Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 4, 16–23
DOI: 10.7546/nntdm.2019.25.4.16-23

Certain generating functions for the quadruple hypergeometric series K_{10}

Praveen Agarwal[1,2], Jihad A. Younis[3] and Taekyun Kim[4]

[1] Department of Mathematics, Anand International College of Engineering
Jaipur-303012, India

[2] Department of Mathematics, Harish-Chandra Research Institute (HRI)
Allahbad-211 019, UP, India
e-mail: goyal.praveen2011@gmail.com

[3] Department of Mathematics, Aden University
Aden, Yemen
e-mail: jihadalsaqqaf@gmail.com

[4] Department of Mathematics, College of Natural Science
Kwangwoon University, Seoul 139-704, South Korea
e-mail: tkkim@kw.ac.kr

Received: 2 September 2019 **Accepted:** 14 November 2019

Abstract: In this work we aim at establishing certain generating relations, involving the Exton function of four variables K_{10}. Some special cases of the main results here are also considered.
Keywords: Laplace integral, Quadruple hypergeometric series, Generating functions.
2010 Mathematics Subject Classification: 33C50, 33C56.

1 Introduction

The Gaussian hypergeometric function is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (|x| < 1), \tag{1.1} $$

(see [9, 10]), where $(a)_n$ denotes the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \quad (n \in \mathbb{N} := \{1, 2, 3, \cdots\})$$

and

$$(a)_0 = 1.$$

An interesting further generalization of the series ${}_2F_1$ is due to Exton [5] who defined the following hypergeometric series in four variables:

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, a_2, a_2, a_3, a_4; c_1, c_2, c_3, c_4, ; x, y, z, u) \\ &\quad = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p+q}(a_2)_{m+n}(a_3)_p(a_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}. \end{aligned} \tag{1.2}$$

By suitable adjustment of parameters and variables in K_{10}, we can easily find that K_{10} is unification and generalization of Lauricella's functions $F_A^{(3)}$ and F_E [6, 9] and Appell's functions F_2 and F_4 [9, 10]. The following is the Laplace integral representation of the function K_{10} (see [5]):

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, a_2, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u) = \frac{1}{\Gamma(a_1)} \int_0^{\infty} e^{-s} \\ &\times s^{a_1 - 1} \Psi_2(a_2; c_1, c_2; sx, sy) {}_1F_1(a_3; c_3; sz) {}_1F_1(a_4; c_4; su) ds, \ (Re(a_1 > 0), \end{aligned} \tag{1.3}$$

where ${}_1F_1$ is Kummer's function and Ψ_2 is Humbert function defined, respectively, by

$${}_1F_1(a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \quad |x| < \infty \tag{1.4}$$

and

$$\Psi_2(a, b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(b)_m (c)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| < \infty, \ \ |y| < \infty. \tag{1.5}$$

In various areas of applied mathematics and mathematical physics, generating functions play an important role in the investigation of various useful properties of the sequences which they generate. Generating functions are used to find certain properties and formulas for numbers and polynomials in a wide range of research subjects such as modern combinatorics (see, e.g., [10]). In fact, a remarkably large number of generating functions involving a variety of hypergeometric functions have been developed by many authors (for example, [1, 2, 3, 8]), here, we aim at establishing ten new generating functions for the Exton hypergeometric function K_{10}. Some special cases of the main results here are also considered.

2 Generating functions

Here certain generating relations for the Exton hypergeometric function K_{10} are presented.

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)\\&\quad=(1-z-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-z-u)}\Big)^{k}K_{10}(a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_2,\\&\quad\quad c_3-a_3,c_4-a_4;c_1,c_2,c_3,c_4;\lambda_1x,\lambda_1y,\lambda_2z,\lambda_2u),\end{aligned}\tag{2.1}$$

where $\lambda_1=\frac{1}{1-z-u}$, $\lambda_2=\frac{1}{z+u-1}$.

