Diophantine equations related to reciprocals of linear recurrence sequences

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Abstract: In this article we consider the equation
\[
\sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^k} = \sum_{k=0}^{\infty} \frac{U_k(P_2, Q_2)}{y^k},
\]
in integers \((x, y)\), where \(U_n(P, Q)\) is a Lucas sequence defined by \(U_0 = 0, U_1 = 1, U_n = PU_{n-1} - QU_{n-2}\) for \(n > 1\). We also deal with a similar equation related to the generalized Tribonacci sequence.

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1 Introduction

Let \(P\) and \(Q\) be non-zero integers. The Lucas sequence \(U_n(P, Q)\) is defined by \(U_0 = 0, U_1 = 1, U_n = PU_{n-1} - QU_{n-2}\) for \(n > 1\).

Stancliff [7] noted an interesting property of the Fibonacci sequence \(F_n = U_n(1, -1)\), namely
\[
\frac{1}{F_{11}} = \frac{1}{89} = 0.0112358\ldots = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.
\]
De Weger \[10\] determined all \( x \geq 2 \) in case of \((P, Q) = (1, -1)\) of the equation

\[
\frac{1}{U_n(P, Q)} = \sum_{k=0}^{\infty} \frac{U_k(P, Q)}{x^{k+1}}.
\]

The solutions are as follows

\[
\frac{1}{F_1} = 1, \quad \frac{1}{F_2} = 1, \quad \frac{1}{F_5} = \frac{1}{5}, \quad \frac{1}{F_{10}} = \frac{1}{55}, \quad \frac{1}{F_{11}} = \frac{1}{89},
\]

Tengely \[9\] provided methods to determine similar identities in case of Lucas sequences. As an example he proved that

\[
\frac{1}{U_{10}} = \frac{1}{416020} = \sum_{k=0}^{\infty} \frac{U_k}{647^{k+1}},
\]

where \( U_0 = 0, U_1 = 1 \) and \( U_n = 4U_{n-1} + U_{n-2}, n \geq 2 \).

Hashim and Tengely \[3\] obtained results related to the equation

\[
\frac{1}{U_n(P_2, Q_2)} = \sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^{k+1}},
\]

for certain pairs \((P_1, Q_1) \neq (P_2, Q_2)\).

There are many other nice results in the literature dealing with Diophantine equations related to base \( b \) representations and binary linear recurrence sequences. Bravo and Luca \[1\] completely solved the equation \( F_m + F_n = 2^a \). Chim and Ziegler \[2\] generalized their result, they solved the equation \( F_{n_1} + F_{n_2} = 2^{m_1} + 2^{m_2} + 2^{m_3} \) in non-negative integers \((n_1, n_2, m_1, m_2, m_3)\). Luca \[5\] proved that 55 is the largest Fibonacci number whose decimal expansion uses only one distinct digit.

In this article we study the integral solutions \((x, y)\) of the equation

\[
\sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(P_2, Q_2)}{y^{k+1}}.
\]

(1)

Using elementary number theory we have the following results. For a given polynomial \( f(x) \) over the integers let \( m(f) = \text{max}\{|x| : f(x) = 0\}\).

**Theorem 1.** Let \( P_1, Q_1, P_2, Q_2 \) be non-zero integers such that \((P_1, Q_1) \neq (P_2, Q_2)\). If \((P_2^2 - P_1^2) + 4(Q_1 - Q_2) = d_1d_2 \neq 0 \) and \( d_1 - d_2 \equiv -2P_1 \pmod{4}, d_1 + d_2 \equiv -2P_2 \pmod{4} \), then the positive integral solutions \( x, y \) of

\[
\sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{U_k(P_2, Q_2)}{y^{k+1}}
\]

