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Composition of happy functions

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Abstract: For positive integers $e \geq 1$ and $b \geq 2$, let $S_{e,b} : \mathbb{N} \to \mathbb{N}$ be defined by

$$S_{e,b}(x) = a_k^e + a_{k-1}^e + \cdots + a_1^e$$

if $x = (a_k a_{k-1} \cdots a_1)_b = a_k b^{k-1} + a_{k-1} b^{k-2} + \cdots + a_2 b + a_1$ is the expansion of x in base b. We call $S_{e,b}$ an (e, b)-happy function. Let g be a composition of various (e, b)-happy functions. We show that, for any given $x \in \mathbb{N}$, the iteration sequence $(g^{(n)}(x))_{n \geq 0}$ either converges to a fixed point or eventually becomes a cycle. Here $g^{(0)}$ is the identity function mapping x to x for all x and $g^{(n)}$ is the n-fold composition of g. In addition, we prove that the number of all possible fixed points and cycles is finite. Examples are also given.
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1 Introduction and main results

Let $S_2 : \mathbb{N} \to \mathbb{N}$ be the function that takes a positive integer to the sum of the squares of its decimal digits. Then, S_2 is called the *happy function* and if $S_2^{(n)}(x) = 1$ for some $n \geq 1$, Then, x is called a *happy number* (see [1] and [5, Chapter E34]). For $n \geq 1$ and a function $f : \mathbb{N} \to \mathbb{N}$, $f^{(0)}$ is the identity function and $f^{(n)}$ is the n-fold composition of f. For example, the sequence $(S_2^{(n)}(7))_{n\geq 0}$ is $(7, 49, 97, 130, 10, 1, 1, \ldots)$ which converges to the fixed point 1 and the sequence $(S_2^{(n)}(2))_{n\geq 0}$ is $(2, 4, 16, 37, 58, 89, 145, 42, 20, 4, \ldots)$ which is eventually the cycle $(4, 16, \ldots, 20)$. It is well-known (see [1] or [5]) that for any $x \in \mathbb{N}$, $(S_2^{(n)}(x))_{n\geq 0}$ either converges to 1 or becomes the cycle $(4, 16, \ldots, 20)$. As usual, $(a_1, a_2, \ldots, a_k)$ and any cyclic permutation such as $(a_{k-1}, a_k, a_1, a_2, \ldots, a_{k-2})$ are considered the same cycle. By the above, we see that 7 is happy but 2 is not. See also Sequence A007770 in OEIS [8] for a list of happy numbers and other information. More generally, for positive integers $e \geq 1$ and $b \geq 2$, we define $S_{e,b} : \mathbb{N} \to \mathbb{N}$ by

$$S_{e,b}(x) = a_k^e + a_{k-1}^e + \cdots + a_1^e, \tag{1}$$

if $x = (a_k a_{k-1} \ldots a_1)_b = a_k b^{k-1} + a_{k-1} b^{k-2} + \cdots + a_2 b + a_1$ is the expansion of x in base b. We call $S_{e,b}$ an (e, b)-happy function and if there exists $n \geq 1$ such that $S_{e,b}^{(n)}(x) = 1$, then x is said to be an (e, b)-happy number (see [3] and [6]). For convenience, if we write a number without specifying a base, it is always written in base 10. Grundman and Teeple [4] obtain a result which implies that if x, e, b are given, then the sequence

$$(S_{e,b}^{(n)}(x))_{n\geq 0} \text{ converges to a fixed point or eventually becomes a cycle.} \tag{2}$$

In this article, we generalize (1) and (2) to the following form.

Definition 1.1. *For each* $\underline{\boldsymbol{e}} = (e_1, \ldots, e_k)$ *and* $\underline{\boldsymbol{b}} = (b_1, \ldots, b_k)$ *with* $e_i \geq 1$ *and* $b_i \geq 2$ *for all* $i = 1, 2, \ldots, k$, *define* $S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}} : \mathbb{N} \to \mathbb{N}$ *by*

$$S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}(x) = (S_{e_1,b_1} \circ S_{e_2,b_2} \circ \cdots \circ S_{e_k,b_k})(x) \quad \text{for all } x \in \mathbb{N}. \tag{3}$$