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)\\&\quad=(1-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-u)}\Big)^{k}K_{10}(a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_2,a_3,\\&\quad\quad c_4-a_4;c_1,c_2,c_3,c_4;\lambda_1x,\lambda_1y,\lambda_1z,\lambda_2u),\end{aligned}\tag{2.2}$$

where $\lambda_1=\frac{1}{1-u}$, $\lambda_2=\frac{1}{u-1}$.

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_2,a_3,c_4;c_1,c_2,c_3,c_4;x,y,z,u)\\&\quad=(1-z-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-z-u)}\Big)^{k}F_E(a_1+k,a_1+k,a_1+k,\\&\quad\quad c_3-a_3,a_2,a_2;c_3,c_1,c_2;\lambda_2z,\lambda_1x,\lambda_1y),\end{aligned}\tag{2.3}$$

where $\lambda_1=\frac{1}{1-z-u}$, $\lambda_2=\frac{1}{z+u-1}$.

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_2,a_3,c_4;c_1,c_2,c_3,c_4;x,y,z,u)\\&\quad=(1-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-u)}\Big)^{k}F_E(a_1+k,a_1+k,a_1+k,a_3,a_2,a_2;\\&\quad\quad c_3,c_1,c_2;\lambda z,\lambda x,\lambda y),\end{aligned}\tag{2.4}$$

where $\lambda=\frac{1}{1-u}$.

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k,a_1+k,a_1+k,a_1+k,c_1,c_1,a_3,a_4;c_1,c_1,c_3,c_4;x,y,z,u)\\&\quad=(1-x-y-z-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-x-y-z-u)}\Big)^{k}\end{aligned}$$

$$\times X_8\Big(a_1+k, c_3-a_3, c_4-a_4; c_1, c_3, c_4; \frac{xy}{(1-x-y-z-u)^2}, \frac{z}{(x+y+z+u-1)}, \frac{u}{(x+y+z+u-1)}\Big), \tag{2.5}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k, a_1+k, a_1+k, a_1+k, c_1, c_1, a_3, a_4; c_1, c_1, c_3, c_4; x, y, z, u) = (1-x-y)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-x-y)}\Big)^k X_8\Big(a_1+k, a_3, a_4; c_1, c_3, c_4; \frac{xy}{(1-x-y)^2}, \frac{z}{(1-x-y)}, \frac{u}{(1-x-y)}\Big), \tag{2.6}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k, a_1+k, a_1+k, a_1+k, c_1, c_1, c_3, a_4; c_1, c_1, c_3, c_4; x, y, z, u) = (1-x-y-z-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-x-y-z-u)}\Big)^k H_4\Big(a_1+k, c_4-a_4; c_1, c_4; \frac{xy}{(1-x-y-z-u)^2}, \frac{u}{(x+y+z+u-1)}\Big), \tag{2.7}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k, a_1+k, a_1+k, a_1+k, c_1, c_1, c_3, a_4; c_1, c_1, c_3, c_4; x, y, z, u) = (1-x-y-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-x-y-z)}\Big)^k H_4\Big(a_1+k, a_4; c_1, c_4; \frac{xy}{(1-x-y-z)^2}, \frac{u}{(1-x-y-z)}\Big), \tag{2.8}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k, a_1+k, a_1+k, a_1+k, a_2, a_2, c_3, c_4; c_1, c_2, c_3, c_4; x, y, z, u) = (1-z-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-z-u)}\Big)^k F_4\Big(a_1+k, a_2; c_1, c_2; \frac{x}{(1-z-u)}, \frac{y}{(1-z-u)}\Big), \tag{2.9}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}K_{10}(a_1+k, a_1+k, a_1+k, a_1+k, c_1, c_1, c_3, c_4; c_1, c_1, c_3, c_4; x, y, z, u) = (1-x-y-z-u)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{w}{(1-x-y-z-u)}\Big)^k \times {}_2F_1\Big(\frac{a_1+k}{2}, \frac{a_1+k+1}{2}; c_1; \frac{4xy}{(1-x-y-z-u)^2}\Big), \tag{2.10}$$

where X_8 is the Exton function and H_4 is the Horn function defined, respectively, by

$$X_8(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\times\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.11}$$

and

$$H_4(a,b;c,d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_m}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{2.12}$$