satisfy

\[
x = \frac{d_1 - d_2 + 2P_1}{4} > m(x^2 - P_1x + Q_1), \quad y = \frac{d_1 + d_2 + 2P_2}{4} > m(x^2 - P_2x + Q_2).
\]
If \((P_2^2 - P_1^2) + 4(Q_1 - Q_2) = 0\) and \(P_1 \equiv P_2 \pmod{2}\), then the positive integral solutions \(x, y\) of
\[
\sum_{k=0}^{\infty} U_k(P_1, Q_1) x^{k+1} = \sum_{k=0}^{\infty} U_k(P_2, Q_2) y^{k+1}
\]
satisfy
\[
x > m(x^2 - P_1x + Q_1), \quad y = \pm x + \frac{P_2 + P_1}{2} > m(x^2 - P_2x + Q_2),
\]
where \(Q_2 = Q_1 + \frac{P_2^2 - P_1^2}{4} \).

Consider the equation
\[
\sum_{k=0}^{\infty} T_k(a_2, a_1, a_0) x^{k+1} = \sum_{k=0}^{\infty} T_k(b_2, b_1, b_0) y^{k+1},
\]
where \(T_n\) denotes the generalized Tribonacci sequence defined by \(T_0(p, q, r) = T_1(p, q, r) = 0, T_2(p, q, r) = 1\) and
\[
T_n(p, q, r) = pT_{n-1}(p, q, r) + qT_{n-2}(p, q, r) + rT_{n-3}(p, q, r) \quad \text{if } n \geq 3.
\]

**Theorem 2.** If \((x, y)\) is an integral solution of (2) for given \((a_2, a_1, a_0) \neq (b_2, b_1, b_0)\), then either
\[
9(a_2^2 - b_2^2 + 3a_1 - 3b_1)y + 2a_2^2 - 3a_2b_2 + b_2^2 + 9a_1a_2 - 9a_1b_2 + 27a_0 - 27b_0 = 0
\]
or in case of \(|y| > B\) we have
\[
|3x - 3y - a_2 + b_2| < C,
\]
where \(B, C\) are constants depending only on \(a_i, b_i, i = 0, 1, 2\).

## 2 Auxiliary results

In the proofs we will use the following two results of Köhler [4].

**Theorem A.** Let \(A_0, A_1, a_0, a_1\) be arbitrary complex numbers. Define the sequence \(\{a_n\}\) by the recursion \(a_{n+1} = A_0a_n + A_1a_{n-1}\). Then the formula
\[
\sum_{k=0}^{\infty} z^{k+1} = \frac{a_0z - A_0a_0 + a_1}{z^2 - A_0z - A_1}
\]
holds for all complex \(z\) such that \(|z|\) is larger than the absolute values of the zeros of \(z^2 - A_0z - A_1\).

**Theorem B.** Let arbitrary complex numbers \(A_0, A_1, \ldots, A_m, a_0, a_1, \ldots, a_m\) be given. Define the sequence \(\{a_n\}\) by the recursion
\[
a_{n+1} = A_0a_n + A_1a_{n-1} + \ldots + A_m a_{n-m}
\]
Then for all complex \(z\) such that \(|z|\) is larger than the absolute values of all zeros of \(q(z) = z^{m+1} - A_0z^m - A_1z^{m-1} - \ldots - A_m\), the formula
\[
\sum_{k=1}^{\infty} \frac{a_k}{z^k} = \frac{p(z)}{q(z)}
\]
holds with \(p(z) = a_0z^m + b_1z^{m-1} + \ldots + b_m\), where \(b_k = a_k - \sum_{i=0}^{k-1} A_ia_{k-1-i}\) for \(1 \leq k \leq m\).
3 Proofs of the results

Proof of Theorem 1. By applying Theorem A to equation (1) we get that

\[
\frac{1}{x^2 - P_1 x + Q_1} = \frac{1}{y^2 - P_2 y + Q_2}.
\]

By algebraic manipulations we obtain the equation

\[
(2y + 2x - P_1 - P_2)(2y - 2x + P_1 - P_2) = P_2^2 - P_1^2 + 4(Q_1 - Q_2).
\]