If $x \in \mathbb{N}$ *and* $S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}^{(n)}(x) = 1$ *for some* $n \geq 1$, *then* x *is said to be* $(\underline{\boldsymbol{e}}, \underline{\boldsymbol{b}})$*-happy.*

So if $e_1 = e_2 = \cdots = e_k = e$ and $b_1 = b_2 = \cdots = b_k = b$, then the iteration sequence $(S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}^{(n)}(x))_{n\geq 0}$ is a subsequence of $(S_{e,b}^{(n)}(x))_{n\geq 0}$ but if e_i or b_i are not all equal, then $(S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}^{(n)}(x))_{n\geq 0}$ may be a totally different sequence. For instance, suppose $\underline{\boldsymbol{e}} = (3, 2)$, $\underline{\boldsymbol{b}} = (4, 10)$, and $x = 7$. Then, $S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}(x) = (S_{3,4} \circ S_{2,10})(7) = S_{3,4}(S_{2,10}(7)) = S_{3,4}(7^2) = S_{3,4}(49) = S_{3,4}((301)_4) = 3^3 + 0^3 + 1^3 = 28$. $S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}(28) = (S_{3,4} \circ S_{2,10})(28) = S_{3,4}(2^2 + 8^2) = S_{3,4}(68) = S_{3,4}((1010)_4) = 1^3 + 0^3 + 1^3 + 0^3 = 2$. So the sequence $(S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}^{(n)}(7))_{n\geq 0}$ is $(7, 28, 2, 1, 1, \ldots)$, and so 7 and 2 are $(\underline{\boldsymbol{e}}, \underline{\boldsymbol{b}})$-happy numbers. Our purpose is to show that (2) also holds when $S_{e,b}$ is replaced by $S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}$. The proof can be obtained from a general method as follows. For a function $f : \mathbb{N} \to \mathbb{N}$, define the following two conditions:

(A) There exists $N_f \in \mathbb{N}$ such that $f(x) < x$ for all $x \geq N_f$.

(B) For each $x \in \mathbb{N}$, the sequence $(f^{(n)}(x))_{n\geq 0}$ converges to a fixed point of f or eventually enters into a cycle. In addition, the number of all such fixed points and cycles is finite.

Then, the generalization of (2) to $S_{\underline{e},\underline{b}}$ follows from the following two theorems.

Theorem 1.2. *If $f : \mathbb{N} \to \mathbb{N}$ satisfies (A), then f satisfies (B).*

Theorem 1.3. *If $f_1, f_2, \ldots, f_k : \mathbb{N} \to \mathbb{N}$ satisfy (A), then their composition $f_1 \circ f_2 \circ \cdots \circ f_k$ also satisfies (A).*

Remark that the idea of Theorem 1.2 is not new; for example, it is used in the proof of the main result in [4] in the case $f = S_{e,b}$. Nevertheless, Theorem 1.2 for a general function f seems to be new, and as far as we are aware Theorem 1.3 is also new and it leads us to the following theorem.

Theorem 1.4. *The function $S_{\underline{e},\underline{b}}$ defined by* (3) *satisfies (A) and (B).*

For more details about happy numbers and happy functions, see for example [2, 7, 9–11] and the references therein.

2 Proof of the main results

Proof of Theorem 1.2. For convenience, we write N instead of N_f and we assert that

$$\text{for every } y \in \mathbb{N}\text{, there exists } n \in \mathbb{N} \text{ such that } f^{(n)}(y) < N. \tag{4}$$

Suppose that there exists $y \in \mathbb{N}$ such that $f^{(n)}(y) \geq N$ for every $n \in \mathbb{N}$. Since $f(y) \geq N$, we obtain by (A) that $f(f(y)) < f(y)$. Since $f^{(2)}(y) \geq N$, we apply (A) again and obtain $f^{(3)}(y) < f^{(2)}(y) < f(y)$. Let $k \in \mathbb{N}$ be any positive integer. Since $f^{(n)}(y) \geq N$ for every $n \in \mathbb{N}$, we can repeat the above argument k times and obtain a strictly decreasing sequence of positive integers $f(y) > f^{(2)}(y) > f^{(3)}(y) > \cdots > f^{(k)}(y)$. Since these are integers, we have

$$f(y) \geq f^{(2)}(y) + 1 \geq f^{(3)}(y) + 2 \geq \cdots \geq f^{(k)}(y) + k - 1. \tag{5}$$

Since (5) holds for any k, we can choose $k = f(y) + 1$, and obtain $f^{(k)}(y) \leq f(y) - (k-1) = 0$, which is a contradiction. Hence, (4) is proved.

