Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 2, 68–75
DOI: 10.7546/nntdm.2019.25.2.68-75

The $3 \times 3 \times \ldots \times 3$ Points Problem solution

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Received: 6 August 2018 **Revised:** 16 February 2019 **Accepted:** 18 February 2019

Abstract: We offer the first proof of Ripà's $3 \times 3 \times 3 \times 3$ Dots Problem, providing a general solution of the 3^k case (3^k points arranged in a $3 \times 3 \times \ldots \times 3$ grid), for any $k \in \mathbb{N} - \{0\}$. We give also new bounds for the $n \times n \times n$ problem, improving many of the previous results.
Keywords: Graph theory, Topology, Combinatorics, Four-dimensional, Nine dots, Creative thinking, Segment, Connectivity, Outside the box, Upper bound, Lower bound, Point, Game.
2010 Mathematics Subject Classification: 91A43, 05C40.

1 Introduction

As stated by the classic *nine dots problem* appeared in Samuel Loyd's *Cyclopedia of Puzzles* [2], we have to "[…] draw a continuous line through the center of all the eggs so as to mark them off in the fewest number of strokes" (see [1, 3]). However, this time we are considering n^k points located in a k-dimensional space, for any $k \geq 1$.

Despite the fact that we already know the minimum number of straight lines connected to their endpoints to join n^k other points, for each $n \in \mathbb{N} - \{0\}$ and $k \in \{1, 2\}$, there are still many unanswered questions to account for:

- How many straight lines connected at their endpoints we need to join $n \times n \times n$ points arranged in n equidistant grids, formed by n rows and n columns each?
- How many straight lines connected at their endpoints we need to join

  $$3^k := 3 \times 3 \times \ldots \times 3$$

  dots, arranged in the same way as above, in a k-dimensional space?

2 The new bounds

In [5], Ripà extended the trite $n \times n$ dots game to a three-dimensional space [7] providing non-trivial bounds for this problem.

For $n > 3$, the $n \times n \times n$ lower bound is given by [8] as:

$$h_l(n) = n^2 + \left\lceil \frac{3 \cdot n^2 - 4 \cdot n + 2}{2 \cdot (n-1)} \right\rceil \tag{1}$$

and the upper bound

$$h_u(n) = \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor + n - 1 \tag{2}$$

In this section, we aim to further reduce the aforementioned upper bound.

2.1 A new $n \times n \times n$ upper bound

A slightly improved upper bound (for many $n \geq 6$) has been proved by Ripà and Bencini in [6], switching between two different standard patterns:

$$h_u(n) = \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 2. \tag{3}$$

Although it is a thinking outside the box approach, the new result only saves 0 to 2 lines (for any $n \geq 6$), and the current upper bounds are shown in Table 1.

n	**Best Upper Bound Currently Discovered**
1	**1**
2	**7**
3	**14**
4	23
5	37
6	57
7	78
8	103
9	128
10	157

n	**Best Upper Bound Currently Discovered**
11	190
12	227
13	264
14	305
15	350
16	399
17	448
18	501
19	558
20	619

n	**Best Upper Bound Currently Discovered**
21	680
22	745
23	814
24	887
25	960
26	1037
27	1118
28	1203
29	1288
30	1377

Table 1: $n \times n \times n$ points puzzle upper bounds, following the *two basic paths* described in [6]. The upper bounds for $n \in \{1, 2, 3\}$ are equal to the lower bounds, definitely solving the $n \times n \times n$ problem for these trivial cases.

Nota Bene. The upper bounds for $n \geq 3$ represent outside of the box solutions, including the covering paths shown in Figure 1.