(3)

If \( P_2^2 - P_1^2 + 4(Q_1 - Q_2) \neq 0 \), then for all \( d_1 \) and \( d_2 \) we consider the following system of equations

\[
\begin{align*}
2y + 2x - P_1 - P_2 &= d_1, \\
2y - 2x + P_1 - P_2 &= d_2 = \frac{P_2^2 - P_1^2 + 4(Q_1 - Q_2)}{d_1}.
\end{align*}
\]

We obtain integral solutions if \( d_1 - d_2 \equiv -2P_1 \pmod{4} \) and \( d_1 + d_2 \equiv -2P_2 \pmod{4} \). In this case the solutions are given by

\[
x = \frac{d_1 - d_2 + 2P_1}{4} \quad \text{and} \quad y = \frac{d_1 + d_2 + 2P_2}{4}.
\]

If \( P_2^2 - P_1^2 + 4(Q_1 - Q_2) = 0 \), then

\[
Q_2 = Q_1 + \frac{P_2^2 - P_1^2}{4},
\]

and \( Q_2 \) is an integer if \( P_1 \equiv P_2 \pmod{2} \). There are two possible cases, either \( 2y + 2x - P_1 - P_2 = 0 \) or \( 2y - 2x + P_1 - P_2 = 0 \). In the former case we have \( y = -x + \frac{P_1 + P_2}{2} \) and in the latter one we get \( y = x + \frac{P_2 - P_1}{2} \).

Proof of Theorem 2. Using Köhler’s results given in Theorem B, equation (2) yields that

\[
\frac{1}{x^3 - a_2 x^2 - a_1 x - a_0} = \frac{1}{y^3 - b_2 y^2 - b_1 y - b_0}.
\]

Hence

\[
H(x, y) = x^3 - a_2 x^2 - a_1 x - a_0 - y^3 + b_2 y^2 + b_1 y + b_0 = 0.
\]

This equation satisfies Runge’s condition [6] therefore in case when \( H(x, y) \) is irreducible over the rationals there exist only finitely many integral solutions \((x, y)\). We obtain that

\[
0 = 27H(x, y) = (3x - 3y - a_2 + b_2)G(x, y) - I(y),
\]

where \( G(x, y) = 9x^2 + 9xy + 9y^2 - 3(2a_2 + b_2)x - 3(a_2 + 2b_2)y - 2a_2^2 + a_2 b_2 + b_2^2 - 9a_1 \) and \( I(y) = 9(a_2^2 - b_2^2 + 3a_1 - 3b_1)y + 2a_2^3 - 3a_2^2 b_2 + b_2^3 + 9a_1 a_2 - 9a_1 b_2 + 27a_0 - 27b_0 \). Here we may have that \( I(y) \) is identically equal to 0, then \( H(x, y) \) is reducible over the rationals. In this case solutions can be obtained from \( 3x - 3y - a_2 + b_2 = 0 \) or \( G(x, y) = 0 \). Now assume that \( I(y) \neq 0 \). We obtain that

\[
3x - 3y - a_2 + b_2 = \frac{I(y)}{G(x, y)} = \frac{4I(y)}{27y^2 - 18b_2 y - 12a_2^2 + 3b_2^2 - 36a_1 + (6x + 3y - (2a_2 + b_2))^2}
\]

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Here one can determine a bound $B$ for $|y|$ such that if $|y| > B$, then
\[ 27y^2 - 18b_2y - 12a_2^2 + 3b_2^2 - 36a_1 + (6x + 3y - (2a_2 + b_2))^2 > 26y^2. \]
Thus
\[ |3x - 3y - a_2 + b_2| < \frac{4I(y)}{26y^2}. \]
Since $I(y)$ is linear we get that
\[ \frac{4|I(y)|}{26y^2} < C \]
for some positive constant $C$ and the statement follows.

