = p^r - 1$ and $n = q^s - 1$, for some prime p and q and an integers r, $s \geq 1$.	$K_{m,n}$ may be realized as $\Gamma(R)$ if and only if $m = 1$ and $n = p^r - 1$, for some prime p and an integers $r \geq 1$.

Table 1. Table showing variations between $\Gamma'(R)$ and $\Gamma(R)$

Acknowledgements

The first author's work is supported by the Research Grant from DST [MTR/2018/000607] under Mathematical Research Impact Centric Support for a period of 3 years (2019 – 2022). The second author is thankful to the University Grant Commission (UGC) for providing the research grant vide sanctioned letter number: 1054 (CSIR-UGC NET JUNE 2017).

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