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Convolution identities for Tetranacci numbers

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Abstract: Convolution identities for various numbers (e.g., Bernoulli, Euler, Genocchi, Catalan, Cauchy and Stirling numbers) have been studied by many authors. Recently, several convolution identities have been studied for Fibonacci and Tribonacci numbers too. In this paper, we give convolution identities with and without binomial (multinomial) coefficients for Tetranacci numbers, and convolution identities with binomial coefficients for Tetranacci and Tetranacci-type numbers.
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1 Introduction

Convolution identities (or the sums of products) for various types of numbers (or polynomials) have been studied, with or without binomial coefficients, including the Bernoulli, Euler, Genocchi, Catalan, Cauchy and Stirling numbers (see, e.g., [1–3, 5, 6, 9, 10] and the references therein). One typical well-known formula is due to Euler, given by

$$\sum_{k=0}^n\binom{n}{k}\mathcal{B}_k\mathcal{B}_{n-k}=-n\mathcal{B}_{n-1}-(n-1)\mathcal{B}_n\quad(n\ge 0)\,,$$

where $\mathcal{B}_n$ are the Bernoulli numbers, defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^\infty\mathcal{B}_n\frac{x^n}{n!}\quad(|x|<2\pi)\,.$$

Some convolution identities for the Fibonacci numbers F_n were established by Komatsu, Masakova and Pelantova [9]. Recently, several convolution identities with and without binomial coefficients for Tribonacci numbers T_n were obtained by Komatsu [7]. By using symmetric formulae, more various identities for Tribonacci and Tribonacci-type numbers were given by Komatsu and Li [8].

The *Tetranacci numbers* $\mathfrak{T}_n$ are defined by

$$\mathfrak{T}_n = \mathfrak{T}_{n-1} + \mathfrak{T}_{n-2} + \mathfrak{T}_{n-3} + \mathfrak{T}_{n-4} \quad (n \geq 4) \quad \text{with} \quad \mathfrak{T}_0 = 0,\ \mathfrak{T}_1 = \mathfrak{T}_2 = 1,\ \mathfrak{T}_3 = 2 \tag{1}$$

and their sequence is given by

$$\{\mathfrak{T}_n\}_{n\geq 0} = 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, \dots$$

([12, A000078]).

The generating function without factorials is given by

$$\mathfrak{T}(x) := \frac{x}{1 - x - x^2 - x^3 - x^4} = \sum_{n=0}^{\infty} \mathfrak{T}_n x^n \tag{2}$$

because of the recurrence relation (1).

On the other hand, the generating function with binomial coefficients is given by

$$\mathfrak{t}(x) := c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x} + c_4 e^{\delta x} = \sum_{n=0}^{\infty} \mathfrak{T}_n \frac{x^n}{n!}\,, \tag{3}$$

where α, β, γ and δ are the roots of $x^4 - x^3 - x^2 - x - 1 = 0$ and

$$\begin{aligned}
c_1 :&= \frac{2 - (\beta + \gamma + \delta) + (\beta\gamma + \gamma\delta + \delta\beta)}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} \\
&= \frac{1}{-\alpha^3 + 6\alpha - 1}\,, \\
c_2 :&= \frac{2 - (\alpha + \gamma + \delta) + (\alpha\gamma + \gamma\delta + \delta\alpha)}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} \\
&= \frac{1}{-\beta^3 + 6\beta - 1}\,, \\
c_3 :&= \frac{2 - (\alpha + \beta + \delta) + (\alpha\beta + \beta\delta + \delta\alpha)}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} \\
&= \frac{1}{-\gamma^3 + 6\gamma - 1}\,, \\
c_4 :&= \frac{2 - (\alpha + \beta + \gamma) + (\alpha\beta + \beta\gamma + \gamma\alpha)}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} \\
&= \frac{1}{-\delta^3 + 6\delta - 1}\,.
\end{aligned}$$

Notice that

$$\begin{aligned}
c_1 + c_2 + c_3 + c_4 &= 0\,, \\
c_1\alpha + c_2\beta + c_3\gamma + c_4\delta &= 1\,, \\
c_1\alpha^2 + c_2\beta^2 + c_3\gamma^2 + c_4\delta^2 &= 1\,, \\
c_1\alpha^3 + c_2\beta^3 + c_3\gamma^3 + c_4\delta^3 &= 2,
\end{aligned}$$

because $\mathfrak{T}_n$ has a Binet-type formula:

$$\mathfrak{T}_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + c_4\gamma^n \quad (n \ge 0)\,.$$

There have not been many papers about the Tetranacci numbers, though they are a special case of the so-called Fibonacci s-step numbers or s-generalized Fibonacci numbers $F_n^{(s)}$, defined by

$$F_n^{(s)} = F_{n-1}^{(s)} + F_{n-2}^{(s)} + \cdots + F_{n-s}^{(s)}$$

with some initial values. Clearly, for $s = 4$, we have the Tetranacci numbers $\mathfrak{T}_n = F_n^{(4)}$.

De Moivre-type identities for the Tetranacci numbers[1] and several related identities were given by Lin [11]. Some properties of the Tetranacci sequence modulo m and certain identities involving them were established by Waddill [13, 14]. Some congruences for lacunary sequences were derived as an analogous result for the Tetranacci numbers by Young [15]. By using matrix methods, explicit formulas for the Tribonacci and Tetranacci numbers were given by Kiric [4].

In this paper, we give convolution identities with and without binomial (multinomial) coefficients for the Tetranacci numbers and convolution identities with binomial coefficients for the Tetranacci and Tetranacci-type numbers. Some results are based upon symmetric formulae.

It may be possible to consider convolution identities for more general, Fibonacci s-step numbers, including pentanacci numbers ($s = 5$, [12, A001591]), hexanacci numbers ($s = 6$, [12, A001592]) and heptanacci numbers ($s = 7$, [12, A066178]). However, when $s \ge 5$, by the Abel–Ruffini theorem (in 1799, 1824), there is no algebraic expression for general quintic equations over the rationals in terms of radicals. This result also holds for equations of higher degrees. Therefore, to consider the identities exactly for the Tetranacci numbers would be meaningful.

2 Convolution identities without binomial coefficients

By (2), we have

$$\mathfrak{T}'(x) = \frac{1 + x^2 + 2x^3 + 3x^4}{(1 - x - x^2 - x^3 - x^4)^2}\,.$$

Hence,

$$(1 + x^2 + 2x^3 + 3x^4)\mathfrak{T}(x)^2 = x^2\mathfrak{T}'(x)\,. \tag{4}$$

The left-hand side of (4) is

$$\begin{aligned}
&(1 + x^2 + 2x^3 + 3x^4)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathfrak{T}_k\mathfrak{T}_{n-k}x^n\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathfrak{T}_k\mathfrak{T}_{n-k}x^n + \sum_{n=2}^{\infty}\sum_{k=0}^{n-2}\mathfrak{T}_k\mathfrak{T}_{n-k-2}x^n\\
&\quad + 2\sum_{n=3}^{\infty}\sum_{k=0}^{n-3}\mathfrak{T}_k\mathfrak{T}_{n-k-3}x^n + 3\sum_{n=4}^{\infty}\sum_{k=0}^{n-4}\mathfrak{T}_k\mathfrak{T}_{n-k-4}x^n\\
&= \sum_{n=4}^{\infty}\sum_{k=0}^{n-4}\mathfrak{T}_k(\mathfrak{T}_{n-k} + \mathfrak{T}_{n-k-2} + 2\mathfrak{T}_{n-k-3} + 3\mathfrak{T}_{n-k-4})x^n
\end{aligned}$$

[1] In [11], it is called Tetrabonacci numbers.

$$+\sum_{n=4}^{\infty}(\mathfrak{T}_{n-1}+\mathfrak{T}_{n-2}+3\mathfrak{T}_{n-3})x^n+x^2+2x^3\,.$$

The right-hand side of (4) is

$$x^2\sum_{n=0}^{\infty}(n+1)\mathfrak{T}_{n+1}x^n=\sum_{n=2}^{\infty}(n-1)\mathfrak{T}_{n-1}x^n\,.$$

Therefore, we get the following result.

Theorem 2.1. *For $n\ge 4$, we have*

$$\sum_{k=0}^{n-4}\mathfrak{T}_k(\mathfrak{T}_{n-k}+\mathfrak{T}_{n-k-2}+2\mathfrak{T}_{n-k-3}+3\mathfrak{T}_{n-k-4})=(n-2)\mathfrak{T}_{n-1}-\mathfrak{T}_{n-2}-3\mathfrak{T}_{n-3}\,.$$

The identity (4) can be written as

$$\mathfrak{T}(x)^2=\frac{x^2}{1+x^2+2x^3+3x^4}\mathfrak{T}'(x)\,. \tag{5}$$

Since

$$\begin{aligned}\frac{1}{1+x^2+2x^3+3x^4}&=\sum_{l=0}^{\infty}(-1)^l x^{2l}(1+2x+3x^2)^l\\&=\sum_{l=0}^{\infty}(-1)^l x^{2l}\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}}\binom{l}{i,j,k}1^i(2x)^j(3x^2)^k\\&=\sum_{m=0}^{\infty}\sum_{\substack{j,k=0\\ 3j+2k\le m\\ j+4k\le m}}(-1)^{\frac{m-j-2k}{2}}\frac{1+(-1)^{m-j-2k}}{2}\\&\quad\times\binom{\frac{m-j-2k}{2}}{\frac{m-j-2k}{2}-j-k,j,k}2^j3^kx^m,\end{aligned}$$

and

$$\mathfrak{T}'(x)=\sum_{n=0}^{\infty}(n+1)\mathfrak{T}_{n+1}x^n\,,$$

the right-hand side of (5) is

$$\begin{aligned}x^2A\sum_{l=0}^{\infty}(l+1)\mathfrak{T}_{l+1}x^l&=x^2\sum_{n=0}^{\infty}\sum_{l=0}^{n}B(l+1)\mathfrak{T}_{l+1}x^n\\&=\sum_{n=2}^{\infty}\sum_{l=0}^{n-2}C(l+1)\mathfrak{T}_{l+1}x^n\,,\end{aligned}$$

where

$$A=\sum_{m=0}^{\infty}\sum_{\substack{j,k=0\\ 3j+2k\le m\\ j+4k\le m}}(-1)^{\frac{m-j-2k}{2}}\frac{1+(-1)^{m-j-2k}}{2}\binom{\frac{m-j-2k}{2}}{\frac{m-j-2k}{2}-j-k,j,k}2^j3^kx^m,$$

$$B=\sum_{\substack{j,k=0}}^{\substack{3j+2k\le n-l\\ j+4k\le n-l}}(-1)^{\frac{n-l-j-2k}{2}}\frac{1+(-1)^{n-l-j-2k}}{2}\binom{\frac{n-l-j-2k}{2}}{\frac{n-l-3j-4k}{2},j,k}2^j3^k,$$

$$C=\sum_{\substack{j,k=0}}^{\substack{3j+2k\le n-2-l\\ j+4k\le n-2-l}}(-1)^{\frac{n-2-l-j-2k}{2}}\frac{1+(-1)^{n-2-l-j-2k}}{2}\binom{\frac{n-2-l-j-2k}{2}}{\frac{n-2-l-3j-4k}{2},j,k}2^j3^k.$$

Since the left-hand side of (5) is

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathfrak{T}_k\mathfrak{T}_{n-k}x^n\,,$$

comparing the coefficients on both sides, we obtain the following result without binomial coefficient.

