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Diophantine equations related to reciprocals of linear recurrence sequences

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Abstract: In this article we consider the equation

$$\sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(P_2, Q_2)}{y^{k+1}},$$

in integers (x, y), where $U_n(P, Q)$ is a Lucas sequence defined by $U_0 = 0, U_1 = 1, U_n = PU_{n-1} - QU_{n-2}$ for $n > 1$. We also deal with a similar equation related to the generalized Tribonacci sequence.

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1 Introduction

Let P and Q be non-zero integers. The Lucas sequence $U_n(P, Q)$ is defined by $U_0 = 0, U_1 = 1, U_n = PU_{n-1} - QU_{n-2}$ for $n > 1$.

Stancliff [7] noted an interesting property of the Fibonacci sequence $F_n = U_n(1, -1)$, namely

$$\frac{1}{F_{11}} = \frac{1}{89} = 0.0112358\ldots = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

De Weger [10] determined all $x \geq 2$ in case of $(P, Q) = (1, -1)$ of the equation

$$\frac{1}{U_n(P,Q)} = \sum_{k=0}^{\infty} \frac{U_k(P,Q)}{x^{k+1}}.$$

The solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=0}^{\infty} \frac{F_k}{2^{k+1}}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=0}^{\infty} \frac{F_k}{3^{k+1}},$$
$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=0}^{\infty} \frac{F_k}{8^{k+1}}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

Tengely [9] provided methods to determine similar identities in case of Lucas sequences. As an example he proved that

$$\frac{1}{U_{10}} = \frac{1}{416020} = \sum_{k=0}^{\infty} \frac{U_k}{647^{k+1}},$$

where $U_0 = 0, U_1 = 1$ and $U_n = 4U_{n-1} + U_{n-2}, n \geq 2$.

Hashim and Tengely [3] obtained results related to the equation

$$\frac{1}{U_n(P_2,Q_2)} = \sum_{k=0}^{\infty} \frac{U_k(P_1,Q_1)}{x^{k+1}},$$

for certain pairs $(P_1, Q_1) \neq (P_2, Q_2)$.

There are many other nice results in the literature dealing with Diophantine equations related to base b representations and binary linear recurrence sequences. Bravo and Luca [1] completely solved the equation $F_m + F_n = 2^a$. Chim and Ziegler [2] generalized their result, they solved the equation $F_{n_1} + F_{n_2} = 2^{m_1} + 2^{m_2} + 2^{m_3}$ in non-negative integers $(n_1, n_2, m_1, m_2, m_3)$. Luca [5] proved that 55 is the largest Fibonacci number whose decimal expansion uses only one distinct digit.

In this article we study the integral solutions (x, y) of the equation

$$\sum_{k=0}^{\infty} \frac{U_k(P_1,Q_1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(P_2,Q_2)}{y^{k+1}}. \tag{1}$$

Using elementary number theory we have the following results. For a given polynomial $f(x)$ over the integers let $m(f) = \max\{|x| : f(x) = 0\}$.

Theorem 1. *Let P_1, Q_1, P_2, Q_2 be non-zero integers such that $(P_1, Q_1) \neq (P_2, Q_2)$. If $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = d_1 d_2 \neq 0$ and $d_1 - d_2 \equiv -2P_1 \pmod 4, d_1 + d_2 \equiv -2P_2 \pmod 4$, then the positive integral solutions x, y of*

$$\sum_{k=0}^{\infty} \frac{U_k(P_1,Q_1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(P_2,Q_2)}{y^{k+1}}$$

satisfy

$$x = \frac{d_1 - d_2 + 2P_1}{4} > m(x^2 - P_1 x + Q_1), \quad y = \frac{d_1 + d_2 + 2P_2}{4} > m(x^2 - P_2 x + Q_2).$$

If $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = 0$ *and* $P_1 \equiv P_2 \pmod 2$, *then the positive integral solutions* x, y *of*

$$\sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(P_2, Q_2)}{y^{k+1}}$$

satisfy

$$x > m(x^2 - P_1 x + Q_1), \quad y = \pm x + \frac{P_2 \mp P_1}{2} > m(x^2 - P_2 x + Q_2),$$

where $Q_2 = Q_1 + \frac{P_2^2 - P_1^2}{4}$.