Now let $x \in \mathbb{N}$ and suppose that $(f^{(n)}(x))_{n\geq 0}$ does not converge to a fixed point of f. By (4), there exists $n_1 \in \mathbb{N}$ such that $f^{(n_1)}(x) < N$. Again by (4), there exists $n_2 \in \mathbb{N}$ such that $f^{(n_2)}(f^{(n_1)}(x)) < N$. Repeating this process N times, we obtain the set of positive integers

$$f^{(n_1)}(x), f^{(n_1+n_2)}(x), \ldots, f^{(n_1+n_2+\cdots+n_N)}(x),$$

which are less than N. By the pigeonhole principle, some of them are the same, say

$$f^{(n_1+n_2+\cdots+n_j)}(x) = f^{(n_1+n_2+\cdots+n_j+\cdots+n_\ell)}(x) \quad \text{for some } \ell > j \geq 1.$$

Let $y = f^{(n_1+n_2+\cdots+n_j)}(x)$. Then, the tail of the sequence $(f^{(n)}(x))_{n\geq 0}$ eventually becomes

$$(y, f(y), f^{(2)}(y), \ldots, f^{(n_{j+1}+n_{j+2}+\cdots+n_\ell-1)}(y), y, \ldots),$$

which is a cycle. This proves the first part of (B). Next, we show that the set U_f of fixed points and cycles is finite. More precisely, we will show that

$$U_f := \{x \in \mathbb{N} \mid \exists n \in \mathbb{N}, f^{(n)}(x) = x\} \subseteq [1, M], \tag{6}$$

where $M = \max\{N, f(1), f(2), \ldots, f(N)\}$. First of all, by (A), if x is a fixed point of f, then $x < N$ and so $x \in [1, M]$. Suppose that x is an element in a cycle arising from the iteration $(f^{(n)}(y))_{n\geq 0}$ for some $y \in \mathbb{N}$. If $x < N$, then $x \in [1, M]$ and we are done. So suppose $x \geq N$. By (4), there exists $n \in \mathbb{N}$ such that $f^{(n)}(x) < N$. Since x is in a cycle, after some iterations, it must come back to x. That is, there exists $k \in \mathbb{N}$ such that $f^{(k)}(f^{(n)}(x)) = x$. If $k = 1$ or $f^{(n+k-1)}(x) \leq N$, then $x = f(f^{(n+k-1)}(x)) \leq M$ and we are done. So suppose $k \geq 2$ and $f^{(n+k-1)}(x) > N$. Let ℓ be the smallest positive integer such that $f^{(n+k-\ell)}(x) < N$. Then, $\ell > 1$ and for each $1 \leq i < \ell$, $f^{(n+k-i)}(x) \geq N$. So

$$f^{(n+k-\ell+1)}(x) > f^{(n+k-\ell+2)}(x) > \cdots > f^{(n+k-1)}(x) > f^{(n+k)}(x) = x.$$

So $x < f^{(n+k-\ell+1)}(x) = f(f^{(n+k-\ell)}(x)) \leq M$. Therefore, (6) is verified and the proof is complete. □

Proof of Theorem 1.3. We prove this by induction on k. When $k = 1$, the result is obvious. Assume that $k \in \mathbb{N}$ and the result holds for k. Suppose that $f_1, f_2, \ldots, f_{k+1} : \mathbb{N} \to \mathbb{N}$ satisfy (A). Let $f = f_1 \circ f_2 \circ \cdots \circ f_{k+1}$ and $g = f_1 \circ f_2 \circ \cdots \circ f_k$. Then, there are $m_1, m_2 \in \mathbb{N}$ such that

$$g(x) < x \quad \text{for all } x \geq m_1, \text{ and } f_{k+1}(x) < x \quad \text{for all } x \geq m_2. \tag{7}$$

