Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 2, 30–35
DOI: 10.7546/nntdm.2019.25.2.30-35

Inequalities between the arithmetic functions φ, ψ and σ. Part 2

József Sándor[1] and Krassimir Atanassov[2]

[1] Babes-Bolyai University of Cluj, Romania
e-mail: jsandor@math.ubbcluj.ro

[2] Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering – Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8000, Bulgaria
e-mail: krat@bas.bg

Received: 19 June 2018 **Accepted:** 12 April 2019

Abstract: We prove inequalities related to $\varphi(n)^{\varphi(n)}.\psi(n)^{\psi(n)}$ or $\varphi(n)^{\psi(n)}.\psi(n)^{\varphi(n)}$ and related powers, where φ and ψ denote the Euler, resp. Dedekind arithmetic functions. More general theorem for the arithmetical functions f, g and h is formulated and proved.
Keywords and phrases: Arithmetic function, Inequality.
2010 Mathematics Subject Classification: 11A25, 26D15.

1 Introduction

Suggested by a paper [1], V. Kannan and R. Srikanth [2] have discovered the following inequality

$$\varphi(n)^{\varphi(n)}.\psi(n)^{\psi(n)} > n^{\varphi(n)+\psi(n)} \tag{1}$$

for $n > 1$, where φ and ψ denote the Euler totient function, resp. Dedekind arithmetic function.

In what follows, we shall give inequalities of type (1), with strong refinements, as well as their converses and analogues relations.

Recall that for $n > 1$ one has

$$\frac{\varphi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right), \ \frac{\psi(n)}{n} = \prod_{p|n}\left(1 + \frac{1}{p}\right) \tag{2}$$

where p is a prime, and $\varphi(1) = \psi(1) = 1$.

Similarly, one has for $n > 1$:

$$\sigma(n) = \sum_{d|n} d = \prod_{p^\alpha|n} \frac{p^\alpha - 1}{p - 1}, \tag{3}$$

where p^α are the maximal prime power divisors of n. One has by definition $\sigma(1) = 1$.

These arithmetic functions satisfy many important properties (see, e.g., [4]). For example, the following inequalities are well-known:

$$\varphi(n) \le \psi(n) \le \sigma(n), \tag{4}$$

$$\varphi(n) + \sigma(n) \ge 2n \tag{5}$$

with equality in the left side of (4) only if $n = 1$; in the right side of (4) if $n = 1$ or n is prime; and in (5) only if $n = 1$ or n is prime.

It follows at once from (2) that

$$\varphi(n).\psi(n) \le n^2 \tag{6}$$

for $n > 1$. For the same n, a stronger version of (6) is

$$\varphi(n).\sigma(n) \le n^2. \tag{7}$$

2 Auxilary results

Lemma 1. Let $x_i > 0, \lambda_i > 0$ $(i = 1, 2, ..., n)$ be real numbers such that $\lambda_1 + \lambda_2 + ... + \lambda_n = 1$. Then one has

$$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} \le \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n. \tag{8}$$

There is inequality only for $x_1 = ... = x_n = 1$.

Proof. This is the well-known weighted arithmetic mean-geometric mean inequality, see, e.g. [3]. □

Lemma 2. Let $x, y > 0$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$. Then

$$x^\lambda y^\mu \le \lambda x + \mu y \tag{9}$$

and

$$x^\lambda y^\mu \ge \frac{1}{\frac{\lambda}{x} + \frac{\mu}{y}}. \tag{10}$$

Proof. Let $n = 2, x_1 = x, x_2 = y$ in (8). Then (9) follows. Now, letting $n = 2, x_1 = \frac{1}{x}, x_2 = \frac{1}{y}$ in (8), we get (10). $\square$

Remark 1. (9) is called also as the weighted arithmetic-geometric inequality for two numbers; while (10) as the weighted geometric-harmonic inequality for two numbers.