Theorem 2.2. *For $n\ge 2$,*

$$\sum_{k=0}^{n}\mathfrak{T}_k\mathfrak{T}_{n-k}=\sum_{l=0}^{n-2}(l+1)\mathfrak{T}_{l+1}D,$$

where

$$D=\sum_{\substack{j,k=0}}^{\substack{3j+2k\le n-2-l\\ j+4k\le n-2-l}}(-1)^{\frac{n-2-l-j-2k}{2}}\frac{1+(-1)^{n-2-l-j-2k}}{2}\binom{\frac{n-2-l-j-2k}{2}}{\frac{n-2-l-3j-4k}{2},j,k}2^j3^k.$$

3 Some preliminary lemmas

For convenience, we shall introduce modified Tetranacci numbers $\mathfrak{T}_n^{(s_0,s_1,s_2,s_3)}$, satisfying the recurrence relation

$$\mathfrak{T}_n^{(s_0,s_1,s_2,s_3)}=\mathfrak{T}_{n-1}^{(s_0,s_1,s_2,s_3)}+\mathfrak{T}_{n-2}^{(s_0,s_1,s_2,s_3)}+\mathfrak{T}_{n-3}^{(s_0,s_1,s_2,s_3)}+\mathfrak{T}_{n-4}^{(s_0,s_1,s_2,s_3)}\quad(n\ge 4)$$

with given initial values $\mathfrak{T}_0^{(s_0,s_1,s_2,s_3)}=s_0$, $\mathfrak{T}_1^{(s_0,s_1,s_2,s_3)}=s_1$, $\mathfrak{T}_2^{(s_0,s_1,s_2,s_3)}=s_2$, and $\mathfrak{T}_3^{(s_0,s_1,s_2,s_3)}=s_3$. Hence, $\mathfrak{T}_n=\mathfrak{T}_n^{(0,1,1,2)}$ are the ordinary Tetranacci numbers.

First, we shall prove the following four lemmata.

Lemma 3.1. *We have*

$$c_1^2e^{\alpha x}+c_2^2e^{\beta x}+c_3^2e^{\gamma x}+c_4^2e^{\delta x}=\frac{1}{563}\sum_{n=0}^{\infty}\mathfrak{T}_n^{(40,64,215,344)}\frac{x^n}{n!}\,.$$

Proof. For Tetranacci-type numbers s_n, satisfying the recurrence relation $s_n=s_{n-1}+s_{n-2}+s_{n-3}+s_{n-4}$ $(n\ge 4)$ with given initial values s_0, s_1, s_2 and s_3, we have

$$d_1e^{\alpha x}+d_2e^{\beta x}+d_3e^{\gamma x}+d_4e^{\delta x}=\sum_{n=0}^{\infty}s_n\frac{x^n}{n!}\,.\tag{6}$$

Since d_1, d_2, d_3 and d_4 satisfy the system of the equations

$$\begin{aligned} d_1 + d_2 + d_3 + d_4 &= s_0\,, \\ d_1\alpha + d_2\beta + d_3\gamma + d_4\gamma &= s_1\,, \\ d_1\alpha^2 + d_2\beta^2 + d_3\gamma^2 + d_4\gamma^2 &= s_2\,, \\ d_1\alpha^3 + d_2\beta^3 + d_3\gamma^3 + d_4\gamma^3 &= s_3\,, \end{aligned}$$

we have

$$d_1 = \frac{\begin{vmatrix} s_0 & 1 & 1 & 1 \\ s_1 & \beta & \gamma & \delta \\ s_2 & \beta^2 & \gamma^2 & \delta^2 \\ s_3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\beta\gamma\delta + s_2(\beta+\gamma+\delta) - s_3 - s_1(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\,,$$

$$d_2 = \frac{\begin{vmatrix} 1 & s_0 & 1 & 1 \\ \alpha & s_1 & \gamma & \delta \\ \alpha^2 & s_2 & \gamma^2 & \delta^2 \\ \alpha^3 & s_3 & \gamma^3 & \delta^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\gamma\delta\alpha + s_2(\gamma+\delta+\alpha) - s_3 - s_1(\gamma\delta+\gamma\alpha+\delta\alpha)}{(\gamma-\beta)(\delta-\beta)(\alpha-\beta)}\,,$$

$$d_3 = \frac{\begin{vmatrix} 1 & 1 & s_0 & 1 \\ \alpha & \beta & s_1 & \delta \\ \alpha^2 & \beta^2 & s_2 & \delta^2 \\ \alpha^3 & \beta^3 & s_3 & \delta^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\delta\alpha\beta + s_2(\delta+\alpha+\beta) - s_3 - s_1(\delta\alpha+\delta\beta+\alpha\beta)}{(\delta-\gamma)(\alpha-\gamma)(\beta-\gamma)}\,,$$

$$d_4 = \frac{\begin{vmatrix} 1 & 1 & 1 & s_0 \\ \alpha & \beta & \gamma & s_1 \\ \alpha^2 & \beta^2 & \gamma^2 & s_2 \\ \alpha^3 & \beta^3 & \gamma^3 & s_3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\alpha\beta\gamma + s_2(\alpha+\beta+\gamma) - s_3 - s_1(\alpha\beta+\alpha\gamma+\beta\gamma)}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

When $s_0 = 40$, $s_1 = 64$, $s_2 = 215$ and $s_3 = 344$, by $\alpha+\beta+\gamma+\delta = 1$, $\beta\gamma+\beta\delta+\gamma\delta = -1-(\alpha\beta+\alpha\gamma+\alpha\delta) = \alpha^2-\alpha-1$, $\alpha\beta\gamma\delta = 1$ and $\alpha^4 = \alpha^3+\alpha^2+\alpha+1$, we have

$$d_1 = \frac{40\beta\gamma\delta + 215(\beta+\gamma+\delta) - 344 - 64(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} = 563c_1^2.$$

Similarly, we have $d_2 = 563c_2^2$, $d_3 = 563c_3^2$ and $d_4 = 563c_4^2$. □

Lemma 3.2. *We have*

$$\sum_{n=0}^{\infty} t_n \frac{x^n}{n!} = c_1c_2e^{(\alpha+\beta)x} + c_1c_3e^{(\alpha+\gamma)x} + c_1c_4e^{(\alpha+\delta)x} + c_2c_3e^{(\beta+\gamma)x} + c_2c_4e^{(\beta+\delta)x} + c_3c_4e^{(\gamma+\delta)x},$$

where

$$t_n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k\mathfrak{T}_{n-k} - \frac{2^n}{563}\mathfrak{T}_n^{(40,64,215,344)}\right).$$

Proof. Since

$$\begin{aligned}&(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})^2\\ &= c_1^2e^{\alpha x}+c_2^2e^{\beta x}+c_3^2e^{\gamma x}+c_4^2e^{\delta x}+2(c_1c_2e^{(\alpha+\beta)x}+c_1c_3e^{(\alpha+\gamma)x}+c_1c_4e^{(\alpha+\delta)x}\\ &\quad + c_2c_3e^{(\beta+\gamma)x}+c_2c_4e^{(\beta+\delta)x}+c_3c_4e^{(\gamma+\delta)x}),\end{aligned}$$

we can obtain the following identity:

$$\begin{aligned}\left(\sum_{n=0}^{\infty}\mathfrak{T}_n\frac{x^n}{n!}\right)^2 &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k\mathfrak{T}_{n-k}\frac{x^n}{n!}\\ &= \frac{1}{563}\sum_{n=0}^{\infty}\mathfrak{T}_n^{(40,64,215,344)}\frac{(2x)^n}{n!} + 2\sum_{n=0}^{\infty}t_n\frac{x^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. □

Lemma 3.3. *We have*

$$c_2c_3c_4e^{\alpha x}+c_3c_4c_1e^{\beta x}+c_4c_1c_2e^{\gamma x}+c_1c_2c_3e^{\delta x}=-\frac{1}{563}\sum_{n=0}^\infty \mathfrak{T}_n^{(-5,2,13,32)}\frac{x^n}{n!}\,.$$

Proof. In the proof of Lemma 3.1, we put $s_0=-5$, $s_1=2$, $s_2=13$ and $s_3=32$, instead. We have

$$d_1=\frac{-5\beta\gamma\delta+13(\beta+\gamma+\delta)-32-2(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}=-563c_2c_3c_4.$$

Similarly, we have $d_2=-563c_3c_4c_1,\quad d_3=-563c_4c_1c_2$ and $d_4=-563c_1c_2c_3$. □

Lemma 3.4. *We have*

$$c_1c_2c_3c_4=-\frac{1}{563}\,.$$

Proof. By $\alpha+\beta+\gamma+\delta=1$, $\beta\gamma+\beta\delta+\gamma\delta=-1-(\alpha\beta+\alpha\gamma+\alpha\delta)=\alpha^2-\alpha-1$, $\alpha\beta\gamma\delta=1$ and $\alpha^4=\alpha^3+\alpha^2+\alpha+1$, we have

$$\begin{aligned}
&c_1c_2c_3c_4\\
&=\frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\frac{\beta^2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\\
&\quad\times\frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\frac{\delta^2}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\\
&=\frac{\alpha^2\beta^2\gamma^2\delta^2}{(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\gamma-\beta)^2(\beta-\delta)^2}\\
&=\frac{1}{(4\alpha^3-3\alpha^2-2\alpha-1)^2(39\alpha^3-58\alpha^2-23\alpha-23)}\\
&=-\frac{1}{563}.
\end{aligned}$$

□

4 Convolution identities for three and four Tetranacci numbers

Before giving more convolution identities, we shall give some elementary algebraic identities in symmetric form. It is not so difficult to determine the relations among coefficients.

Lemma 4.1. *The following equality holds:*

$$\begin{aligned}
&(a+b+c+d)^3\\
&=A(a^3+b^3+c^3+d^3)+B(abc+abd+acd+bcd)\\
&\quad+C(a^2+b^2+c^2+d^2)(a+b+c+d)\\
&\quad+D(ab+ac+ad+bc+bd+cd)(a+b+c+d),
\end{aligned}$$

where $A=D-2,\quad B=-3D+6,\quad C=-D+3.$

Lemma 4.2. *The following equality holds:*

$$
\begin{aligned}
&(a+b+c+d)^4 \\
&= A(a^4+b^4+c^4+d^4) + Babcd + C(a^3+b^3+c^3+d^4)(a+b+c+d) \\
&\quad + D(a^2+b^2+c^2+d^2)^2 + E(a^2+b^2+c^2+d^2)(ab+ac+ad+bc+bd+cd) \\
&\quad + F(ab+ac+ad+bc+bd+cd)^2 + G(a^2+b^2+c^2+d^2)(a+b+c+d)^2 \\
&\quad + H(ab+ac+ad+bc+bd+cd)(a+b+c)^2 \\
&\quad + I(abc(a+b+c)+abd(a+b+d)+bcd(b+c+d)+acd(a+c+d)) \\
&\quad + J(abc+abd+bcd+acd)(a+b+c+d),
\end{aligned}
$$

where $A=-D+E+G+H-3$, $\quad B=12D+12G-4J-12$, $\quad C=-E-2G-H+4$, $F=-2D-2G-2H+6$, $\quad I=4D-E+2G-H-J$.