Consider the equation

$$\sum_{k=0}^{\infty} \frac{T_k(a_2, a_1, a_0)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k(b_2, b_1, b_0)}{y^{k+1}}, \tag{2}$$

where T_n denotes the generalized Tribonacci sequence defined by $T_0(p,q,r) = T_1(p,q,r) = 0, T_2(p,q,r) = 1$ and

$$T_n(p,q,r) = pT_{n-1}(p,q,r) + qT_{n-2}(p,q,r) + rT_{n-3}(p,q,r) \quad \text{if } n \geq 3.$$

Theorem 2. *If* (x, y) *is an integral solution of* (2) *for given* $(a_2, a_1, a_0) \neq (b_2, b_1, b_0)$, *then either*

$$9\left(a_2^2 - b_2^2 + 3\,a_1 - 3\,b_1\right)y + 2\,a_2^3 - 3\,a_2^2 b_2 + b_2^3 + 9\,a_1 a_2 - 9\,a_1 b_2 + 27\,a_0 - 27\,b_0 = 0$$

or in case of $|y| > B$ *we have*

$$|3x - 3y - a_2 + b_2| < C,$$

where B, C *are constants depending only on* $a_i, b_i, i = 0, 1, 2$.

2 Auxiliary results

In the proofs we will use the following two results of Köhler [4].

Theorem A. *Let* A_0, A_1, a_0, a_1 *be arbitrary complex numbers. Define the sequence* $\{a_n\}$ *by the recursion* $a_{n+1} = A_0 a_n + A_1 a_{n-1}$. *Then the formula*

$$\sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}} = \frac{a_0 z - A_0 a_0 + a_1}{z^2 - A_0 z - A_1}$$

holds for all complex z *such that* $|z|$ *is larger than the absolute values of the zeros of* $z^2 - A_0 z - A_1$.

Theorem B. *Let arbitrary complex numbers* $A_0, A_1, \ldots, A_m, a_0, a_1, \ldots, a_m$ *be given. Define the sequence* $\{a_n\}$ *by the recursion*

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \ldots + A_m a_{n-m}$$

Then for all complex z *such that* $|z|$ *is larger than the absolute values of all zeros of* $q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \ldots - A_m$, *the formula*

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{z^k} = \frac{p(z)}{q(z)}$$

holds with $p(z) = a_0 z^m + b_1 z^{m-1} + \ldots + b_m$, *where* $b_k = a_k - \sum_{i=0}^{k-1} A_i a_{k-1-i}$ *for* $1 \leq k \leq m$.

3 Proofs of the results

Proof of Theorem 1. By applying Theorem A to equation (1) we get that

$$\frac{1}{x^2-P_1x+Q_1}=\frac{1}{y^2-P_2y+Q_2}.$$

By algebraic manipulations we obtain the equation

$$(2y+2x-P_1-P_2)(2y-2x+P_1-P_2)=P_2^2-P_1^2+4(Q_1-Q_2). \tag{3}$$

If $P_2^2-P_1^2+4(Q_1-Q_2)\neq 0$, then for all $d_1|P_2^2-P_1^2+4(Q_1-Q_2)$ we consider the following system of equations

$$\begin{aligned} 2y+2x-P_1-P_2 &= d_1,\\ 2y-2x+P_1-P_2 &= d_2=\frac{P_2^2-P_1^2+4(Q_1-Q_2)}{d_1}. \end{aligned}$$

We obtain integral solutions if $d_1-d_2\equiv -2P_1 \pmod 4$ and $d_1+d_2\equiv -2P_2 \pmod 4$. In this case the solutions are given by

$$x=\frac{d_1-d_2+2P_1}{4} \text{ and } y=\frac{d_1+d_2+2P_2}{4}.$$

If $P_2^2-P_1^2+4(Q_1-Q_2)=0$, then

$$Q_2=Q_1+\frac{P_2^2-P_1^2}{4},$$

and Q_2 is an integer if $P_1\equiv P_2 \pmod 2$. There are two possible cases, either $2y+2x-P_1-P_2=0$ or $2y-2x+P_1-P_2=0$. In the former case we have $y=-x+\frac{P_1+P_2}{2}$ and in the latter one we get $y=x+\frac{P_2-P_1}{2}$. □