Proposition 1. For any $a, b > 0$ one has

$$\left(\frac{a+b}{2}\right)^{a+b} \leq a^a.b^b \leq \left(\frac{a^2+b^2}{a+b}\right)^{a+b} \tag{11}$$

with equality only for $a = b$.

Proof. Put $x = a, y = b, \lambda = \frac{a}{a+b}, \mu = \frac{b}{a+b}$ in (9). Then the right side of (11) follows. Apply now inequality (10) for the same numbers. The left side of (11) follows. $\square$

Proposition 2. For any $a, b > 0$ one has

$$\left(\frac{ab(a+b)}{a^2+b^2}\right)^{a+b} \leq a^b.b^a \leq \left(\frac{2ab}{a+b}\right)^{a+b}, \tag{12}$$

with equality only in $a = b$.

Proof. Let now $x = b, y = a, \lambda = \frac{a}{a+b}, \mu = \frac{b}{a+b}$ in (9). Then we get

$$a^{\frac{b}{a+b}}.b^{\frac{a}{a+b}} \leq \frac{a}{a+b}.b + \frac{b}{a+b}.a = \frac{2ab}{a+b},$$

and the right side of (12) is proved. By applying inequality (10) with the same selections, we get

$$a^{\frac{b}{a+b}}.b^{\frac{a}{a+b}} \geq \frac{1}{\frac{a}{b(a+b)} + \frac{b}{a(a+b)}} = \frac{ab(a+b)}{a^2+b^2},$$

and we are done with the left side of (12). $\square$

Remark 2. As

$$\frac{2ab}{a+b} \leq \sqrt{ab},$$

clearly one has

$$\left(\frac{2ab}{a+b}\right)^{a+b} \leq (ab)^{\frac{a+b}{2}}. \tag{13}$$

3 Main results

Theorem 1. For any integer $n > 1$ one has

$$n^{\varphi(n)+\psi(n)} < \left(\frac{\varphi(n)+\psi(n)}{2}\right)^{\varphi(n)+\psi(n)} < \varphi(n)^{\varphi(n)}.\psi(n)^{\psi(n)}$$

$$< \left(\frac{\varphi(n)^2+\psi(n)^2}{2}\right)^{\frac{\varphi(n)+\psi(n)}{2}} < \psi(n)^{\varphi(n)+\psi(n)}. \tag{14}$$

Proof. Let $a = \varphi(n), b = \psi(n)$ in Proposition 1. Then, by using inequalities (4) and (5), we get the first three inequalities of (14). The last inequality follows only from the left side of relation (4). $\square$

Remark 3. The weaker inequality on the left side of (14) is exactly relation (1). This proof of (1) appears also in the recent book of the first author ([5], pp 50 - 52).

Theorem 2. For any integer $n > 1$ one has

$$\left(\frac{\varphi(n)\psi(n)(\varphi(n)+\psi(n))}{\varphi(n)^2+\psi(n)^2}\right)^{\varphi(n)+\psi(n)} < \varphi(n)^{\psi(n)}.\psi(n)^{\varphi(n)}$$

$$< \left(\frac{2\varphi(n)\psi(n)}{\varphi(n)+\psi(n)}\right)^{\varphi(n)+\psi(n)} < (\varphi(n)\psi(n))^{\frac{\varphi(n)+\psi(n)}{2}} < n^{\varphi(n)+\psi(n)}. \tag{15}$$

Proof. We apply Proposition 2, combined with relation (13) for $a = \varphi(n), b = \psi(n)$. The last inequality of (15) follows by (6). $\square$

Remark 4. The weaker inequality on the right side of (15) gives the inequality

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} < n^{\varphi(n)+\psi(n)} \tag{16}$$

for $n > 1$, which is an analogue of (1).

Remark 5. All inequalities of Theorems 1 and 2 may be replaced with function σ instead of function ψ.