Lemma 4.3. *The following equality holds:*

$$
\begin{aligned}
&(a+b+c+d)^5 \\
&= A(a^5+b^5+c^5+d^5) \\
&\quad + B(abc(ab+bc+ca)+abd(ab+bd+ad)+acd(ac+ad+cd)+bcd(bc+bd+cd)) \\
&\quad + C(abc(a^2+b^2+c^2)+abd(b^2+c^2+d^2) \\
&\qquad + acd(a^2+c^2+d^2)+bcd(b^2+c^2+d^2)) \\
&\quad + D(abc(a+b+c)^2+abd(a+b+d)^2+acd(a+c+d)^2+bcd(b+c+d)^2) \\
&\quad + E(a^4+b^4+c^4+d^4)(a+b+c+d) + F(a+b+c+d)abcd \\
&\quad + G(a+b+c+d) \\
&\qquad \times (abc(a+b+c)+abd(a+b+d)+bcd(b+c+d)+acd(a+c+d)) \\
&\quad + H(a^3+b^3+c^3+d^3)(a^2+b^2+c^2+d^2) \\
&\quad + I(a^3+b^3+c^3+d^3)(ab+ac+ad+bc+bd+cd) \\
&\quad + J(abc+abd+acd+bcd)(a^2+b^2+c^2+d^2) \\
&\quad + K(abc+abd+acd+bcd)(ab+ac+ad+bc+bd+cd) \\
&\quad + L(a^3+b^3+c^3+d^3)(a+b+c+d)^2 \\
&\quad + M(abc+abd+acd+bcd)(a+b+c+d)^2 \\
&\quad + N(a^2+b^2+c^2+d^2)^2(a+b+c+d) \\
&\quad + P(ab+ac+ad+bc+bd+cd)^2(a+b+c+d) \\
&\quad + Q(a^2+b^2+c^2+d^2)(ab+ac+ad+bc+bd+cd)(a+b+c+d) \\
&\quad + R(a^2+b^2+c^2+d^2)(a+b+c+d)^3 \\
&\quad + S(ab+ac+ad+bc+bd+cd)(a+b+c+d)^3,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I+2L+2N+P+2Q+6R+4S-14, \\
B &= -2D-2G-K-2M-2N-5P-2Q-6R-12S+30,
\end{aligned}
$$

$$
\begin{aligned}
C &= -D - G - I - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\\
E &= -I - 2L - N - Q - 3R - S + 5,\\
F &= -3G - J - 3K - 7M - 12P - 3Q - 6R - 27S + 60,\\
H &= -L - 2N - P - Q - 4R - 3S + 10.
\end{aligned}
$$

Now, let us consider the sum of three products with trinomial coefficients.

Lemma 4.4. *We have*

$$
c_1^3 e^{\alpha x} + c_2^3 e^{\beta x} + c_3^3 e^{\gamma x} + c_4^3 e^{\delta x} = \frac{1}{563} \sum_{n=0}^{\infty} \mathfrak{T}_n^{(15,27,48,107)} \frac{x^n}{n!} .
$$

Proof. In the proof of Lemma 3.1, we put $s_0 = 15$, $s_1 = 27$, $s_2 = 48$ and $s_3 = 107$, instead. We can obtain that

$$
d_1 = \frac{15\beta\gamma\delta + 48(\beta+\gamma+\delta) - 107 - 27(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} = 563 c_1^3 .
$$

. Similarly, we have $d_2 = 563c_2^3$, $d_3 = 563c_3^3$ and $d_4 = 563c_4^3$. □

By using Lemmata 3.1, 3.2, 3.3, 4.1 and 4.4, we get the following result.

Theorem 4.5. *For $n \ge 0$,*

$$
\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} \mathfrak{T}_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\\
&= \frac{A}{563} 3^n \mathfrak{T}_n^{(15,27,48,107)} - \frac{B}{563}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k^{(-5,2,13,32)}(-1)^k\\
&\quad + \frac{C}{563}\sum_{k=0}^{n}\binom{n}{k} 2^{n-k}\mathfrak{T}_{n-k}^{(40,64,215,344)}\mathfrak{T}_k + D\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k t_{n-k} .
\end{aligned}
$$

where $A = D - 2$, $B = -3D + 6$, $C = -D + 3$, and t_n in given in Lemma 3.2.

Remark. If we take $D = 0$, we have for $n \ge 0$,

$$
\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} \mathfrak{T}_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\\
&= -\frac{2}{563} 3^n \mathfrak{T}_n^{(15,27,48,107)} - \frac{6}{563}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k^{(-5,2,13,32)}(-1)^k\\
&\quad + \frac{3}{563}\sum_{k=0}^{n}\binom{n}{k} 2^{n-k}\mathfrak{T}_{n-k}^{(40,64,215,344)}\mathfrak{T}_k .
\end{aligned}
$$

Proof. First, by Lemmata 3.1, 3.2, 3.3, 4.1 and 4.4, we have

$$
\begin{aligned}
&(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})^3\\
&=A(c_1^3e^{3\alpha x}+c_2^3e^{3\beta x}+c_3^3e^{3\gamma x}+c_4^3e^{3\delta x})\\
&\quad+B(c_1c_2c_3e^{(\alpha+\beta+\gamma)x}+c_2c_3c_4e^{(\beta+\gamma+\delta)x}+c_1c_2c_4e^{(\alpha+\beta+\delta)x}+c_1c_3c_4e^{(\alpha+\gamma+\delta)x})\\
&\quad+C(c_1^2e^{2\alpha x}+c_2^2e^{2\beta x}+c_3^2e^{2\gamma x}+c_4^2e^{2\delta x})(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})\\
&\quad+D(c_1c_2e^{(\alpha+\beta)x}+c_1c_3e^{(\alpha+\gamma)x}+c_1c_4e^{(\alpha+\delta)x}+c_2c_3e^{(\beta+\gamma)x}+c_2c_4e^{(\beta+\delta)x}+c_3c_4e^{(\gamma+\delta)x})\\
&\quad\times(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})\\
&=\frac{A}{563}\sum_{n=0}^{\infty}\mathfrak{T}_n^{(15,27,48,107)}\frac{(3x)^n}{n!}-\frac{B}{563}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k^{(-5,2,13,32)}(-1)^k\frac{x^n}{n!}\\
&\quad+\frac{C}{563}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\mathfrak{T}_{n-k}^{(40,64,215,344)}\mathfrak{T}_k\frac{x^n}{n!}+D\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_kt_{n-k}\frac{x^n}{n!}\,.
\end{aligned}
$$

On the other hand,

$$
\left(\sum_{n=0}^{\infty}\mathfrak{T}_n\frac{x^n}{n!}\right)^3=\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge0}}\binom{n}{k_1,k_2,k_3}\mathfrak{T}_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\frac{x^n}{n!}\,.
$$

Comparing the coefficients on both sides, we get the desired result. □

Next, we shall consider the sum of the products of four Tetranacci numbers. We need the following supplementary result. The proof is similar to that of Lemma 4.4 and, hence, omitted.

Lemma 4.6. *We have*

$$
c_1^4e^{\alpha x}+c_2^4e^{\beta x}+c_3^4e^{\gamma x}+c_4^4e^{\delta x}=\frac{1}{563^2}\sum_{n=0}^{\infty}\mathfrak{T}_n^{(3052,4658,8804,16451)}\frac{x^n}{n!}\,.
$$

By using Lemmata 3.1, 3.2, 3.3, 4.2, 4.4, and 4.6, letting $I=0$ in Lemma 4.2, comparing the coefficients on both sides, we can get the following theorem.

Theorem 4.7. *For $n\ge0$,*

$$
\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\ge0}}\binom{n}{k_1,k_2,k_3,k_4}\mathfrak{T}_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\mathfrak{T}_{k_4}\\
&=\frac{A}{563^2}4^n\mathfrak{T}_n^{(3052,4658,8804,16451)}-\frac{B}{563}+\frac{C}{563}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}\mathfrak{T}_{n-k}^{(15,27,48,107)}\mathfrak{T}_k\\
&\quad+\frac{D}{563^2}\sum_{k=0}^{n}\binom{n}{k}2^n\mathfrak{T}_{n-k}^{(40,64,215,344)}\mathfrak{T}_k^{(40,64,215,344)}\\
&\quad+\frac{E}{563}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\mathfrak{T}_{n-k}^{(40,64,215,344)}t_k+F\sum_{k=0}^{n}\binom{n}{k}t_{n-k}t_k
\end{aligned}
$$

$$+\frac{G}{563}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}\mathfrak{T}_{k_1}^{(40,64,215,344)}2^{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}$$
$$+H\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}t_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}$$
$$-\frac{J}{563}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}\mathfrak{T}_{k_1}^{(-5,2,13,32)}(-1)^{k_1}\mathfrak{T}_{k_2},$$

where $A=-D+E+G+H-3$*,* $B=-4D+4E+4G+4H-12$*,* $C=-E-2G-H+4$*,* $F=-2D-2G-2H+6$*,* $J=4D-E+2G-H$*, and* t_n *in given in Lemma 3.2.*

Remark. If $D=E=G=H=0$, then by $A=-3$, $B=-12$, $C=4$, $F=6$ and $J=0$, we have for $n\geq 0$,

$$\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}\mathfrak{T}_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\mathfrak{T}_{k_4}$$
$$=-\frac{3}{563^2}4^n\mathfrak{T}_n^{(3052,4658,8804,16451)}+\frac{12}{563}+\frac{4}{563}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}\mathfrak{T}_{n-k}^{(15,27,48,107)}\mathfrak{T}_k$$
$$+6\sum_{k=0}^{n}\binom{n}{k}t_{n-k}t_k\,.$$

Let

$$\sum_{n=0}^{\infty}t_n^{(1)}\frac{x^n}{n!}$$
$$=c_1c_2c_3e^{(\alpha+\beta+\gamma)x}(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x})+c_2c_3c_4e^{(\beta+\gamma+\delta)x}(c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})$$
$$+c_1c_2c_4e^{(\alpha+\beta+\delta)x}(c_1e^{\alpha x}+c_2e^{\beta x}+c_4e^{\delta x})+c_1c_3c_4e^{(\alpha+\gamma+\delta)x}(c_1e^{\alpha x}+c_3e^{\gamma x}+c_4e^{\delta x}).$$

By using Lemmata 3.1, 3.2, 3.3, 4.2, 4.4, and 4.6, comparing the coefficients on both sides, we can get the following theorem.