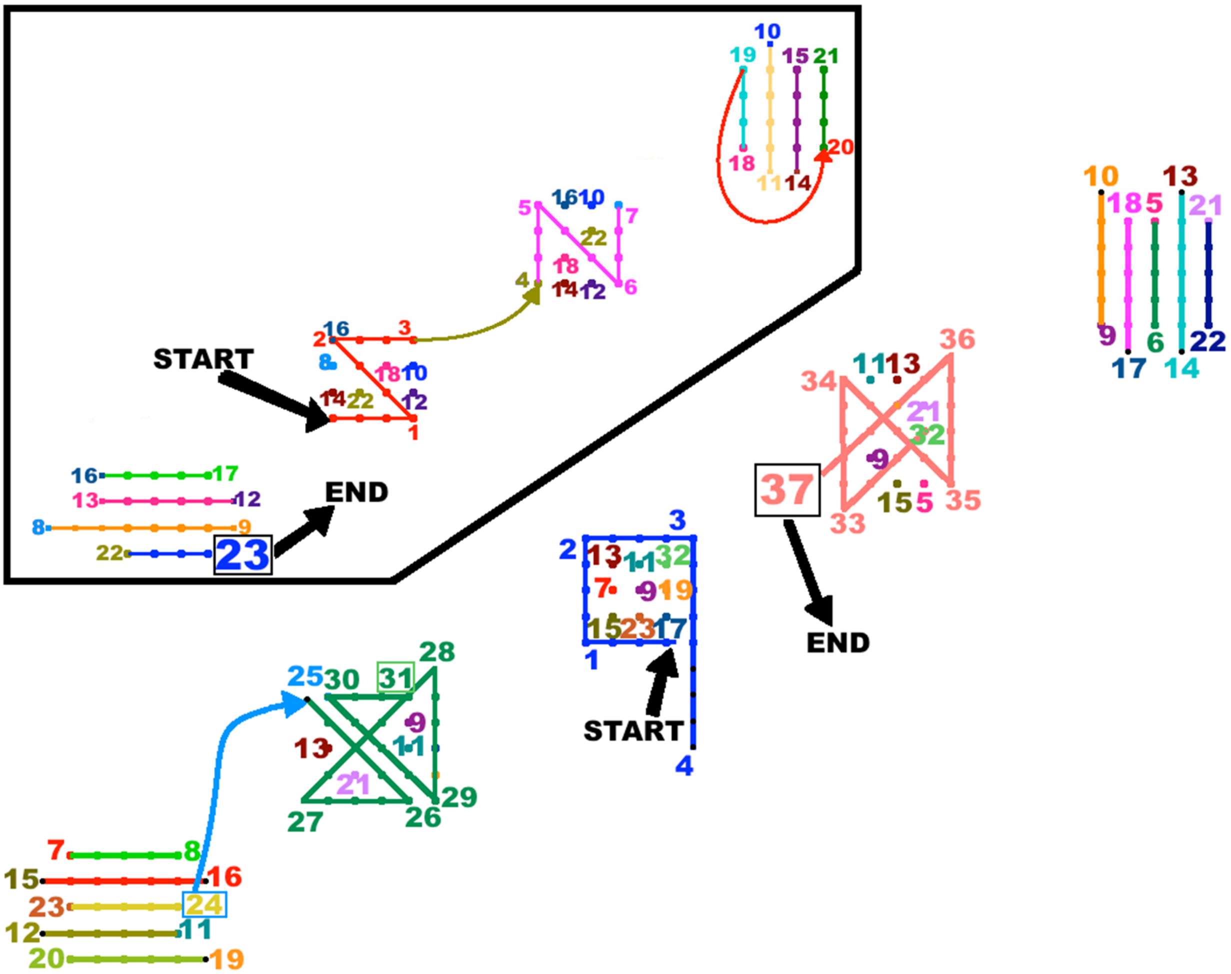

Figure 1. The outside the box solutions for $n = 4$ and $n = 5$
(23 and 37 lines respectively).

2.2 The $n \times n \times \ldots \times n$ current bounds

Let $n \geq 6$, we can improve the upper limit for the k-dimensions $n \times n \times \ldots \times n$ dots problem ($k \geq 4$) by simply defining

$$t := \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 2.$$

Therefore [5, 6], the current bounds are given by:

$$\left\lceil \frac{n^k + \left(\frac{k}{2} - 1\right) \cdot n^2 + (3 - 2 \cdot k) \cdot n + 2 \cdot k - 4}{n - 1} \right\rceil + 1$$

$$\leq h(n, k) \leq (t + 1) \cdot n^{k-3} - 1. \quad (4)$$

For any $n \geq 3$, it follows that:

$$\left\lceil \frac{n^k + \frac{k}{2} \cdot (n-2)^2 - n^2 + 3 \cdot n - 4}{n-1} \right\rceil + 1 \leq h(n,k)$$

$$\leq \left(\left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor - \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + n - 1 \right) \cdot n^{k-3} - 1. \quad (5)$$

3 The $3 \times 3 \times \ldots \times 3$ problem finally solved

In this section, we present the solution for the $3 \times 3 \times 3 \times 3$ dots problem $(k = 4)$ with the minimum number of straight lines connected at their endpoints. Later, we extend this result to the 3^k points problem for any $k \in \mathbb{N}$.