3 Proofs of the results

To prove the above relations, we need the following formulae (cf. [4, 7, 9, 10]):

$$\Gamma(z)=s^z\int_0^{\infty}e^{-st}t^{z-1}dt,\quad Re(z)>0, \tag{3.1}$$

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\quad a\neq 0,-,-2,\cdots, \tag{3.2}$$

$$_1F_1(-;-;x)=e^x, \tag{3.3}$$

$$_1F_1(a;b;x)=e^x\,{}_1F_1(b-a;b;-x), \tag{3.4}$$

$$\Psi_2(c,c;c;x,y)=\exp(x+y)_0F_1(-;c;xy). \tag{3.5}$$

Proof of (2.1)
Let us denote the left-hand side of (2.1) by ω, using (1.3)

$$\omega=\sum_{k=0}^{\infty}\frac{w^k}{k!\Gamma(a_1+k)}\int_0^{\infty}e^{-s}s^{a_1+k-1}\Psi_2(a_2;c_1,c_2;sx,sy)_1F_1(a_3;c_3;sz)$$
$$\times\,{}_1F_1(a_4,c_4,su)ds,$$

by using (3.4), we get

$$\omega=\sum_{k=0}^{\infty}\frac{w^k}{k!\Gamma(a_1+k)}\int_0^{\infty}e^{-s(1-z-u)}s^{a_1+k-1}$$
$$\times\,\Psi_2(a_2;c_1,c_2;sx,sy)_1F_1(c_3-a_3;c_2;-sz)_1F_1(c_4-a_4,c_4,-su)ds.$$

The functions Ψ_2 and $_2F_1$ in the integrand can be written in its series form and, then interchanging the order of the summation and integral sign, which is permissible here, we get

$$\omega=\sum_{k,m,n,p=0}^{\infty}\frac{(a_2)_{m+n}(c_3-a_3)_p(c_4-a_4)_q w^k x^m y^n(-z)^p(-u)^q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q k!m!n!p!q!\Gamma(a_1+k)}$$
$$\times\int_0^{\infty}e^{-s(1-z-u)}s^{a_1+k+m+n+p+q-1}ds.$$

Now, use of (3.1) and (3.2) in the above equation, simplified with a series of manipulations, completes the proof of relation (2.1). The proofs of all remaining relations run in the same way, considering the appropriate integral representation and Laplace transform during the proof. □

4 Special cases

On the other hand, if we set $k = 0$, in (2.1), (2.4), (2.6), (2.7), (2.9) and (2.10), we get the following interesting transformations formulas after little simplification:

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, a_2, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u) \\ &= (1 - z - u)^{-a_1} \times K_{10}\Big(a_1, a_1, a_1, a_1, a_2, a_2, c_3 - a_3, c_4 - a_4; c_1, c_2, c_3, c_4; \\ &\frac{x}{(1 - z - u)}, \frac{y}{(1 - z - u)}, \frac{z}{(z + u - 1)}, \frac{u}{(z - u + 1)}\Big), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, a_2, a_2, a_3, c_4; c_1, c_2, c_3, c_4; x, y, z, u) \\ &= (1 - u)^{-a_1} \times F_E\Big(a_1, a_1, a_1, a_3, a_2, a_2; c_3, c_1, c_2; \frac{z}{(1 - u)}, \frac{x}{(1 - u)}, \frac{y}{(1 - u)}\Big), \end{aligned} \tag{4.2}$$

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, c_1, c_1, a_3, a_4; c_1, c_1, c_3, c_4; x, y, z, u) \\ &= (1 - x - y)^{-a_1} \times X_8\Big(a_1, a_3, a_4; c_1, c_3, c_4; \frac{xy}{(1 - x - y)^2}, \frac{z}{(1 - x - y)}, \frac{u}{(1 - x - y)}\Big), \end{aligned} \tag{4.3}$$