Theorem 4.8. *For* $n\geq 0, I\neq 0$

$$It_n^{(1)}=\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}\mathfrak{T}_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\mathfrak{T}_{k_4}$$
$$-\frac{A}{563^2}4^n\mathfrak{T}_n^{(3052,4658,8804,16451)}+\frac{B}{563}-\frac{C}{563}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}\mathfrak{T}_{n-k}^{(15,27,48,107)}\mathfrak{T}_k$$
$$-\frac{D}{563^2}\sum_{k=0}^{n}\binom{n}{k}2^n\mathfrak{T}_{n-k}^{(40,64,215,344)}\mathfrak{T}_k^{(40,64,215,344)}$$
$$-\frac{E}{563}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\mathfrak{T}_{n-k}^{(40,64,215,344)}t_k-F\sum_{k=0}^{n}\binom{n}{k}t_{n-k}t_k$$

$$
\begin{aligned}
&- \frac{G}{563} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} \mathfrak{T}_{k_1}^{(40,64,215,344)} 2^{k_1} \mathfrak{T}_{k_2} \mathfrak{T}_{k_3} \\
&- H \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} t_{k_1} \mathfrak{T}_{k_2} \mathfrak{T}_{k_3} \\
&+ \frac{J}{563} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} \mathfrak{T}_{k_1}^{(-5,2,13,32)} (-1)^{k_1} \mathfrak{T}_{k_2},
\end{aligned}
$$

where $A = -D + E + G + H - 3$, $B = 12D + 12G - 4J - 12$, $C = -E - 2G - H + 4$, $F = -2D - 2G - 2H + 6$, $I = 4D - E + 2G - H - J$, *and* t_n *is given in Lemma 3.2.*

Remark. If $D = E = G = H = 0$, $J = -1$, then by $A = -3$, $B = -8$, $C = 4$, $F = 6$ and $I = 1$, we have for $n \ge 0$,

$$
\begin{aligned}
\mathfrak{T}_n^{(1)} = &\sum_{\substack{k_1+k_2+k_3+k_4=n \\ k_1,k_2,k_3,k_4\ge 0}} \binom{n}{k_1,k_2,k_3,k_4} \mathfrak{T}_{k_1} \mathfrak{T}_{k_2} \mathfrak{T}_{k_3} \mathfrak{T}_{k_4} \\
&+ \frac{3}{563^2} 4^n \mathfrak{T}_n^{(3052,4658,8804,16451)} - \frac{8}{563} - \frac{4}{563} \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} \mathfrak{T}_{n-k}^{(15,27,48,107)} \mathfrak{T}_k \\
&- 6 \sum_{k=0}^{n} \binom{n}{k} t_{n-k} t_k - \frac{1}{563} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} \mathfrak{T}_{k_1}^{(-5,2,13,32)} (-1)^{k_1} \mathfrak{T}_{k_2}.
\end{aligned}
$$

5 Convolution identities for five Tetranacci numbers

We shall consider the sum of the products of five Tetranacci numbers. We need the following supplementary result. The proof is similar to that of Lemma 4.4 and, hence, omitted.

Lemma 5.1.

$$
c_1^5 e^{\alpha x} + c_2^5 e^{\beta x} + c_3^5 e^{\gamma x} + c_4^5 e^{\delta x} = \frac{1}{563^2} \sum_{n=0}^{\infty} \mathfrak{T}_n^{(500,1423,2598,4986)} \frac{x^n}{n!}.
$$

By using Lemmata 3.1, 3.2, 3.3, 4.3, 4.4, 4.6 and 5.1, comparing the coefficients on both sides, we can get the following theorems.

5.1 Case 1

Let $B = C = D = 0$, we can obtain the following theorem.

Theorem 5.2. *For* $n \ge 0$,

$$
\sum_{\substack{k_1+\cdots+k_5=n \\ k_1,\ldots,k_5\ge 0}} \binom{n}{k_1,\ldots,k_5} \mathfrak{T}_{k_1} \cdots \mathfrak{T}_{k_5}
$$

$$
\begin{aligned}
&= \frac{A}{563^2} 5^n \mathfrak{T}_n^{(500,1423,2598,4986)} + \frac{E}{563^2} \sum_{k=0}^{n} \binom{n}{k} 4^{n-k} \mathfrak{T}_{n-k}^{(3052,4658,8804,16451)} \mathfrak{T}_k \\
&\quad - \frac{F}{563} \sum_{k=0}^{n} \binom{n}{k} \mathfrak{T}_k + G \sum_{k=0}^{n} \binom{n}{k} t_k^{(1)} \mathfrak{T}_{n-k} + \frac{H}{563^2} \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} 2^k \mathfrak{T}_{n-k}^{(15,27,48,107)} \mathfrak{T}_k^{(40,64,215,344)} \\
&\quad + \frac{I}{563} \sum_{k=0}^{n} \binom{n}{k} 3^k \mathfrak{T}_k^{(15,27,48,107)} t_{n-k} \\
&\quad - \frac{J}{563^2} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \geq 0}} \binom{n}{k_1, k_2, k_3} (-1)^{k_1} 2^{k_2} \mathfrak{T}_{k_1}^{(-5,2,13,32)} \mathfrak{T}_{k_2}^{(40,64,215,344)} \\
&\quad - \frac{K}{563} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \geq 0}} \binom{n}{k_1, k_2, k_3} (-1)^{k_1} \mathfrak{T}_{k_1}^{(-5,2,13,32)} t_{k_2} \\
&\quad + \frac{L}{563} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \geq 0}} \binom{n}{k_1, k_2, k_3} 3^{k_1} \mathfrak{T}_{k_1}^{(15,27,48,107)} \mathfrak{T}_{k_2} \mathfrak{T}_{k_3} \\
&\quad - \frac{M}{563} \sum_{\substack{k_1+k_2+k_3+k_4=n \\ k_1,k_2,k_3,k_4 \geq 0}} \binom{n}{k_1, k_2, k_3, k_4} (-1)^{k_1} \mathfrak{T}_{k_1}^{(-5,2,13,32)} \mathfrak{T}_{k_2} \mathfrak{T}_{k_3} \\
&\quad + \frac{N}{563^2} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \geq 0}} \binom{n}{k_1, k_2, k_3} 2^{k_1} \mathfrak{T}_{k_1}^{(40,64,215,344)} 2^{k_2} \mathfrak{T}_{k_2}^{(40,64,215,344)} \mathfrak{T}_{k_3} \\
&\quad + P \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \geq 0}} \binom{n}{k_1, k_2, k_3} t_{k_1} t_{k_2} \mathfrak{T}_{k_3} \\
&\quad + \frac{Q}{563} \sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3 \geq 0}} \binom{n}{k_1, k_2, k_3} 2^{k_1} \mathfrak{T}_{k_1}^{(40,64,215,344)} t_{k_2} \mathfrak{T}_{k_3} \\
&\quad + \frac{R}{563} \sum_{\substack{k_1+k_2+k_3+k_4=n \\ k_1,k_2,k_3,k_4 \geq 0}} \binom{n}{k_1, k_2, k_3, k_4} 2^{k_1} \mathfrak{T}_{k_1}^{(40,64,215,344)} \mathfrak{T}_{k_2} \mathfrak{T}_{k_3} \mathfrak{T}_{k_4} \\
&\quad + S \sum_{\substack{k_1+k_2+k_3+k_4=n \\ k_1,k_2,k_3,k_4 \geq 0}} \binom{n}{k_1, k_2, k_3, k_4} t_{k_1} \mathfrak{T}_{k_2} \mathfrak{T}_{k_3} \mathfrak{T}_{k_4},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I + 2L + 2N + P + 2Q + 6R + 4S - 14, \\
E &= -I - 2L - N - Q - 3R - S + 5, \\
F &= 4G + I + 2L + 6N + 5P + 6Q + 18R + 16S - 50, \\
H &= -L - 2N - P - Q - 4R - 3S + 10, \\
J &= -G - I - 2L - M - 2P - 3Q - 6R - 7S + 20, \\
K &= -2G - 2M - 2N - 5P - 2Q - 6R - 12S + 30,
\end{aligned}
$$

t_n *and* $t_n^{(1)}$ *are same as those in Lemma 3.2 and Theorem 4.8, respectively.*

Remark. If $G = I = L = M = N = P = Q = R = S = 0$, then by $A = -14$, $E = 5$, $F = -50$, $H = 10$, $J = 20$ and $K = 30$, we have for $n \geq 0$,

$$
\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_5=n\\ k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5} \mathfrak{T}_{k_1}\cdots\mathfrak{T}_{k_5}\\
&= -\frac{14}{563^2}5^n\mathfrak{T}_n^{(500,1423,2598,4986)} + \frac{5}{563^2}\sum_{k=0}^n\binom{n}{k}4^{n-k}\mathfrak{T}_{n-k}^{(3052,4658,8804,16451)}\mathfrak{T}_k\\
&\quad + \frac{50}{563}\sum_{k=0}^n\binom{n}{k}\mathfrak{T}_k + \frac{10}{563^2}\sum_{k=0}^n\binom{n}{k}3^{n-k}2^k\mathfrak{T}_{n-k}^{(15,27,48,107)}\mathfrak{T}_k^{(40,64,215,344)}\\
&\quad - \frac{20}{563^2}\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}2^{k_2}\mathfrak{T}_{k_1}^{(-5,2,13,32)}\mathfrak{T}_{k_2}^{(40,64,215,344)}\\
&\quad - \frac{30}{563}\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}\mathfrak{T}_{k_1}^{(-5,2,13,32)}t_{k_2}\,.
\end{aligned}
$$

5.2 Case 2

Let $B \neq 0$, $C = D = 0$, we can obtain the following theorem.
Let

$$
\begin{aligned}
&\sum_{n=0}^\infty t_n^{(2)}\frac{x^n}{n!}\\
&= c_1c_2c_3e^{(\alpha+\beta+\gamma)x}\big(c_1c_2e^{(\alpha+\beta)x} + c_2c_3e^{(\beta+\gamma)x} + c_3c_1e^{(\gamma+\alpha)x}\big) + \cdots\\
&\quad + c_2c_3c_4e^{(\beta+\gamma+\delta)x}\big(c_2c_3e^{(\beta+\gamma)x} + c_3c_4e^{(\gamma+\delta)x} + c_4c_2e^{(\delta+\beta)x}\big).
\end{aligned}
$$

Theorem 5.3. *For $n \geq 0$,*

$$
\begin{aligned}
Bt_n^{(2)} &= \sum_{\substack{k_1+\cdots+k_5=n\\ k_1,\dots,k_5\geq 0}}\binom{n}{k_1,\dots,k_5}\mathfrak{T}_{k_1}\cdots\mathfrak{T}_{k_5} - \frac{A}{563^2}5^n\mathfrak{T}_n^{(500,1423,2598,4986)} - \cdots\\
&\quad - S\sum_{\substack{k_1+k_2+k_3+k_4=n\\ k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}t_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\mathfrak{T}_{k_4},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= -G - J - M + 2N - P - Q - 3S + 6,\\
B &= -2G - K - 2M - 2N - 5P - 2Q - 6R - 12S + 30,\\
E &= G + J + M - N + 2P + 2Q + 3R + 6S - 15,\\
F &= -3G - J - 3K - 7M - 12P - 3Q - 6R - 27S + 60,\\
H &= -L - 2N - P - Q - 4R - 3S + 10,\\
I &= -G - J - 2L - M - 2P - 3Q - 6R - 7S + 20\,,
\end{aligned}
$$

t_n and $t_n^{(1)}$ are same as those in Lemma 3.2 and Theorem 4.8, respectively.

