The $3 \times 3 \times \ldots \times 3$ Points Problem solution

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Abstract: We offer the first proof of Ripà’s $3 \times 3 \times 3 \times 3$ Dots Problem, providing a general solution of the $3^k$ case ($3^k$ points arranged in a $3 \times 3 \times \ldots \times 3$ grid), for any $k \in \mathbb{N} - \{0\}$. We give also new bounds for the $n \times n \times n$ problem, improving many of the previous results.

Keywords: Graph theory, Topology, Combinatorics, Four-dimensional, Nine dots, Creative thinking, Segment, Connectivity, Outside the box, Upper bound, Lower bound, Point, Game.

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1 Introduction

As stated by the classic nine dots problem appeared in Samuel Loyd’s Cyclopaedia of Puzzles [2], we have to “[…] draw a continuous line through the center of all the eggs so as to mark them off in the fewest number of strokes” (see [1, 3]). However, this time we are considering $n^k$ points located in a $k$-dimensional space, for any $k \geq 1$.

Despite the fact that we already know the minimum number of straight lines connected to their endpoints to join $n^k$ other points, for each $n \in \mathbb{N} - \{0\}$ and $k \in \{1, 2\}$, there are still many unanswered questions to account for:

- How many straight lines connected at their endpoints we need to join $n \times n \times n$ points arranged in $n$ equidistant grids, formed by $n$ rows and $n$ columns each?
- How many straight lines connected at their endpoints we need to join $3^k := 3 \times 3 \times \ldots \times 3$ dots, arranged in the same way as above, in a $k$-dimensional space?
2 The new bounds

In [5], Ripà extended the trite $n \times n$ dots game to a three-dimensional space [7] providing non-trivial bounds for this problem.

For $n > 3$, the $n \times n \times n$ lower bound is given by [8] as:

$$h_l(n) = n^2 + \left\lfloor \frac{3n^2 - 4n + 2}{2(n-1)} \right\rfloor$$

and the upper bound

$$h_u(n) = \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor + n - 1$$

In this section, we aim to further reduce the aforementioned upper bound.

2.1 A new $n \times n \times n$ upper bound

A slightly improved upper bound (for many $n \geq 6$) has been proved by Ripà and Bencini in [6], switching between two different standard patterns:

$$h_u(n) = \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 2.$$  

Although it is a thinking outside the box approach, the new result only saves 0 to 2 lines (for any $n \geq 6$), and the current upper bounds are shown in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Best Upper Bound Currently Discovered</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>3</td>
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<td>4</td>
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<tr>
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<td>10</td>
<td>157</td>
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</table>

Table 1: $n \times n \times n$ points puzzle upper bounds, following the two basic paths described in [6]. The upper bounds for $n \in \{1, 2, 3\}$ are equal to the lower bounds, definitely solving the $n \times n \times n$ problem for these trivial cases.
**Nota Bene.** The upper bounds for \( n \geq 3 \) represent outside of the box solutions, including the covering paths shown in Figure 1.

![Figure 1. The outside the box solutions for \( n = 4 \) and \( n = 5 \) (23 and 37 lines respectively).](image)

### 2.2 The \( n \times n \times \ldots \times n \) current bounds

Let \( n \geq 6 \), we can improve the upper limit for the \( k \)-dimensions \( n \times n \times \ldots \times n \) dots problem \( (k \geq 4) \) by simply defining

\[
t := \left[ \frac{3}{2} \cdot n^2 \right] - \left[ \frac{n-1}{4} \right] + \left[ \frac{n+1}{4} \right] - \left[ \frac{n+2}{4} \right] + \left[ \frac{n}{4} \right] + n - 2.
\]

Therefore [5, 6], the current bounds are given by:

\[
\left\lfloor \frac{n^k + \left( \frac{k}{2} - 1 \right) \cdot n^2 + (3 - 2 \cdot k) \cdot n + 2 \cdot k - 4}{n - 1} \right\rfloor + 1
\]

\[
\leq h(n, k) \leq (t + 1) \cdot n^{k-3} - 1.
\]
For any \( n \geq 3 \), it follows that:

\[
\left\lfloor \frac{n^k + \frac{k}{2} \cdot (n - 2)^2 - n^2 + 3 \cdot n - 4}{n - 1} \right\rfloor + 1 \leq h(n, k)
\]

\[
\leq \left( \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 1 \right) \cdot n^{k-3} - 1. \tag{5}
\]

### 3 The \( 3 \times 3 \times ... \times 3 \) problem finally solved

In this section, we present the solution for the \( 3 \times 3 \times 3 \) dots problem \( (k = 4) \) with the minimum number of straight lines connected at their endpoints. Later, we extend this result to the \( 3^k \) points problem for any \( k \in \mathbb{N} \).