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, c_1, c_1, c_3, a_4; c_1, c_1, c_3, c_4; x, y, z, u) \\ &= (1 - x - y - z - u)^{-a_1} H_4\Big(a_1, c_4 - a_4; c_1, c_4; \frac{xy}{(1 - x - y - z - u)^2}, \\ &\frac{u}{(x + y + z + u - 1)}\Big), \end{aligned} \tag{4.4}$$

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, a_2, a_2, c_3, c_4; c_1, c_2, c_3, c_4; x, y, z, u) \\ &= (1 - z - u)^{-a_1} \times F_4\Big(a_1, a_2; c_1, c_2; \frac{x}{(1 - z - u)}, \frac{y}{(1 - z - u)}\Big), \end{aligned} \tag{4.5}$$

$$\begin{aligned} &K_{10}(a_1, a_1, a_1, a_1, c_1, c_1, c_3, c_4; c_1, c_1, c_3, c_4; x, y, z, u) \\ &= (1 - x - y - z - u)^{-a_1} {}_2F_1\Big(\frac{a_1}{2}, \frac{a_1 + 1}{2}; c_1; \frac{4xy}{(1 - x - y - z - u)^2}\Big). \end{aligned} \tag{4.6}$$

By setting $x = y = 0$ in (4.2), we obtain the known result [9, p. 306 (109)]. Put $x = 0$ in (2.10), we have generating function for Lauricella function $F_A^{(3)}$

$$(1 - y - z - u)^{a_1} e^{\left(\frac{w}{1-y-z-u}\right)} = \sum_{k=0}^{\infty} \frac{w^k}{k!} F_A^{(3)}(a_1 + k, c_1, c_3, c_4; c_1, c_3, c_4; y, z, u). \tag{4.7}$$

Setting $u = 0$ in (4.7), we shall obtain a generating function involving Appell function F_2

$$(1 - y - z)^{a_1} e^{\left(\frac{w}{1-y-z}\right)} = \sum_{k=0}^{\infty} \frac{w^k}{k!} F_2(a_1 + k, c_1, c_3; c_1, c_3; y, z), \tag{4.8}$$

which, for $z=0$, we get

$$(1-y)^{a_1} e^{\left(\frac{w}{1-y}\right)} = \sum_{k=0}^{\infty} \frac{w^k}{k!}\, {}_2F_1(a_1+k, c_1; c_1; y). \tag{4.9}$$

Finally, substituting $u=0$ in (2.5) and (2.8), we get the generating relations

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{w^k}{k!} F_E(a_1+k, a_1+k, a_1+k, a_3, c_1, c_1; c_3, c_1, c_1; z, x, y) \\ &= (1-x-y-z)^{-a_1} \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\frac{w}{(1-x-y-z)}\Big)^k H_4\Big(a_1+k, c_3-a_3; c_1, c_3; \\ &\quad \frac{xy}{(1-x-y-z)^2}, \frac{z}{(x+y+z-1)}\Big) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{w^k}{k!} F_E(a_1+k, a_1+k, a_1+k, c_3, c_1, c_1; c_3, c_1, c_1; z, x, y) \\ &= (1-x-y-z)^{-a_1} \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\frac{w}{(1-x-y-z)}\Big)^k {}_2F_1\Big(\frac{a_1+k}{2}, \frac{a_1+k+1}{2}; c_1; \\ &\quad \frac{4xy}{(1-x-y-z)^2}\Big), \end{aligned} \tag{4.11}$$

respectively.

Acknowledgements

We are thankful to the editor and referees for their careful reading and valuable suggestions to make the article friendly readable. The research work of Praveen Agarwal is supported by Science and Engineering Research Board under Teachers Associateship for Research Excellence (TAR/2018/000001).

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