Remark. If $G = J = K = L = M = N = P = Q = R = S = 0$, then by $A = 6$, $B = 30$, $E = -15$, $F = 60$, $H = 10$ and $I = 20$, we have for $n \geq 0$,

$$
\begin{aligned}
30t_n^{(2)} &= \sum_{\substack{k_1+\cdots+k_5=n\\k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5}\mathfrak{T}_{k_1}\cdots\mathfrak{T}_{k_5} - \frac{6}{563^2}5^n\mathfrak{T}_n^{(500,1423,2598,4986)}\\
&\quad + \frac{15}{563^2}\sum_{k=0}^{n}\binom{n}{k}4^{n-k}\mathfrak{T}_{n-k}^{(3052,4658,8804,16451)}\mathfrak{T}_k + \frac{60}{563}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k\\
&\quad - \frac{10}{563^2}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}2^k\mathfrak{T}_{n-k}^{(15,27,48,107)}\mathfrak{T}_k^{(40,64,215,344)} - \frac{20}{563}\sum_{k=0}^{n}\binom{n}{k}3^k\mathfrak{T}_k^{(15,27,48,107)}t_{n-k}\,.
\end{aligned}
$$

5.3 Case 3

Let $C \neq 0$, $B = D = 0$, we can obtain the following theorem.
Let

$$
\begin{aligned}
&\sum_{n=0}^{\infty} t_n^{(3)}\frac{x^n}{n!}\\
&= c_1c_2c_3e^{(\alpha+\beta+\gamma)x}(c_1^2e^{2\alpha x} + c_2^2e^{2\beta x} + c_3^2e^{2\gamma x}) + \cdots\\
&\quad + c_2c_3c_4e^{(\beta+\gamma+\delta)x}(c_2^2e^{2\beta x} + c_3^2e^{2\gamma x} + c_4^2e^{2\delta x}).
\end{aligned}
$$

Theorem 5.4. *For $n \geq 0$,*

$$
\begin{aligned}
Ct_n^{(3)} &= \sum_{\substack{k_1+\cdots+k_5=n\\k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5}\mathfrak{T}_{k_1}\cdots\mathfrak{T}_{k_5} - \frac{A}{563^2}5^n\mathfrak{T}_n^{(500,1423,2598,4986)} - \cdots\\
&\quad - S\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}} \binom{n}{k_1,k_2,k_3,k_4}t_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\mathfrak{T}_{k_4},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I + 2L + 2N + P + 2Q + 6R + 4S - 14,\\
C &= -G - I - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\\
E &= -I - 2L - N - Q - 3R - S + 5,\\
F &= 3G - J - M + 6N + 3P + 3Q + 12R + 9S - 30,\\
H &= -L - 2N - P - Q - 4R - 3S + 10,\\
K &= -2G - 2M - 2N - 5P - 2Q - 6R - 12S + 30,
\end{aligned}
$$

t_n and $t_n^{(1)}$ are same as those in Lemma 3.2 and Theorem 4.8, respectively.

Remark. If $G = I = J = L = M = N = P = Q = R = S = 0$, then by $A = -14$, $C = 20$, $E = 5$, $F = -30$, $H = 10$ and $K = 30$, we have for $n \geq 0$,

$$
20t_n^{(3)} = \sum_{\substack{k_1+\cdots+k_5=n\\k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5}\mathfrak{T}_{k_1}\cdots\mathfrak{T}_{k_5} + \frac{14}{563^2}5^n\mathfrak{T}_n^{(500,1423,2598,4986)}
$$

$$
\begin{aligned}
&- \frac{5}{563^2}\sum_{k=0}^{n}\binom{n}{k}4^{n-k}\mathfrak{T}_{n-k}^{(3052,4658,8804,16451)}\mathfrak{T}_k + \frac{30}{563}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k \\
&- \frac{10}{563^2}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}2^k\mathfrak{T}_{n-k}^{(15,27,48,107)}\mathfrak{T}_k^{(40,64,215,344)} \\
&+ \frac{30}{563}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}\mathfrak{T}_{k_1}^{(-5,2,13,32)}t_{k_2}\,.
\end{aligned}
$$

5.4 Case 4

Let $D \neq 0$, $B = C = 0$, we can obtain the following theorem.
Let

$$
\begin{aligned}
&\sum_{n=0}^{\infty} t_n^{(4)}\frac{x^n}{n!} \\
&= c_1c_2c_3e^{(\alpha+\beta+\gamma)x}(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x})^2+\cdots \\
&\quad + c_2c_3c_4e^{(\beta+\gamma+\delta)x}(c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})^2.
\end{aligned}
$$

Theorem 5.5. *For $n \ge 0$,*

$$
\begin{aligned}
Dt_n^{(4)} = &\sum_{\substack{k_1+\cdots+k_5=n\\k_1,\dots,k_5\ge 0}}\binom{n}{k_1,\dots,k_5}\mathfrak{T}_{k_1}\cdots\mathfrak{T}_{k_5} - \frac{A}{563^2}5^n\mathfrak{T}_n^{(500,1423,2598,4986)} - \cdots \\
&- S\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\ge 0}}\binom{n}{k_1,k_2,k_3,k_4}t_{k_1}\mathfrak{T}_{k_2}\mathfrak{T}_{k_3}\mathfrak{T}_{k_4},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I+2L+2N+P+2Q+6R+4S-14,\\
D &= -G-I-J-2L-M-2P-3Q-6R-7S+20,\\
E &= -I-2L-N-Q-3R-S+5,\\
F &= -3G-6I-7J-7M+6N-12L-9P-15Q-24R-33S+90,\\
H &= -L-2N-P-Q-4R-3S+10,\\
K &= 2I+2J-2N+4L-P+4Q+6R+2S-10,
\end{aligned}
$$

t_n and $t_n^{(1)}$ are given in Lemma 3.2 and Theorem 4.8, respectively.

Remark. If $G = I = J = L = M = N = P = Q = R = S = 0$, then by $A = -14$, $D = 20$, $E = 5$, $F = 90$, $H = 10$ and $K = -10$, we have for $n \ge 0$,

$$
\begin{aligned}
20t_n^4 = &\sum_{\substack{k_1+\cdots+k_5=n\\k_1,\dots,k_5\ge 0}}\binom{n}{k_1,\dots,k_5}\mathfrak{T}_{k_1}\cdots\mathfrak{T}_{k_5} + \frac{14}{563^2}5^n\mathfrak{T}_n^{(500,1423,2598,4986)} \\
&- \frac{5}{563^2}\sum_{k=0}^{n}\binom{n}{k}4^{n-k}\mathfrak{T}_{n-k}^{(3052,4658,8804,16451)}\mathfrak{T}_k + \frac{90}{563}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{T}_k
\end{aligned}
$$

$$- \frac{10}{563^2}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}2^k\mathfrak{T}_{n-k}^{(15,27,48,107)}\mathfrak{T}_k^{(40,64,215,344)}$$
$$- \frac{10}{563}\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}\mathfrak{T}_{k_1}^{(-5,2,13,32)}t_{k_2}\,.$$

6 More general results

We shall consider a general case of Lemmata 3.1, 4.4 and 4.6. Similarly to the proof of Lemma 3.1, for Tetranacci-type numbers $s_{1,n}^{(n)}$, satisfying the recurrence relation $s_{1,k}^{(n)} = s_{1,k-1}^{(n)}+s_{1,k-2}^{(n)}+s_{1,k-3}^{(n)}+s_{1,k-4}^{(n)}$ ($k\ge 4$) with given initial values $s_{1,0}^{(n)}$, $s_{1,1}^{(n)}$, $s_{1,2}^{(n)}$ and $s_{1,3}^{(n)}$, we have the form

$$d_1^{(n)}e^{\alpha x}+d_2^{(n)}e^{\beta x}+d_3^{(n)}e^{\gamma x}+d_4^{(n)}e^{\delta x}=\sum_{k=0}^{\infty}s_{1,k}^{(n)}\frac{x^k}{k!}\,.$$

Theorem 6.1. *For $n\ge 1$, we have*

$$c_1^n e^{\alpha x}+c_2^n e^{\beta x}+c_3^n e^{\gamma x}+c_4^n e^{\delta x}=\frac{1}{A_1^{(n)}}\sum_{k=0}^{\infty}\mathfrak{T}_{1,k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\frac{x^k}{k!}\,,$$

where $s_{1,0}^{(n)}$, $s_{1,1}^{(n)}$, $s_{1,2}^{(n)}$, $s_{1,3}^{(n)}$ and $A_1^{(n)}$ satisfy the recurrence relation:

$$s_{1,0}^{(n)}=\pm\mathrm{lcm}(b_1,b_2,b_3),\quad s_{1,1}^{(n)}=Ms_{1,0}^{(n)},\quad s_{1,2}^{(n)}=Ns_{1,0}^{(n)},\quad s_{1,3}^{(n)}=Ps_{1,0}^{(n)},$$
$$A_1^{(n)}=\frac{A_1^{(n-1)}}{s_{1,2}^{(n-1)}}(4s_{1,3}^{(n)}-3s_{1,2}^{(n)}-2s_{1,1}^{(n)}-s_{1,0}^{(n)}),$$

b_1, b_2, b_3, M, N and P are determined in the proof.