Proof of Theorem 2. Using Köhler's results given in Theorem B, equation (2) yields that

$$\frac{1}{x^3-a_2x^2-a_1x-a_0}=\frac{1}{y^3-b_2y^2-b_1y-b_0}.$$

Hence

$$H(x,y)=x^3-a_2x^2-a_1x-a_0-y^3+b_2y^2+b_1y+b_0=0.$$

This equation satisfies Runge's condition [6] therefore in case when $H(x,y)$ is irreducible over the rationals there exist only finitely many integral solutions (x,y). We obtain that

$$0=27H(x,y)=(3x-3y-a_2+b_2)G(x,y)-I(y),$$

where $G(x,y)=9\,x^2+9\,xy+9\,y^2-3\,(2\,a_2+b_2)x-3\,(a_2+2\,b_2)y-2\,a_2^2+a_2b_2+b_2^2-9\,a_1$ and $I(y)=9\,(a_2^2-b_2^2+3\,a_1-3\,b_1)y+2\,a_2^3-3\,a_2^2b_2+b_2^3+9\,a_1a_2-9\,a_1b_2+27\,a_0-27\,b_0$. Here we may have that $I(y)$ is identically equal to 0, then $H(x,y)$ is reducible over the rationals. In this case solutions can be obtained from $3x-3y-a_2+b_2=0$ or $G(x,y)=0$. Now assume that $I(y)\neq 0$. We obtain that

$$3x-3y-a_2+b_2=\frac{I(y)}{G(x,y)}=\frac{4I(y)}{27\,y^2-18\,b_2y-12\,a_2^2+3\,b_2^2-36\,a_1+(6x+3y-(2a_2+b_2))^2}$$

Here one can determine a bound B for $|y|$ such that if $|y| > B$, then

$$27\,y^2 - 18\,b_2 y - 12\,a_2^2 + 3\,b_2^2 - 36\,a_1 + (6x + 3y - (2a_2 + b_2))^2 > 26y^2.$$

Thus

$$|3x - 3y - a_2 + b_2| < \frac{4I(y)}{26y^2}.$$

Since $I(y)$ is linear we get that

$$\frac{4|I(y)|}{26y^2} < C$$

for some positive constant C and the statement follows. □

4 Applications of the results

Example 1. As an application of Theorem 1 consider the following example. Let $(P_1, Q_1) = (1, -1)$ and $(P_2, Q_2) = (18, 1)$. We have that $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = (18^2 - 1) + 4(-1-1) = 315$. We obtain a system of equations given by

$$\begin{aligned} 2y + 2x - 19 &= d_1, \\ 2y - 2x - 17 &= \frac{315}{d_1}, \end{aligned}$$

where

$$d_1 \in \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 15, \pm 21, \pm 35, \pm 45, \pm 63, \pm 105, \pm 315\}.$$

The solutions are as follows

$$\begin{aligned} (x,y) \in \{ & (79,-70), (26,-18), (15,-8), (10,-4), (7,-2), (2,0), (-1,0), (-6,-2), \\ & (-9,-4), (-14,-8), (-25,-18), (-78,-70), (-78,88), (-25,36), (-14,26), \\ & (-9,22), (-6,20), (-1,18), (2,18), (7,20), (10,22), (15,26), (26,36), (79,88)\}. \end{aligned}$$

Here we have $m(x^2 - x - 1) \approx 1.618$ so $x \geq 2$ and $m(x^2 - 18x + 1) \approx 17.944$ hence $y \geq 18$. Thus the solutions are as follows

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{U_k(1,-1)}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(18,1)}{18^{k+1}} &= 1, \\ \sum_{k=0}^{\infty} \frac{U_k(1,-1)}{7^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(18,1)}{20^{k+1}} &= \frac{1}{41}, \\ \sum_{k=0}^{\infty} \frac{U_k(1,-1)}{10^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(18,1)}{22^{k+1}} &= \frac{1}{89}, \\ \sum_{k=0}^{\infty} \frac{U_k(1,-1)}{15^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(18,1)}{26^{k+1}} &= \frac{1}{209}, \\ \sum_{k=0}^{\infty} \frac{U_k(1,-1)}{26^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(18,1)}{36^{k+1}} &= \frac{1}{649}, \\ \sum_{k=0}^{\infty} \frac{U_k(1,-1)}{79^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(18,1)}{88^{k+1}} &= \frac{1}{6161}. \end{aligned}$$

Example 2. As a next example consider the case with $(P_1, Q_1) = (1, -1)$ and $(P_2, Q_2) = (2t+1, t^2+t-1)$. We get that

$$\sum_{k=0}^{\infty} \frac{U_k(1,-1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(2t+1, t^2+t-1)}{(x+t)^{k+1}} = \frac{1}{x^2-x-1}$$

for $x \geq 2$.