4 Applications of the results

Example 1. As an application of Theorem 1 consider the following example. Let $(P_1, Q_1) = (1, -1)$ and $(P_2, Q_2) = (18, 1)$. We have that $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = (18^2 - 1) + 4(-1 - 1) = 315$. We obtain a system of equations given by
\[
\begin{align*}
2y + 2x - 19 &= d_1, \\
2y - 2x - 17 &= \frac{315}{d_1},
\end{align*}
\]
where
\[ d_1 \in \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 15, \pm 21, \pm 35, \pm 45, \pm 63, \pm 105, \pm 315\}. \]
The solutions are as follows
\[
(x, y) \in \{(79, -70), (26, -18), (15, -8), (10, -4), (7, -2), (2, 0), (-1, 0), (-6, -2), (-9, -4), (-14, -8), (-25, -18), (-78, -70), (-78, 88), (-25, 36), (-14, 26), (-9, 22), (-6, 20), (-1, 18), (2, 18), (7, 20), (10, 22), (15, 26), (26, 36), (79, 88)\}.
\]
Here we have $m(x^2 - x - 1) \approx 1.618$ so $x \geq 2$ and $m(x^2 - 18x + 1) \approx 17.944$ hence $y \geq 18$. Thus the solutions are as follows
\[
\begin{align*}
\sum_{k=0}^{\infty} \frac{U_k(1, -1)}{2^{k+1}} &= \sum_{k=0}^{\infty} \frac{U_k(18, 1)}{18^{k+1}} = 1, \\
\sum_{k=0}^{\infty} \frac{U_k(1, -1)}{7^{k+1}} &= \sum_{k=0}^{\infty} \frac{U_k(18, 1)}{20^{k+1}} = \frac{1}{41}, \\
\sum_{k=0}^{\infty} \frac{U_k(1, -1)}{10^{k+1}} &= \sum_{k=0}^{\infty} \frac{U_k(18, 1)}{22^{k+1}} = \frac{1}{89}, \\
\sum_{k=0}^{\infty} \frac{U_k(1, -1)}{15^{k+1}} &= \sum_{k=0}^{\infty} \frac{U_k(18, 1)}{26^{k+1}} = \frac{1}{209}, \\
\sum_{k=0}^{\infty} \frac{U_k(1, -1)}{26^{k+1}} &= \sum_{k=0}^{\infty} \frac{U_k(18, 1)}{36^{k+1}} = \frac{1}{649}, \\
\sum_{k=0}^{\infty} \frac{U_k(1, -1)}{79^{k+1}} &= \sum_{k=0}^{\infty} \frac{U_k(18, 1)}{88^{k+1}} = \frac{1}{6161}. 
\end{align*}
\]
**Example 2.** As a next example consider the case with \((P_1, Q_1) = (1, -1)\) and \((P_2, Q_2) = (2t + 1, t^2 + t - 1)\). We get that

\[
\sum_{k=0}^{\infty} U_k(1, -1) x^{k+1} = \sum_{k=0}^{\infty} U_k(2t + 1, t^2 + t - 1) (x + t)^{k+1} = \frac{1}{x^2 - x - 1}
\]

for \(x \geq 2\).

**Example 3.** Consider the positive integral solutions \(x, y\) of the equation

\[
\sum_{k=0}^{\infty} T_k(-1, 7, 3) x^{k+1} = \sum_{k=0}^{\infty} T_k(5, -5, -3) y^{k+1}.
\]

Theorem B implies that

\[
\frac{1}{x^3 + x^2 - 7x - 3} = \frac{1}{y^3 - 5y^2 + 5y + 3},
\]

therefore we get

\[
x^3 + x^2 - 7x - 3 = y^3 - 5y^2 + 5y + 3.
\]

Following the proof of Theorem 2 we have

\[
H(x, y) = x^3 + x^2 - 7x - y^3 + 5y^2 - 5y - 6 = 0.
\]