Proof. By $d_1^{(n)}=A_1^{(n)}c_1^n$, $d_1^{(n-1)}=A_1^{(n-1)}c_1^{n-1}$,

$$d_1^{(n)}=\frac{s_{1,0}^{(n)}\beta\gamma\delta+s_{1,2}^{(n)}(\beta+\gamma+\delta)-s_{1,3}^{(n)}-s_{1,1}^{(n)}(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)},$$
$$c_1=\frac{2-(\beta+\gamma+\delta)+(\beta\gamma+\gamma\delta+\delta\beta)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}=\frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}=\frac{1}{-\alpha^3+6\alpha-1},$$

we can obtain the following recurrence relation:

$$A=-3s_{1,1}^{(n-1)}-s_{1,2}^{(n-1)}+s_{1,3}^{(n-1)},\quad B=5s_{1,1}^{(n-1)}+5s_{1,2}^{(n-1)}-5s_{1,3}^{(n-1)},$$
$$C=5s_{1,1}^{(n-1)}-2s_{1,2}^{(n-1)}+2s_{1,3}^{(n-1)},\quad D=5s_{1,1}^{(n-1)}-s_{1,2}^{(n-1)}+s_{1,3}^{(n-1)},$$
$$E=s_{1,0}^{(n-1)}-s_{1,1}^{(n-1)},\quad F=-5s_{1,0}^{(n-1)},\quad G=2s_{1,0}^{(n-1)}+6s_{1,1}^{(n-1)},$$
$$H=s_{1,0}^{(n-1)}-s_{1,1}^{(n-1)},\quad I=4s_{1,1}^{(n-1)}+6s_{1,2}^{(n-1)},\quad J=-3s_{1,1}^{(n-1)}-5s_{1,2}^{(n-1)},$$

$$
\begin{aligned}
&K = -2s_{1,1}^{(n-1)} + 2s_{1,2}^{(n-1)}, \quad L = -s_{1,1}^{(n-1)} + s_{1,2}^{(n-1)},\\
&M = \frac{(LA-DI)(FA-BE)-(HA-DE)(JA-BI)}{(GA-CE)(JA-BI)-(KA-CI)(FA-BE)},\\
&N = \frac{M(KA-CI)+(LA-DI)}{BI-JA}, \quad P = -\frac{1}{A}(BN+CM+D),\\
&M = \frac{a_1}{b_1}, \quad N = \frac{a_2}{b_2}, \quad P = \frac{a_3}{b_3}, \quad \text{with} \quad \gcd(a_i,b_i)=1,\\
&s_{1,0}^{(n)} = \pm \mathrm{lcm}(b_1,b_2,b_3), \quad s_{1,1}^{(n)} = Ms_{1,0}^{(n)}, \quad s_{1,2}^{(n)} = Ns_{1,0}^{(n)}, \quad s_{1,3}^{(n)} = Ps_{1,0}^{(n)},\\
&A_1^{(n)} = \frac{A_1^{(n-1)}}{s_{1,2}^{(n-1)}}(4s_{1,3}^{(n)} - 3s_{1,2}^{(n)} - 2s_{1,1}^{(n)} - s_{1,0}^{(n)}).
\end{aligned}
$$

We choose the value of $s_{1,0}^{(n)}$ such that for some k_0, $\forall k \geq k_0$, $\mathfrak{T}_{1,k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}$ is positive. □

Next we shall consider a general case of Lemma 3.3. Similarly to the proof of Lemma 3.3, for Tetranacci-type numbers $s_{1,k}^{(n)}$, satisfying the recurrence relation $s_{1,k}^{(n)} = s_{1,k-1}^{(n)} + s_{1,k-2}^{(n)} + s_{1,k-3}^{(n)} + s_{1,k-4}^{(n)}$ ($k \geq 4$) with given initial values $s_{1,0}^{(n)}$, $s_{1,1}^{(n)}$, $s_{1,2}^{(n)}$ and $s_{1,3}^{(n)}$, we have the form

$$
r_1^{(n)}e^{\alpha x} + r_2^{(n)}e^{\beta x} + r_3^{(n)}e^{\gamma x} + r_4^{(n)}e^{\delta x} = \sum_{k=0}^{\infty} s_{1,k}^{(n)}\frac{x^k}{k!},
$$

where $r_1^{(n)}$, $r_2^{(n)}$, $r_3^{(n)}$ and $r_4^{(n)}$ are determined by solving the system of the equations.

Theorem 6.2.

$$
c_2^n c_3^n c_4^n e^{\alpha x} + c_1^n c_3^n c_4^n e^{\beta x} + c_1^n c_2^n c_4^n e^{\gamma x} + c_1^n c_2^n c_3^n e^{\delta x} = \frac{1}{A_2^{(n)}} \sum_{k=0}^{\infty} \mathfrak{T}_{2,k}^{(s_{2,0}^{(n)},s_{2,1}^{(n)},s_{2,2}^{(n)},s_{2,3}^{(n)})}\frac{x^k}{k!},
$$

where $s_{2,0}^{(n)}$, $s_{2,1}^{(n)}$, $s_{2,2}^{(n)}$, $s_{2,3}^{(n)}$ and $A_2^{(n)}$ satisfy the recurrence relation:

$$
\begin{aligned}
&s_{2,0}^{(n)} = \pm \mathrm{lcm}(b_1,b_2,b_3), \quad s_{2,1}^{(n)} = Ms_{2,0}^{(n)}, \quad s_{2,2}^{(n)} = Ns_{2,0}^{(n)}, \quad s_{2,3}^{(n)} = Ps_{2,0}^{(n)},\\
&A_2^{(n)} = \frac{A_2^{(n-1)}}{2s_{2,2}^{(n-1)} - s_{2,3}^{(n-1)}}(-16s_{2,3}^{(n)} + 103s_{2,2}^{(n)} - 157s_{2,1}^{(n)} - 10s_{2,0}^{(n)}),
\end{aligned}
$$

b_1,b_2, b_3, M, N and P are determined in the proof.

Proof. By $r_1^{(n)} = A_2^{(n)}c_2^n c_3^n c_4^n$, we can obtain the following recurrence relation:

$$
\begin{aligned}
&A = -16s_{2,0}^{(n-1)} - 16s_{2,1}^{(n-1)} + 158s_{2,2}^{(n-1)} - 71s_{2,3}^{(n-1)},\\
&B = 103s_{2,0}^{(n-1)} + 103s_{2,1}^{(n-1)} - 243s_{2,2}^{(n-1)} + 70s_{2,3}^{(n-1)},\\
&C = -157s_{2,0}^{(n-1)} - 157s_{2,1}^{(n-1)} - 209s_{2,2}^{(n-1)} + 183s_{2,3}^{(n-1)},\\
&D = -10s_{2,0}^{(n-1)} - 10s_{2,1}^{(n-1)} - 42s_{2,2}^{(n-1)} + 26s_{2,3}^{(n-1)},\\
&E = 32s_{2,0}^{(n-1)} + 16s_{2,1}^{(n-1)} - 330s_{2,2}^{(n-1)} + 157s_{2,3}^{(n-1)},\\
&F = -206s_{2,0}^{(n-1)} - 103s_{2,1}^{(n-1)} + 365s_{2,2}^{(n-1)} - 131s_{2,3}^{(n-1)},
\end{aligned}
$$

$$G=314s_{2,0}^{(n-1)}+157s_{2,1}^{(n-1)}+351s_{2,2}^{(n-1)}-254s_{2,3}^{(n-1)},$$
$$H=20s_{2,0}^{(n-1)}+10s_{2,1}^{(n-1)}+216s_{2,2}^{(n-1)}-113s_{2,3}^{(n-1)},$$
$$I=32s_{2,1}^{(n-1)}-36s_{2,2}^{(n-1)}+10s_{2,3}^{(n-1)},\quad J=-206s_{2,1}^{(n-1)}+91s_{2,2}^{(n-1)}+6s_{2,3}^{(n-1)},$$
$$K=314s_{2,1}^{(n-1)}+69s_{2,2}^{(n-1)}-113s_{2,3}^{(n-1)},\quad L=20s_{2,1}^{(n-1)}-304s_{2,2}^{(n-1)}+147s_{2,3}^{(n-1)},$$

$$M=\frac{(LA-DI)(FA-BE)-(HA-DE)(JA-BI)}{(GA-CE)(JA-BI)-(KA-CI)(FA-BE)},$$
$$N=\frac{M(GA-CE)+(HA-DE)}{BE-FA},\quad P=-\frac{1}{A}(BN+CM+D),$$
$$M=\frac{a_1}{b_1},\quad N=\frac{a_2}{b_2},\quad P=\frac{a_3}{b_3},\quad \gcd(a_i,b_i)=1,$$
$$s_{2,0}^{(n)}=\pm\mathrm{lcm}(b_1,b_2,b_3),\quad s_{2,1}^{(n)}=Ms_{2,0}^{(n)},\quad s_{2,2}^{(n)}=Ns_{2,0}^{(n)},\quad s_{2,3}^{(n)}=Ps_{2,0}^{(n)},$$
$$A_2^{(n)}=\frac{A_2^{(n-1)}}{2s_{2,2}^{(n-1)}-s_{2,3}^{(n-1)}}(-16s_{2,3}^{(n)}+103s_{2,2}^{(n)}-157s_{2,1}^{(n)}-10s_{2,0}^{(n)}).$$

We choose the value of $s_{2,0}^{(n)}$ such that for some k_0, $\forall k\geq k_0$, $\mathfrak{T}_{2,k}^{(s_{2,0}^{(n)},s_{2,1}^{(n)},s_{2,2}^{(n)},s_{2,3}^{(n)})}$ is positive. □

As application, we compute some values of $s_{2,0}^{(n)}$, $s_{2,1}^{(n)}$, $s_{2,2}^{(n)}$, $A_2^{(n)}$ for some n. For $n=2$, we have

$$A=-170,\quad B=-1228,\quad C=3610,\quad D=316,\quad E=606,\quad F=1377,$$
$$G=-4821,\quad H=-888,\quad I=-84,\quad J=963,\quad K=-2091,\quad L=792,$$
$$M=-\frac{34}{15},\quad N=-6,\quad P=-\frac{44}{15},$$
$$s_{2,0}^2=-15,\quad s_{2,1}^2=34,\quad s_{2,2}^2=90,\quad s_{2,3}^2=44,\quad A_2^2=563^2,$$
$$c_2^2c_3^2c_4^2e^{\alpha x}+c_1^2c_3^2c_4^2e^{\beta x}+c_1^2c_2^2c_4^2e^{\gamma x}+c_1^2c_2^2c_3^2e^{\delta x}=\frac{1}{563^2}\sum_{k=0}^{\infty}\mathfrak{T}_{2,k}^{(-15,34,90,44)}\frac{x^k}{k!}.$$

For $n=3$, we have

$$A=10792,\quad B=-16833,\quad C=13741,\quad D=-2826,\quad E=-22728,\quad F=26674,$$
$$G=21042,\quad H=14508,\quad I=-1712,\quad J=1450,\quad K=11914,\quad L=-20212,$$
$$M=\frac{353}{175},\quad N=-\frac{21}{25},\quad P=\frac{38}{25},$$
$$s_{2,0}^3=175,\quad s_{2,1}^3=353,\quad s_{2,2}^3=-147,\quad s_{2,3}^3=266,\quad A_2^3=-563^3,$$
$$c_2^3c_3^3c_4^3e^{\alpha x}+c_1^3c_3^3c_4^3e^{\beta x}+c_1^3c_2^3c_4^3e^{\gamma x}+c_1^3c_2^3c_3^3e^{\delta x}=-\frac{1}{563^3}\sum_{k=0}^{\infty}\mathfrak{T}_{2,k}^{(175,353,-147,266)}\frac{x^k}{k!}.$$

We can obtain more convolution identities for any fixed n, but we only give some of the results. The proofs of the next eight theorems are similar to the proofs of Theorem (Lemma) 3.2, 4.5, 4.7, 4.8, 5.2, 5.3, 5.4 and 5.5, and, hence, omitted.

Let

$$c_1^n c_2^n e^{(\alpha+\beta)x} + \cdots + c_3^n c_4^n e^{(\gamma+\delta)x} = \sum_{k=0}^{\infty} t_{1,k}^{(n)} \frac{x^k}{k!},$$

then by previous algebraic identities, we can obtain the following theorems.