3.1 The perfect $3 \times 3 \times 3 \times 3$ points problem solution with 41 lines

Given $n = 3$ and $k = 4$, from (5) it follows that $h_l(3,4) = 41$, even though a solution involving 42 lines is known [4]. In order to solve the $3 \times 3 \times 3 \times 3$ dots problem, we need to show how to join the 81 dots using only 41 lines.

Figures 2 to 5 show that a covering path consisting of 41 stright line segments exists.

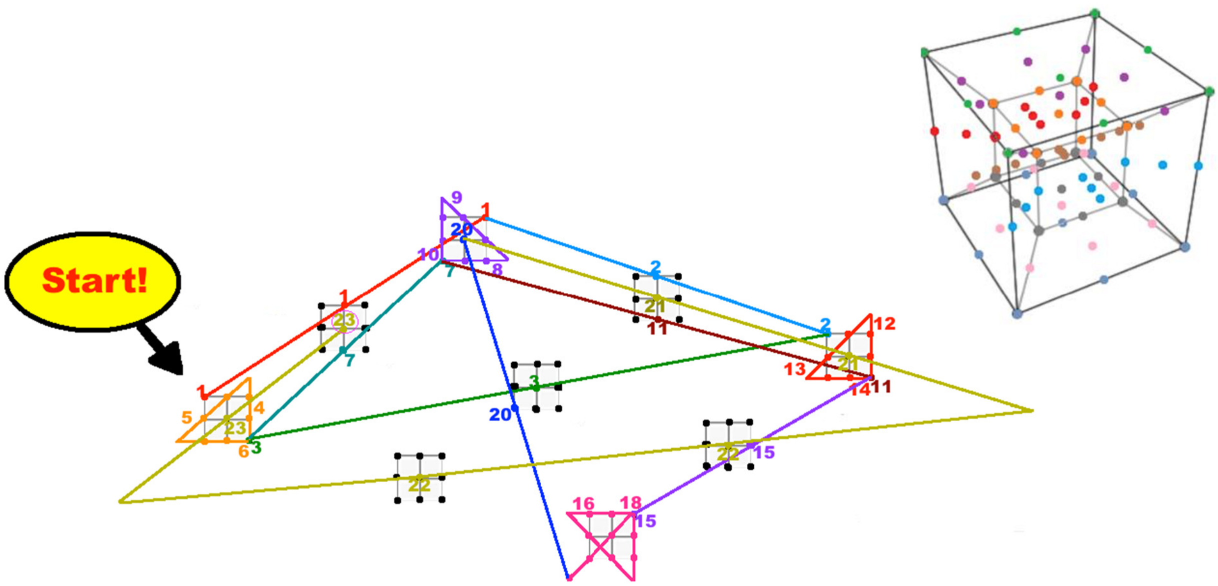

Figure 2. Solving the $3 \times 3 \times 3 \times 3$ dots puzzle: lines 1 to 23.