Let $m_3 = \max\{g(x) \mid 1 \leq x < m_1\}$ and $m = \max\{m_1, m_2, m_3\} + 1$. Let $x \geq m$. We will show that $f(x) < x$. If $f_{k+1}(x) \geq m_1$, then we obtain by (7) that

$$f(x) = g(f_{k+1}(x)) < f_{k+1}(x) < x.$$

On the other hand, if $f_{k+1}(x) < m_1$, then $f(x) = g(f_{k+1}(x)) \leq m_3 < m \leq x$. This completes the proof. □

Proof of Theorem 1.4. Grundman and Teeple [4, Theorem 1] show that if $x \geq b^{e+1}$, then $S_{e,b}(x) < x$. That is, $S_{e,b}$ has property (A) for every $e \geq 1$ and $b \geq 2$. By Theorem 1.3, $S_{\underline{e},\underline{b}}$ also satisfies (A). Then, by Theorem 1.2, we obtain that $S_{\underline{e},\underline{b}}$ satisfies (B), as desired. □

We remind the reader again that if we write a number without specifying a base, it is always written in base 10. We show some explicit calculations in the following examples.

Suppose $\underline{e} = (e_1, e_2, \ldots, e_k)$, $\underline{b} = (b_1, b_2, \ldots, b_k)$, and $f = S_{\underline{e},\underline{b}}$. By Theorem 1.4, f satisfies (A), that is, there exists $N \in \mathbb{N}$ such that $f(x) < x$ for all $x \geq N$. We can find such N by the argument given in the proof of Theorems 1.2, 1.3, and 1.4.

Example 2.1. *Consider $f = S_{4,6} \circ S_{2,5} \circ S_{3,4} \circ S_{5,3}$, which is the last line of Table 1. By the proof of Theorem 1.4, we know that*

$$S_{4,6}(x) < x \quad \textit{for } x \geq 6^5 \quad \textit{and} \quad S_{2,5}(x) < x \quad \textit{for } x \geq 5^3.$$

In the proof of Theorem 1.3, we let $m_1 = 6^5$, $m_2 = 5^3$, $m_3 = \max\{S_{4,6}(x) \mid 1 \leq x < 6^5\} = 5^5$, *and* $m = \max\{m_1, m_2, m_3\} + 1 = 6^5 + 1$. *Then*

$$(S_{4,6} \circ S_{2,5})(x) < x \quad \textit{for } x \geq 6^5 + 1.$$

Again by the proof of Theorem 1.4, we have $S_{3,4}(x) < x$ *for* $x \geq 4^4$. *By the proof of Theorem 1.3, we let* $m_1 = 6^5 + 1$, $m_2 = 4^4$,

$$\begin{aligned} m_3 &= \max\{(S_{4,6} \circ S_{2,5})(x) \mid 1 \leq x < 6^5 + 1\} \\ &\leq \max\{S_{4,6}(x) \mid 1 \leq x \leq 96\} \\ &< S_{4,6}((255)_6) = 1266, \end{aligned}$$

and $m = \max\{m_1, m_2, m_3\} + 1 = 6^5 + 2$. *Then,* $(S_{4,6} \circ S_{2,5} \circ S_{3,4})(x) < x$ *for* $x \geq 6^5 + 2$. *Finally, we know that* $S_{5,3}(x) < x$ *for* $x \geq 3^6$, *so we let* $m_1 = 6^5 + 2$, $m_2 = 3^6$,

$$\begin{aligned} m_3 &= \max\{(S_{4,6} \circ S_{2,5} \circ S_{3,4})(x) \mid 1 \leq x < 6^5 + 2\} \\ &\leq \max\{(S_{4,6} \circ S_{2,5})(x) \mid 1 \leq x \leq 189\} \\ &\leq \max\{(S_{4,6}(x)) \mid 1 \leq x \leq 49\} \\ &\leq S_{4,6}((155)_6) = 1251, \end{aligned}$$

and $m = \max\{m_1, m_2, m_3\} + 1 = 6^5 + 3$. *Hence*

$$f(x) = (S_{4,6} \circ S_{2,5} \circ S_{3,4} \circ S_{5,3})(x) < x \quad \textit{for all } x \geq 6^5 + 3.$$

This shows an algorithm to obtain an N *satisfying the condition (A). This choice of* N *may not be optimal but if it is necessary, we can find the minimal* N *by checking if* $f(x) < x$ *for* $x = N-1, N-2, N-3, \ldots$ *and then we stop when we get the first* x *such that* $f(x) \geq x$. *Then, we use a computer to find all fixed points and cycles of* f *by checking the sequence* $(f^{(n)}(x))_{n \geq 0}$ *where* $x = 1, 2, 3, \ldots, N$.