Theorem 6.3. *For $m \geq 0$, $n \geq 1$,*

$$c_1^n c_2^n e^{(\alpha+\beta)x} + \cdots + c_3^n c_4^n e^{(\gamma+\delta)x} = \sum_{k=0}^{\infty} t_{1,m}^{(n)} \frac{x^m}{m!},$$

where

$$t_{1,m}^{(n)} = \frac{1}{2}\left(\frac{1}{(A_1^{(n)})^2}\sum_{k=0}^{m}\binom{m}{k}\mathfrak{T}_{1,k}^{(s_{1,0}^{n},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} - \frac{2^m}{A_1^{(2n)}}\mathfrak{T}_{1,m}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\right).$$

Theorem 6.4. *For $m \geq 0$, $n \geq 1$,*

$$\begin{aligned}
&\frac{1}{(A_1^{(n)})^3}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\\
&= \frac{A}{A_1^{(3n)}}3^m\mathfrak{T}_{1,m}^{(s_{1,0}^{(3n)},s_{1,1}^{(3n)},s_{1,2}^{(3n)},s_{1,3}^{(3n)})} + \frac{B}{A_2^{(n)}}\sum_{k=0}^{m}\binom{m}{k}\mathfrak{T}_{2,k}^{(s_{2,0}^{(n)},s_{2,1}^{(n)},s_{2,2}^{(n)},s_{2,3}^{(n)})}(-1)^k\\
&\quad + \frac{C}{A_1^{(n)}A_1^{(2n)}}\sum_{k=0}^{m}\binom{m}{k}\mathfrak{T}_{1,k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}2^{m-k}\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(2n)},s_{1,1}^{(2n)},s_{1,2}^{(2n)},s_{1,3}^{(2n)})}\\
&\quad + \frac{D}{A_1^{(n)}}\sum_{k=0}^{m}\binom{m}{k}\mathfrak{T}_{1,k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}t_{1,m-k}^{(n)},
\end{aligned}$$

where

$$A = D - 2, \quad B = -3D + 6, \quad C = -D + 3,$$

$t_{1,m}^{(n)}$ is determined in Theorem 6.3.

Theorem 6.5. *For $m \geq 0$, $n \geq 1$,*

$$\begin{aligned}
&\frac{1}{(A_1^{(n)})^4}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\cdots\mathfrak{T}_{1,k_4}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\\
&= \frac{A}{A_1^{(4n)}}4^m\mathfrak{T}_{1,m}^{(s_{1,0}^{(4n)},\cdots,s_{1,3}^{(4n)})} + B(\frac{-1}{563})^n + \frac{C}{A_1^{(3n)}A_1^{(n)}}\sum_{k=0}^{m}\binom{m}{k}3^{m-k}\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})}\mathfrak{T}_{1,k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&\quad + \frac{D}{(A_1^{(2n)})^2}\sum_{k=0}^{m}\binom{m}{k}2^m\mathfrak{T}_{1,k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}\\
&\quad + \frac{E}{A_1^{(2n)}}\sum_{k=0}^{m}\binom{m}{k}2^{m-k}\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}t_{1,m-k}^{(n)} + F\sum_{k=0}^{m}\binom{m}{k}t_{1,k}^{(n)}t_{1,m-k}^{(n)}\\
&\quad + \frac{G}{A_1^{(2n)}(A_1^{(n)})^2}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}2^{k_1}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}
\end{aligned}$$

$$+\frac{H}{(A_1^{(n)})^2}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}t_{1,k_1}^{(n)}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$+\frac{J}{A_2^{(n)}A_1^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}(-1)^{k_1}\mathfrak{T}_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},$$

where $A=-D+E+G+H-3$, $B=-4D+4E+4G+4H-12$, $C=-E-2G-H+4$, $F=-2D-2G-2H+6$, $J=4D-E+2G-H$, $t_{1,m}^{(n)}$ *is determined in Theorem 6.3.*

Let

$$\sum_{k=0}^{\infty}t_{2,k}^{(n)}\frac{x^k}{k!}=c_1^nc_2^nc_3^ne^{(\alpha+\beta+\gamma)x}(c_1^ne^{\alpha x}+c_2^ne^{\beta x}+c_3^ne^{\gamma x})+\cdots$$
$$+c_2^nc_3^nc_4^ne^{(\alpha+\gamma+\delta)x}(c_2^ne^{\alpha x}+c_3^ne^{\gamma x}+c_4^ne^{\delta x}).$$

Theorem 6.6. *For* $m\geq 0$, $n\geq 1$,$I\neq 0$,

$$It_{2,m}^{(n)}=\frac{1}{(A_1^{(n)})^4}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\cdots\mathfrak{T}_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$-\frac{A}{A_1^{(4n)}}4^m\mathfrak{T}_{1,m}^{(s_{1,0}^{(4n)},\cdots,s_{1,3}^{(4n)})}-\cdots$$
$$-\frac{J}{A_2^{(n)}A_1^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}(-1)^{k_1}\mathfrak{T}_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},$$

where $A=-D+E+G+H-3$, $B=12D+12G-4J-12$, $C=-E-2G-H+4$, $F=-2D-2G-2H+6$, $I=4D-E+2G-H-J$, $t_{1,m}^{(n)}$ *is determined in Theorem 6.3.*

Lemma 4.3 will be discussed in four cases.

6.1 Case 1

$B=C=D=0$.

Theorem 6.7. *For* $m\geq 0$, $n\geq 1$,

$$\frac{1}{(A_1^{(n)})^5}\sum_{\substack{k_1+\cdots+k_5=m\\k_1,\cdots,k_5\geq 0}}\binom{m}{k_1,\cdots,k_5}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\cdots\mathfrak{T}_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$=\frac{A}{A_1^{(5n)}}5^m\mathfrak{T}_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})}+\frac{E}{A_1^{(4n)}A_1^n}\sum_{k=0}^{m}\binom{m}{k}4^{m-k}\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(4n)},\cdots,s_{1,3}^{(4n)})}\mathfrak{T}_{1,k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$+F(\frac{-1}{563})^n\sum_{k=0}^{m}\binom{m}{k}\mathfrak{T}_{1,k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}+\frac{G}{A_1^{(n)}}\sum_{k=0}^{m}\binom{m}{k}t_{2,k}^{(n)}\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$+\frac{H}{A_1^{(3n)}A_1^{(2n)}}\sum_{k=0}^{m}\binom{m}{k}3^{m-k}2^k\mathfrak{T}_{1,m-k}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})}\mathfrak{T}_{1,k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}$$

$$
\begin{aligned}
&+\frac{I}{A_1^{(3n)}}\sum_{k=0}^{m}\binom{m}{k}3^k\mathfrak{T}_{1,k}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})}t_{1,m-k}^{(n)}\\
&+\frac{J}{A_2^{(n)}A_1^{(2n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}(-1)^{k_1}\mathfrak{T}_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})}2^{k_2}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}\\
&+\frac{K}{A_2^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}(-1)^{k_1}\mathfrak{T}_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})}t_{1,k_2}^{(n)}\\
&+\frac{L}{A_1^{(3n)}(A_1^{(n)})^2}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}3^{k_1}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&+\frac{M}{A_2^{(n)}(A_1^{(n)})^2}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}(-1)^{k_1}\mathfrak{T}_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&+\frac{N}{(A_1^{(2n)})^2A_1^{n}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}2^{k_1}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}2^{k_2}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&+\frac{P}{A_1^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}t_{1,k_1}^{(n)}t_{1,k_2}^{(n)}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&+\frac{Q}{A_1^{(2n)}A_1^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}2^{k_1}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}t_{1,k_2}^{(n)}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&+\frac{R}{A_1^{(2n)}(A_1^{(n)})^3}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}2^{k_1}\mathfrak{T}_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&\qquad\qquad\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\mathfrak{T}_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&+\frac{S}{(A_1^{(n)})^3}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}t_{1,k_1}^{(n)}\mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\mathfrak{T}_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I+2L+2N+P+2Q+6R+4S-14,\\
E &= -I-2L-N-Q-3R-S+5,\\
F &= 4G+I+2L+6N+5P+6Q+18R+16S-50,\\
H &= -L-2N-P-Q-4R-3S+10,\\
J &= -G-I-2L-M-2P-3Q-6R-7S+20,\\
K &= -2G-2M-2N-5P-2Q-6R-12S+30,
\end{aligned}
$$

$t_{1,m}^{(n)}$ *and* $t_{2,m}^{(n)}$ *are determined in Theorem 6.3 and 6.6, respectively.*

6.2 Case 2

$B \neq 0$, $C = D = 0$. Let

$$\begin{aligned}&\sum_{k=0}^{\infty} t_{3,k}^{(n)} \frac{x^k}{k!}\\&= c_1^n c_2^n c_3^n e^{(\alpha+\beta+\gamma)x}(c_1^n c_2^n e^{(\alpha+\beta)x} + c_2^n c_3^n e^{(\beta+\gamma)x} + c_3^n c_1^n e^{(\gamma+\alpha)x}) + \cdots\\&\quad + c_2^n c_3^n c_4^n e^{(\beta+\gamma+\delta)x}(c_2^n c_3^n e^{(\beta+\gamma)x} + c_3^n c_4^n e^{(\gamma+\delta)x} + c_4^n c_2^n e^{(\delta+\beta)x}).\end{aligned}$$

Theorem 6.8. *For $m \geq 0$, $n \geq 1$,*

$$\begin{aligned}Bt_{3,m}^{(n)} &= \frac{1}{(A_1^{(n)})^5} \sum_{\substack{k_1+\cdots+k_5=m\\k_1,\cdots,k_5\geq 0}} \binom{m}{k_1,\cdots,k_5} \mathfrak{T}_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots \mathfrak{T}_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\&\quad - \frac{A}{A_1^{(5n)}} 5^m \mathfrak{T}_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})} - \cdots\\&\quad - \frac{S}{(A_1^{(n)})^3} \sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}} \binom{m}{k_1,k_2,k_3,k_4} t_{1,k_1}^{(n)} \mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \mathfrak{T}_{1,k_3}^{(s_{1,0}^{n},\cdots,s_{1,3}^{(n)})} \mathfrak{T}_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},\end{aligned}$$

where

$$\begin{aligned}A &= -G - J - M + 2N - P - Q - 3S + 6,\\B &= -2G - K - 2M - 2N - 5P - 2Q - 6R - 12S + 30,\\E &= G + J + M - N + 2P + 2Q + 3R + 6S - 15,\\F &= -3G - J - 3K - 7M - 12P - 3Q - 6R - 27S + 60,\\H &= -L - 2N - P - Q - 4R - 3S + 10,\\I &= -G - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\end{aligned}$$

$t_{1,m}^{(n)}$ and $t_{2,m}^{(n)}$ are determined in Theorem 6.3 and 6.6, respectively.