We determine \(G(x, y)\) and \(I(y)\), these are given by

\[
G(x, y) = 9x^2 + 9xy + 9y^2 - 9x - 27y - 45, \quad I(y) = 108y - 108.
\]

If \(I(y) = 0\), then \(y = 1\) and it follows that \(x^3 + x^2 - 7x - 3 = 4\), that is \(x = -1\). Hence we do not get positive integral solutions in this case. Assume that \(I(y) \neq 0\). We get that

\[
3x - 3y + 6 = \frac{4(108y - 108)}{9x^2 + 9xy + 9y^2 - 9x - 27y - 45}.
\]

It can be written as

\[
3x - 3y + 6 = \frac{4(108y - 108)}{27y^2 - 90y - 189 + (6x + 3y - 3)^2}.
\]

We have that \(26y^2 < 27y^2 - 90y - 189 + (6x + 3y - 3)^2\) for positive integers if \(y \geq 93\). It follows that \(|3x - 3y + 6| < 1\) if \(y \geq 93\). That is \(3x - 3y + 6 = 0\), so we obtain that \(I(y) = 0\), a contradiction. It remains to deal with the values of \(y\) for which \(3 = m(x^3 - 5x^2 + 5x + 3) \leq y \leq 93\). Using SageMath [8] we obtain that the only integral solutions in this range are given by \(x = -3, y = 3\) and \(x = -2, y = 4\), so we do not get positive integral solutions.

**Example 4.** As a second application of Theorem 2 let us consider the equation

\[
\sum_{k=0}^{\infty} T_k(-4, -5, -6) x^{k+1} = \sum_{k=0}^{\infty} T_k(1, 8, 18) y^{k+1}.
\]
Here we obtain that

\[
\begin{align*}
H(x, y) &= x^3 - y^3 + 4x^2 + y^2 + 5x + 8y + 24, \\
G(x, y) &= 9x^2 + 9xy + 9y^2 + 21x + 6y + 10, \\
I(y) &= -216y - 598.
\end{align*}
\]

The equation \( I(y) = 0 \) does not have integral solutions. We obtain that \( 4G(x, y) > 26y^2 \) if \( y > 18 \) and \( |3x - 3y + 5| < 1 \) if \( y > 30 \). Hence we have that if \( y > 30 \), then \( 3x - 3y + 5 = 0 \), therefore \( I(y) = 0 \), a contradiction. It remains to deal with the cases \( y \in [5, 6, \ldots, 30] \). It follows that the only positive solution is given by \( (x, y) = (9, 11) \), that is we have

\[
\sum_{k=0}^{\infty} \frac{T_k(-4, -5, -6)}{9^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k(1, 8, 18)}{11^{k+1}} = \frac{1}{1104}.
\]

**Example 5.** Finally let us describe an example with identically zero \( I(y) \), in which case we obtain infinitely many solutions. Let \( (a_2, a_1, a_0) = (1, 6, 5) \) and \( (b_2, b_1, b_0) = (4, 1, 1) \). It follows that

\[
\begin{align*}
H(x, y) &= x^3 - y^3 - x^2 + 4y^2 - 6x + y - 4, \\
G(x, y) &= 9x^2 + 9xy + 9y^2 - 18x - 27y - 36, \\
I(y) &= 0.
\end{align*}
\]

We obtain that either \( 3x - 3y + 3 = 0 \) or \( G(x, y) = 0 \). In the former case \( y = x + 1 \) and

\[
\sum_{k=0}^{\infty} \frac{T_k(1, 6, 5)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k(4, 1, 1)}{(x + 1)^{k+1}} = \frac{1}{x^3 - x^2 - 6x - 5}, \quad x \geq 4.
\]

In the latter case we have that

\[
0 = 12G(x, y) = 3(6x + 3y - 6)^2 + (9y - 12)^2 - 684.
\]

We do not get new integral solutions since the equation \((9y - 12)^2 + 3(6x + 3y - 6)^2 = 684\) has no solutions in \( \mathbb{Z} \).

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