We give two more examples to illustrate alternative calculations.

Example 2.2. *Let* $\underline{\boldsymbol{e}} = (3, 2)$, $\underline{\boldsymbol{b}} = (10, 10)$, *and* $f = S_{\underline{\boldsymbol{e}},\underline{\boldsymbol{b}}}$. *That is,* $f = S_{3,10} \circ S_{2,10}$. *Then, for each* $x \in \mathbb{N}$, *the sequence* $(f^{(n)}(x))_{n \geq 0}$ *contains either* 1 *or* 27. *Moreover,* 1 *is the only fixed point of* f *and if the sequence* $(f^{(n)}(x))_{n \geq 0}$ *does not contain* 1, *then it eventually becomes the cycle* $(27, 152)$.

Proof. We first show that

$$f(x) < x \quad \text{for all } x \geq 1467. \tag{8}$$

Let $x \in [1467, 9999] \cap \mathbb{N}$. Then, $x = (abcd)_{10}$ for some $a, b, c, d \in \{0, 1, 2, \ldots, 9\}$ and $a \neq 0$. Therefore, $f(x) = S_{3,10}(S_{2,10}(x)) = S_{3,10}(a^2+b^2+c^2+d^2)$. Since $a^2+b^2+c^2+d^2 \leq 4 \cdot 9^2 = 324$, we see that $f(x) \leq \max\{S_{3,10}(x) \mid 1 \leq x \leq 324\} = S_{3,10}(299) = 2^3 + 9^3 + 9^3 = 1466 < x$. Next suppose that $x \geq 10^4$ and write $x = (a_k a_{k-1} \ldots a_1)_{10}$ with $a_k \neq 0$. So $k \geq 5$ and $f(x) =$

$S_{3,10}(a_k^2 + a_{k-1}^2 + \cdots + a_1^2)$. We have $a_k^2 + a_{k-1}^2 + \cdots + a_1^2 \leq 9^2 + 9^2 + \cdots + 9^2 = 81k$ and it is easy to prove by induction on k that $81k < 10^{k-1}$ for $k \geq 5$. Therefore,

$$f(x) \leq \max\{S_{3,10}(x) \mid 1 \leq x < 10^{k-1}\} = S_{3,10}(\underbrace{99\ldots9}_{k-1 \text{ digits}}) = 9^3 + 9^3 + \cdots + 9^3 = 729(k-1).$$

It is also easy to prove by induction that $729(k-1) < 10^{k-1}$ for all $k \geq 5$. So we obtain $f(x) < 10^{k-1} \leq a_k 10^{k-1} \leq x$, as required. Hence (8) is verified. So we only need to check, for each $x \leq 1466$, whether the sequence $(f^{(n)}(x))_{n\geq 0}$ converges to a fixed point or becomes a cycle. This can be done using a computer. We find that for each positive integer $x \leq 1466$, the sequence $(f^{(n)}(x))_{n\geq 0}$ converges to 1 or eventually becomes the cycle $(27, 152)$. □

Example 2.3. *Let $f = S_{3,7} \circ S_{2,5}$. Then, for each $x \in \mathbb{N}$, the sequence $(f^{(n)}(x))_{n\geq 0}$ contains either* 1, 28 *or* 216. *Moreover,* 1 *and* 28 *are the only fixed points of f and if the sequence $(f^{(n)}(x))_{n\geq 0}$ does not contain* 1 *or* 28, *then it eventually enters into the cycle* $(216, 224)$.