Example 3. Consider the positive integral solutions x, y of the equation

$$\sum_{k=0}^{\infty} \frac{T_k(-1,7,3)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k(5,-5,-3)}{y^{k+1}}.$$

Theorem B implies that

$$\frac{1}{x^3+x^2-7x-3} = \frac{1}{y^3-5y^2+5y+3},$$

therefore we get

$$x^3+x^2-7x-3 = y^3-5y^2+5y+3.$$

Following the proof of Theorem 2 we have

$$H(x,y) = x^3+x^2-7x-y^3+5y^2-5y-6 = 0.$$

We determine $G(x,y)$ and $I(y)$, these are given by

$$G(x,y) = 9x^2+9xy+9y^2-9x-27y-45, \quad I(y) = 108y-108.$$

If $I(y) = 0$, then $y = 1$ and it follows that $x^3+x^2-7x-3 = 4$, that is $x = -1$. Hence we do not get positive integral solutions in this case. Assume that $I(y) \neq 0$. We get that

$$3x-3y+6 = \frac{4(108y-108)}{9x^2+9xy+9y^2-9x-27y-45}.$$

It can be written as

$$3x-3y+6 = \frac{4(108y-108)}{27y^2-90y-189+(6x+3y-3)^2}.$$

We have that $26y^2 < 27y^2-90y-189+(6x+3y-3)^2$ for positive integers if $y \geq 93$. It follows that $|3x-3y+6| < 1$ if $y \geq 93$. That is $3x-3y+6 = 0$, so we obtain that $I(y) = 0$, a contradiction. It remains to deal with the values of y for which $3 = m(x^3-5x^2+5x+3) \leq y \leq 93$. Using SageMath [8] we obtain that the only integral solutions in this range are given by $x = -3, y = 3$ and $x = -2, y = 4$, so we do not get positive integral solutions.

Example 4. As a second application of Theorem 2 let us consider the equation

$$\sum_{k=0}^{\infty} \frac{T_k(-4,-5,-6)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k(1,8,18)}{y^{k+1}}.$$

Here we obtain that

$$\begin{aligned} H(x,y) &= x^3 - y^3 + 4x^2 + y^2 + 5x + 8y + 24, \\ G(x,y) &= 9x^2 + 9xy + 9y^2 + 21x + 6y + 10, \\ I(y) &= -216y - 598. \end{aligned}$$

The equation $I(y) = 0$ does not have integral solutions. We obtain that $4G(x,y) > 26y^2$ if $y > 18$ and $|3x - 3y + 5| < 1$ if $y > 30$. Hence we have that if $y > 30$, then $3x - 3y + 5 = 0$, therefore $I(y) = 0$, a contradiction. It remains to deal with the cases $y \in [5, 6, \ldots, 30]$. It follows that the only positive solution is given by $(x,y) = (9,11)$, that is we have

$$\sum_{k=0}^{\infty} \frac{T_k(-4,-5,-6)}{9^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k(1,8,18)}{11^{k+1}} = \frac{1}{1104}.$$

Example 5. Finally let us describe an example with identically zero $I(y)$, in which case we obtain infinitely many solutions. Let $(a_2, a_1, a_0) = (1, 6, 5)$ and $(b_2, b_1, b_0) = (4, 1, 1)$. It follows that

$$\begin{aligned} H(x,y) &= x^3 - y^3 - x^2 + 4y^2 - 6x + y - 4, \\ G(x,y) &= 9x^2 + 9xy + 9y^2 - 18x - 27y - 36, \\ I(y) &= 0. \end{aligned}$$

We obtain that either $3x - 3y + 3 = 0$ or $G(x,y) = 0$. In the former case $y = x + 1$ and

$$\sum_{k=0}^{\infty} \frac{T_k(1,6,5)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k(4,1,1)}{(x+1)^{k+1}} = \frac{1}{x^3 - x^2 - 6x - 5}, \quad x \geq 4.$$

In the latter case we have that

$$0 = 12G(x,y) = 3(6x + 3y - 6)^2 + (9y - 12)^2 - 684.$$

We do not get new integral solutions since the equation $(9y-12)^2 + 3(6x+3y-6)^2 = 684$ has no solutions in $\mathbb{Z}$.

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