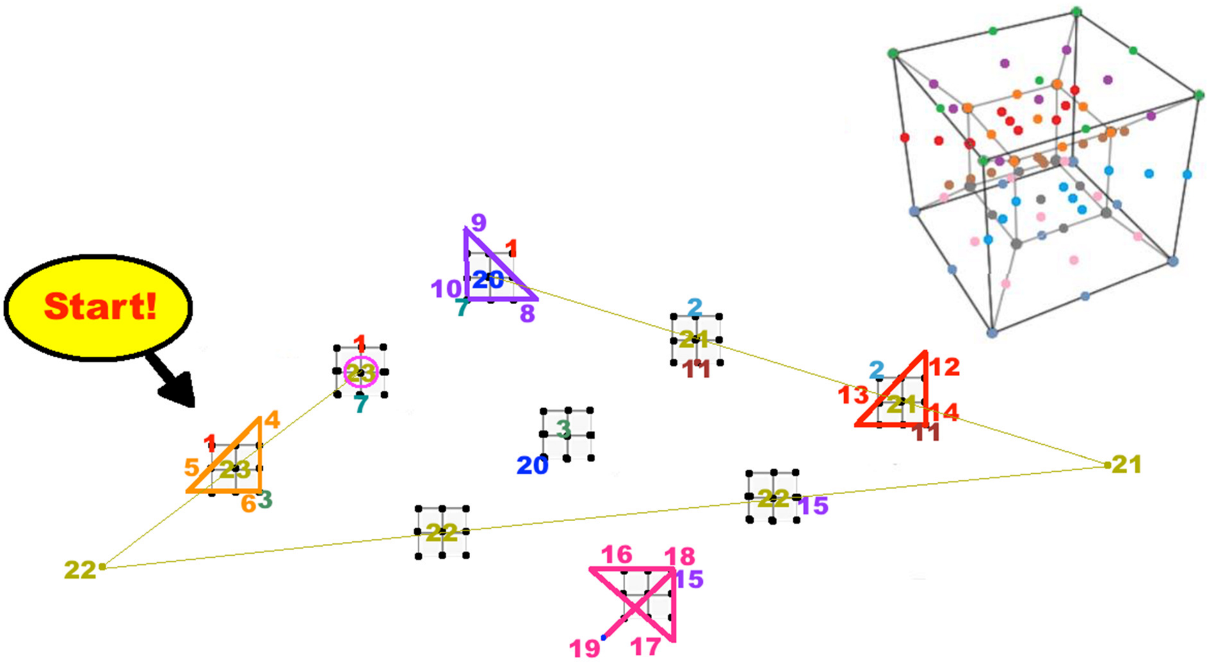

Figure 3. Solving the 3 × 3 × 3 × 3 dots puzzle:
moves 1 to 23 (a few lines are hidden).

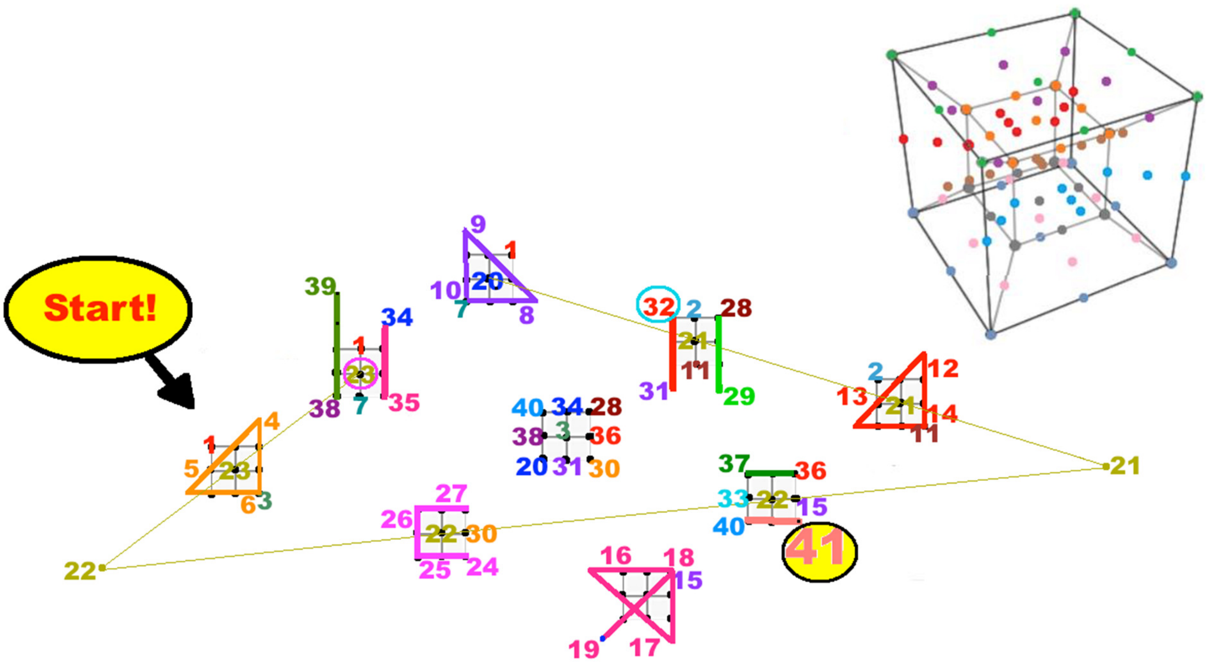

Figure 4. The 3 × 3 × 3 × 3 solution with 41 moves
(a few lines are hidden).