#### 3.1 The perfect \( 3 \times 3 \times 3 \times 3 \) points problem solution with 41 lines

Given \( n = 3 \) and \( k = 4 \), from (5) it follows that \( h(3, 4) = 41 \), even though a solution involving 42 lines is known [4]. In order to solve the \( 3 \times 3 \times 3 \times 3 \) dots problem, we need to show how to join the 81 dots using only 41 lines.

Figures 2 to 5 show that a covering path consisting of 41 stright line segments exists.

![Figure 2. Solving the \( 3 \times 3 \times 3 \times 3 \) dots puzzle: lines 1 to 23.](image1)
Figure 3. Solving the $3 \times 3 \times 3 \times 3$ dots puzzle: moves 1 to 23 (a few lines are hidden).

Figure 4. The $3 \times 3 \times 3 \times 3$ solution with 41 moves (a few lines are hidden).
3.2 Solving the general $3^k$ problem

For any $k \geq 4$, from (4), we express the current upper bound for the general $3^k$ points problem as

$$h_u(3, k) = 42 \cdot 3^{k-4} - 1.$$  \hfill (6)

Some values of $h_u(3, k)$ are shown in Table 2.

<table>
<thead>
<tr>
<th>$k$</th>
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<th>$k$</th>
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</tbody>
</table>

Table 2: The current $3^k$ points puzzle upper bounds. If $k = 0$, we have a singularity and the whole space collapses in a single point, thus $h_u(3, k) = 1$.  

Figure 5. The final path with 41 lines
($k = 4, n = 3; h_l = 41 = h_u$).
The lower bound (4), for many $3 \times 3 \times \ldots \times 3$ configurations, can be also improved. From (5), we have that:

$$h_l(3, k) \geq \left[\frac{1}{2} \cdot \left(2^k + \frac{k}{2}\right)\right] - 1 \tag{7}$$

Let $n = 3$, for any $k \geq 2$, we can assume, without loss of generality, that we need (at least) $k - 2$ lines that join a single point each (as shown in Figure 5, for $k = 4$, by the segments #24 and #33).

In the $3 \times 3 \times 3$ case, we need one segment to reach the central grid, and this segment joins only one new dot (instead of two), while in the $3 \times 3 \times \ldots \times 3$ scenario, we have to reach a grid on the edge, in order to join its 9 dots plus all the dots on the opposite side of the hypercube.

Similarly, we have to repeat the same process for the other edges, spending one more line to join a single dot for any additional dimension beyond the second one.

Hence, in the worst-case scenario, we can assume that the new lower bound for the $3^k$ dots problem ($\forall k \geq 2$) becomes:

$$h_l(3, k) = \left[\frac{3^k-k-1}{2}\right] + k - 1 \tag{8}$$

Thus, we have got an improvement for any $h_l(n = 3, k \geq 5)$, such as $h_l(3, 5) = 123$ (instead of 122) and $h_l(3, 22) = 15690529814$ (instead of 15690529809).

4 Conclusion

Even if the most interesting open problem belonging to the family of the classic nine dots puzzle by Samuel Loyd [2], the $3 \times 3 \times 3$ case, has finally been solved, the research for the best solution to the $n \times n \times n$ dots problem (particularly for $n > 4$) and to the $3 \times 3 \times \ldots \times 3$ dots puzzle (for any $k \geq 4$), is not over yet (see [7, 8]).

We conjecture that, for any $k \geq 2$, $h(3, k) = h_l(3, k) = \left[\frac{3^k+k-3}{2}\right] (8)$, but this inference still needs a proof.

References