6.3 Case 3

$C \neq 0$, $B = D = 0$. Let

$$\begin{aligned}&\sum_{k=0}^{\infty} t_{4,k}^{(n)} \frac{x^k}{k!}\\&= c_1^n c_2^n c_3^n e^{(\alpha+\beta+\gamma)x}(c_1^{2n} e^{2\alpha x} + c_2^{2n} e^{2\beta x} + c_3^{2n} e^{2\gamma x}) + \cdots\\&\quad + c_2^n c_3^n c_4^n e^{(\beta+\gamma+\delta)x}(c_2^{2n} e^{2\beta x} + c_3^{2n} e^{2\gamma x} + c_4^{2n} e^{2\delta x}).\end{aligned}$$

Theorem 6.9. *For $m \geq 0$, $n \geq 1$,*

$$Ct_{4,m}^{(n)} = \frac{1}{(A_1^{(n)})^5} \sum_{\substack{k_1+\cdots+k_5=m\\k_1,\cdots,k_5\geq 0}} \binom{m}{k_1,\cdots,k_5} \mathfrak{T}_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots \mathfrak{T}_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$- \frac{A}{A_1^{(5n)}} 5^m \mathfrak{T}_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})} - \cdots$$

$$- \frac{S}{(A_1^{(n)})^3} \sum_{\substack{k_1+k_2+k_3+k_4=m\\ k_1,k_2,k_3,k_4\geq 0}} \binom{m}{k_1,k_2,k_3,k_4} t_{1,k_1}^{(n)} \mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \mathfrak{T}_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},$$

where

$$\begin{aligned}
A &= I + 2L + 2N + P + 2Q + 6R + 4S - 14,\\
C &= -G - I - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\\
E &= -I - 2L - N - Q - 3R - S + 5,\\
F &= 3G - J - M + 6N + 3P + 3Q + 12R + 9S - 30,\\
H &= -L - 2N - P - Q - 4R - 3S + 10,\\
K &= -2G - 2M - 2N - 5P - 2Q - 6R - 12S + 30,
\end{aligned}$$

$t_{1,m}^{(n)}$ and $t_{2,m}^{(n)}$ are determined in Theorem 6.3 and 6.6, respectively.

6.4 Case 4

$D \neq 0$, $B = C = 0$. Let

$$\begin{aligned}
&\sum_{k=0}^{\infty} t_{5,k}^{(n)} \frac{x^k}{k!}\\
&= c_1^n c_2^n c_3^n e^{(\alpha+\beta+\gamma)x} (c_1^n e^{\alpha x} + c_2^n e^{\beta x} + c_3^n e^{\gamma x})^2 + \cdots\\
&\quad + c_2^n c_3^n c_4^n e^{(\beta+\gamma+\delta)x} (c_2^n e^{\beta x} + c_3^n e^{\gamma x} + c_4^n e^{\delta x})^2.
\end{aligned}$$

Theorem 6.10. *For $m \geq 0$, $n \geq 1$,*

$$Dt_{5,m}^{(n)} = \frac{1}{(A_1^{(n)})^5} \sum_{\substack{k_1+\cdots+k_5=m\\ k_1,\cdots,k_5\geq 0}} \binom{m}{k_1,\cdots,k_5} \mathfrak{T}_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots \mathfrak{T}_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$- \frac{A}{A_1^{(5n)}} 5^m \mathfrak{T}_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})} - \cdots$$

$$- \frac{S}{(A_1^{(n)})^3} \sum_{\substack{k_1+k_2+k_3+k_4=m\\ k_1,k_2,k_3,k_4\geq 0}} \binom{m}{k_1,k_2,k_3,k_4} t_{1,k_1}^{(n)} \mathfrak{T}_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \mathfrak{T}_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \mathfrak{T}_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},$$

where

$$\begin{aligned}
A &= I + 2L + 2N + P + 2Q + 6R + 4S - 14,\\
D &= -G - I - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\\
E &= -I - 2L - N - Q - 3R - S + 5,\\
F &= -3G - 6I - 7J - 7M + 6N - 12L - 9P - 15Q - 24R - 33S + 90,\\
H &= -L - 2N - P - Q - 4R - 3S + 10,\\
K &= 2I + 2J - 2N + 4L - P + 4Q + 6R + 2S - 10,
\end{aligned}$$

$t_{1,m}^{(n)}$ and $t_{2,m}^{(n)}$ are determined in Theorem 6.3 and 6.6, respectively.

7 Some more interesting general expressions

We shall give some more interesting general expressions.

Lemma 7.1. *For $n \geq 1$, we have*

$$
\begin{aligned}
&(c_2c_3 + c_3c_4 + c_4c_2)e^{\alpha x} + (c_3c_4 + c_4c_1 + c_1c_3)e^{\beta x}\\
&\quad + (c_1c_2 + c_1c_4 + c_4c_2)e^{\gamma x} + (c_1c_2 + c_2c_3 + c_1c_3)e^{\delta x}\\
&= \frac{1}{563}\sum_{k=0}^{\infty} \mathfrak{T}_k^{(146,416,581,1080)}\frac{x^k}{k!}.
\end{aligned}
$$

Theorem 7.2.

$$
\begin{aligned}
&(c_2c_3 + c_3c_4 + c_4c_2)^n e^{\alpha x} + (c_3c_4 + c_4c_1 + c_1c_3)^n e^{\beta x}\\
&\quad + (c_1c_2 + c_1c_4 + c_4c_2)^n e^{\gamma x} + (c_1c_2 + c_2c_3 + c_1c_3)^n e^{\delta x}\\
&= \frac{1}{A_3^n}\sum_{k=0}^{\infty} \mathfrak{T}_{3,k}^{(s_{3,0}^n,s_{3,1}^n,s_{3,2}^n,s_{3,3}^n)}\frac{x^k}{k!}\,,
\end{aligned}
$$

where $s_{3,0}^{(n)}$, $s_{3,1}^{(n)}$, $s_{3,2}^{(n)}$, $s_{3,3}^{(n)}$ and $A_3^{(n)}$ satisfy the recurrence relation:

$$
\begin{aligned}
&s_{3,0}^{(n)} = \pm\mathrm{lcm}(b_1, b_2, b_3), \quad s_{3,1}^{(n)} = Ms_{3,0}^{(n)}, \quad s_{3,2}^{(n)} = Ns_{3,0}^{(n)}, \quad s_{3,3}^{(n)} = Ps_{3,0}^{(n)},\\
&A_3^{(n)} = A_3^{(n-1)}\frac{(-16s_{3,3}^{(n)} + 103s_{3,2}^{(n)} - 157s_{3,1}^{(n)} - 10s_{3,0}^{(n)})}{-8s_{3,3}^{(n)} + 10s_{3,2}^{(n)} + 7s_{3,1}^{(n)} - 6s_{3,0}^{(n)}}.
\end{aligned}
$$

b_1,b_2, b_3, M, N and P are determined in the proof.

Proof. Similarly to the proof of Theorem 6.3, we consider the form

$$
h_1^{(n)}e^{\alpha x} + h_2^{(n)}e^{\beta x} + h_3^{(n)}e^{\gamma x} + h_4^{(n)}e^{\delta x} = \sum_{k=0}^{\infty} s_{3,k}^{(n)}\frac{x^k}{k!}\,.
$$

By $h_1^{(n)} = A_3^{(n)}(c_2c_3 + c_3c_4 + c_4c_2)^n$, we can obtain the following recurrence relation:

$$
\begin{aligned}
A &= -650s_{3,0}^{(n-1)} + 385s_{3,1}^{(n-1)} + 854s_{3,2}^{(n-1)} - 664s_{3,3}^{(n-1)},\\
B &= 1862s_{3,0}^{(n-1)} + 231s_{3,1}^{(n-1)} - 1627s_{3,2}^{(n-1)} + 1178s_{3,3}^{(n-1)},\\
C &= -1100s_{3,0}^{(n-1)} - 2380s_{3,1}^{(n-1)} - 417s_{3,2}^{(n-1)} + 522s_{3,3}^{(n-1)},\\
D &= 16s_{3,0}^{(n-1)} - 252s_{3,1}^{(n-1)} - 170s_{3,2}^{(n-1)} + 148s_{3,3}^{(n-1)},\\
E &= 1198s_{3,0}^{(n-1)} - 1083s_{3,1}^{(n-1)} - 1906s_{3,2}^{(n-1)} + 1368s_{3,3}^{(n-1)},\\
F &= -2434s_{3,0}^{(n-1)} + 814s_{3,1}^{(n-1)} + 3473s_{3,2}^{(n-1)} - 1769s_{3,3}^{(n-1)},\\
G &= 988s_{3,0}^{(n-1)} + 1935s_{3,1}^{(n-1)} - 757s_{3,2}^{(n-1)} - 933s_{3,3}^{(n-1)},\\
H &= -518s_{3,0}^{(n-1)} + 801s_{3,1}^{(n-1)} + 920s_{3,2}^{(n-1)} - 834s_{3,3}^{(n-1)},\\
I &= 268s_{3,0}^{(n-1)} + 186s_{3,1}^{(n-1)} - 132s_{3,2}^{(n-1)} - 16s_{3,3}^{(n-1)},
\end{aligned}
$$

$$
\begin{aligned}
J &= -1303s_{3,0}^{(n-1)} - 1690s_{3,1}^{(n-1)} + 146s_{3,2}^{(n-1)} + 666s_{3,3}^{(n-1)},\\
K &= -3980s_{3,0}^{(n-1)} - 3273s_{3,1}^{(n-1)} + 1638s_{3,2}^{n-1} + 620s_{3,3}^{(n-1)},\\
L &= 1012s_{3,0}^{(n-1)} - 869s_{3,1}^{(n-1)} - 1490s_{3,2}^{(n-1)} + 1116s_{3,3}^{(n-1)},\\
M &= \frac{(LA-DI)(FA-BE)-(HA-DE)(JA-BI)}{(GA-CE)(JA-BI)-(KA-CI)(FA-BE)},\\
N &= \frac{M(GA-CE)+(HA-DE)}{BE-FA},\\
P &= -\frac{1}{A}(BN+CM+D),
\end{aligned}
$$

$$
\begin{aligned}
&M = \frac{a_1}{b_1}, \quad N = \frac{a_2}{b_2}, \quad P = \frac{a_3}{b_3}, \quad \gcd(a_i, b_i) = 1,\\
&s_{3,0}^{(n)} = \pm \mathrm{lcm}(b_1, b_2, b_3), \quad s_{3,1}^{(n)} = Ms_{3,0}^{(n)}, \quad s_{3,2}^{(n)} = Ns_{3,0}^{(n)}, \quad s_{3,3}^{(n)} = Ps_{3,0}^{(n)},\\
&A_3^{(n)} = A_3^{(n-1)} \frac{(-16s_{3,3}^{(n)} + 103s_{3,2}^{(n)} - 157s_{3,1}^{(n)} - 10s_{3,0}^{(n)})}{-8s_{3,3}^{(n)} + 10s_{3,2}^{(n)} + 7s_{3,1}^{(n)} - 6s_{3,0}^{(n)}}.
\end{aligned}
$$

We choose the value of $s_{3,0}^{(n)}$ such that for some k_0, $\forall k \geq k_0$, $\mathfrak{T}_{3,k}^{(s_{3,0}^{(n)}, s_{3,1}^{(n)}, s_{3,2}^{(n)}, s_{3,3}^{(n)})}$ is positive. □

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