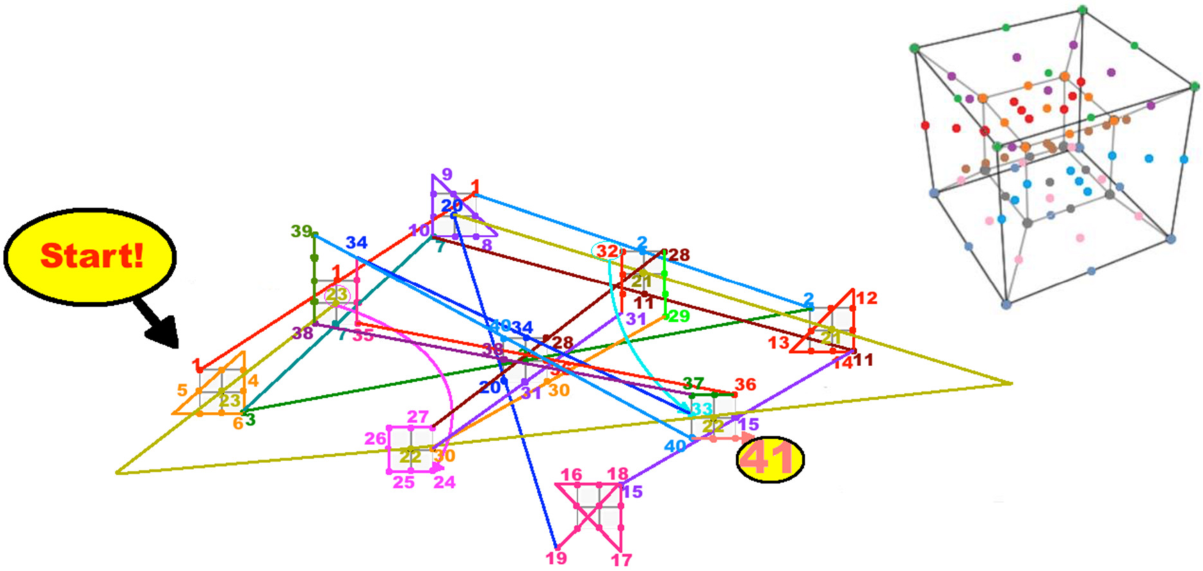

Figure 5. The final path with 41 lines
($k = 4, n = 3; h_l = 41 = h_u$).

3.2 Solving the general 3^k problem

For any $k \geq 4$, from (4), we express the current upper bound for the general 3^k points problem as

$$h_u(3, k) = 42 \cdot 3^{k-4} - 1. \tag{6}$$

Some values of $h_u(3, k)$ are shown in Table 2.

k	Best Upper Bound Currently Discovered	*k*	Best Upper Bound Currently Discovered
0	**1**	10	30617
1	**1**	11	91853
2	**4**	12	275561
3	**14**	13	826685
4	**41**	14	2480057
5	125	15	7440173
6	377	16	22320521
7	1133	17	66961565
8	3401	18	200884697
9	10205	19	602654093

Table 2: The current 3^k points puzzle upper bounds. If $k = 0$, we have a singularity and the whole space collapses in a single point, thus $h_u(3, k) = 1$.

The lower bound (4), for many 3 × 3 × ... × 3 configurations, can be also improved. From (5), we have that:

$$h_l(3,k) \geq \left\lceil \frac{1}{2} \cdot \left(3^k + \frac{k}{2}\right) \right\rceil - 1 \quad (7)$$

Let $n = 3$, for any $k \geq 2$, we can assume, without loss of generality, that we need (at least) $k - 2$ lines that join a single point each (as shown in Figure 5, for $k = 4$, by the segments #24 and #33).

In the 3 × 3 × 3 case, we need one segment to reach the central grid, and this segment joins only one new dot (instead of two), while in the 3 × 3 × ... × 3 scenario, we have to reach a grid on the edge, in order to join its 9 dots plus all the dots on the opposite side of the hypercube.

Similarly, we have to repeat the same process for the other edges, spending one more line to join a single dot for any additional dimension beyond the second one.

Hence, in the *worst-case scenario*, we can assume that the new lower bound for the 3^k dots problem ($\forall k \geq 2$) becomes:

$$h_l(3,k) = \left\lceil \frac{3^k - k - 1}{2} \right\rceil + k - 1 \quad (8)$$

Thus, we have got an improvement for any $h_l(n = 3, k \geq 5)$, such as $h_l(3,5) = 123$ (instead of 122) and $h_l(3,22) = 15690529814$ (instead of 15690529809).

4 Conclusion

Even if the most interesting open problem belonging to the family of the classic *nine dots puzzle* by Samuel Loyd [2], the 3 × 3 × 3 × 3 case, has finally been solved, the research for the best solution to the *n* × *n* × *n* dots problem (particularly for $n > 4$) and to the 3 × 3 × ... × 3 dots puzzle (for any $k \geq 4$), is not over yet (see [7, 8]).

We conjecture that, for any $k \geq 2$, $h(3,k) = h_l(3,k) = \left\lceil \frac{3^k + k - 3}{2} \right\rceil$ (8), but this inference still needs a proof.

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