Proof. In this example, the bases are different (one of them is 5 and the other is 7). So the calculation is slightly different from the previous example. We give two solutions to this problem. We first show that

$$f(x) < x \quad \text{for all } x \geq 7^4. \tag{9}$$

Let $x \in \mathbb{N}$ and $x \geq 7^4$. Since $x \geq 5^4$, we can write $x = (a_k a_{k-1} \ldots a_1)_5$ where $k \geq 5$, $a_k \neq 0$, and $a_i \in \{0, 1, \ldots, 4\}$ for every i. Then, $f(x) = S_{3,7}(S_{2,5}(x)) = S_{3,7}(a_k^2 + a_{k-1}^2 + \cdots + a_1^2)$. We have $a_k^2 + a_{k-1}^2 + \cdots + a_1^2 \leq 4^2 + 4^2 + \cdots + 4^2 = 16k$ and it is easy to prove by induction on k that $16k < 5^{k-1} < 7^{k-1}$ for $k \geq 5$. Now there are two ways we can proceed.

Method 1. Since $16k < 5^{k-1} \leq a_k 5^{k-1} \leq x$, we see that $f(x) \leq \max\{S_{3,7}(y) \mid 1 \leq y < x\}$. Let $\ell \in \mathbb{N}$ be such that $x = (a'_\ell a'_{\ell-1} \ldots a'_1)_7$, $a'_\ell, a'_{\ell-1}, \ldots, a'_1 \in \{0, 1, \ldots, 6\}$, and $a'_\ell \neq 0$. Since $x \geq 7^4$, $\ell \geq 5$ and $7^{\ell-1} \leq x < 7^\ell$. Therefore,

$$f(x) \leq \max\{S_{3,7}(y) \mid 1 \leq y < 7^\ell\} = S_{3,7}((\underbrace{66\ldots6}_{\ell \text{ digits}})_7) = 6^3 + 6^3 + \cdots + 6^3 = 216\ell.$$

Here we remind the reader again that 216ℓ is the product of the numbers 216 and ℓ where $216 = (216)_{10}$. It is also easy to prove by induction that $216\ell < 7^{\ell-1}$ for all $\ell \geq 5$. So we obtain $f(x) < 7^{\ell-1} \leq x$, as required.

Method 2. We know that $16k < 7^{k-1}$, and so

$$f(x) \leq \max\{S_{3,7}(y) \mid 1 \leq y < 7^{k-1}\} = S_{3,7}((\underbrace{66\ldots6}_{k-1 \text{ digits}})_7) = 216(k-1).$$

Since $5^{k-1} \leq a_k 5^{k-1} \leq x$, we obtain $k - 1 \leq \frac{\log x}{\log 5}$. Therefore,

$$f(x) \leq 216(k-1) \leq \frac{216 \log x}{\log 5} = \left(\frac{216}{\log 5}\right)\left(\frac{\log x}{x}\right) x \leq \left(\frac{216}{\log 5}\right)\left(\frac{\log 7^4}{7^4}\right) x < x,$$

where we have used the fact that $x \geq 7^4$ and that the function $y \to \frac{\log y}{y}$ is decreasing on $[3, \infty)$. Hence (9) is verified. Similar to Example 2.2, the rest can be verified using a computer. □

$\underline{e}$	$\underline{b}$	Fixed points of $S_{\underline{e},\underline{b}}$ or cycles in $(S^{(n)}_{\underline{e},\underline{b}}(x))_{n\geq 1}$
$(3,2)$	$(10,10)$	1, $(27,152)$
$(2,3)$	$(10,10)$	1, $(30,53)$
$(3,2)$	$(7,5)$	1, 28, $(216,244)$
$(3,4,2)$	$(5,6,4)$	1, 35, $(17,28)$
$(2,3,5)$	$(6,5,7)$	1, $(10,20,17)$, $(11,41)$
$(2,5,4)$	$(5,6,8)$	1, 16, 19, $(4,12,5,14)$, $(7,13,27,17)$
$(4,3,5,2)$	$(4,3,5,6)$	1, 2, $(98,32)$
$(2,3,5,4)$	$(8,7,5,3)$	1, 4, 75, 98
$(4,2,3,5)$	$(6,5,4,3)$	1, 641, $(257,625)$

Table 1. Fixed points of $S_{\underline{e},\underline{b}}$ or cycles in $(S^{(n)}_{\underline{e},\underline{b}}(x))_{n\geq 1}$

Comments: The origin of this problem is unclear but it appears in Guy's book [5, Chapter E34]. A list of fixed points and cycles of some $S_{\underline{e},\underline{b}}$ is given in Table 1. We also plan to put more data in the third author's ResearchGate account, so the interested reader can freely download it in the future.

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