Indeed, by applying Properties 1 and 2 for $a = \varphi(n), b = \sigma(n)$, by inequalities (4) and (5) one has

$$\varphi(n)\sigma(n) < n^2,$$

$$\varphi(n) + \sigma(n) \geq 2n.$$

Thus, particularly, we get for $n > 1$:

$$\varphi(n)^{\varphi(n)}\sigma(n)^{\sigma(n)} > n^{\varphi(n)+\sigma(n)} \tag{17}$$

and

$$\varphi(n)^{\sigma(n)}\sigma(n)^{\varphi(n)} < n^{\varphi(n)+\sigma(n)}. \tag{18}$$

Theorem 3. Let

$$A = \left\{n \mid n \in \mathcal{N} \;\&\; f(n) \leq \frac{n}{2}\right\},$$

where f is an arithmetic function and for every natural number n $f(n) > 0$. Then, for any arithmetic function g so that $g(n) > 0$ one has

$$f(n)^{g(n)}.g(n)^{f(n)} \leq n^{f(n)+g(n)} \tag{19}$$

for $n \in A$.

Proof. We apply the right side of (12) for $a = f(n), b = g(n)$ and using the fact that the inequality

$$\frac{2f(n)g(n)}{f(n)+g(n)} \leq n$$

may be rewritten as

$$g(n)(2f(n) - n) \leq nf(n).$$

Now, since $f(n) > 0, g(n) > 0$, this is true, if $2f(n) - n \leq 0$. Thus, if $n \in A$, then (19) holds true. □

Remark 6. In particular, we get

$$\varphi(n)^{g(n)}.g(n)^{\varphi(n)} \leq n^{\varphi(n)+g(n)} \tag{20}$$

for every even number n and for any arithmetic function g so that $g(n) > 0$.

If $g(n) \neq \varphi(n)$, then the inequality (20) is strict. Indeed, it is well-known that $\varphi(n) \leq \frac{n}{2}$ for any even number n. So, for A - the set of positive even integers, (20) follows.

Theorem 4. Let the arithmetical functions f, g and h satisfy the following conditions:

(i) $f(n).g(n) < n^2$ for $n > 1$;

(ii) $n + 1 \leq h(n) \leq g(n)$ for $n \geq 2$.

Then one has

$$(g(n))^{f(n)} < (g(n))^{\frac{n^2}{g(n)}} < (h(n))^n \tag{21}$$

Proof. The first inequality of (21) follows by condition (i) and the remark that $g(n) > 1$ by (ii). Now, for the proof of the second inequality , we will use the known fact that for $x > 0$ the real function $F(x) = x^{\frac{1}{x}}$ is strictly decreasing for $x \geq e$ (Eulers constant). Therefore, one has by (ii) that $(g(n))^{\frac{1}{g(n)}} \leq (h(n))^{\frac{1}{h(n)}}$, by remarking that (ii) can be applied, as $n + 1 \geq 3 > e$ for $n \geq 2$. Now, as $h(n) > n$, we get $(h(n))^{\frac{1}{h(n)}} < (h(n))^{\frac{1}{n}}$, and the result follows. □

Remark 7.

1) By selecting $f(n) = \varphi(n), g(n) = \sigma(n)$ and $h(n) = \psi(n)$, we get from (21) the following two inequalities:

$$(\sigma(n))^n < (\psi(n))^{\sigma(n)}$$

and

$$(\sigma(n))^{\varphi(n)} < (\psi(n))^n,$$

which have been proved in Part 1 (see [4])

2) Select $f(n) = \varphi^*(n), g(n) = \sigma^*(n)$, and $h(n) = n + 1$, where φ^* is the unitary analogue of the Euler totient function, and σ^* is the unitary analogue of the sigma function (see [6]).

Now, by ([6], relation (15)), condition (i) is satisfied. The condition (ii) reduces to $\sigma^*(n) \geq n + 1$, which is well-known. Then we get the inequalities:

$$(\sigma^*(n))^n < (n+1)^{\sigma^*(n)} \tag{22}$$

and

$$(\sigma^*(n))^{\varphi^*(n)} < (n+1)^n. \tag{23}$$

Finally, we would like to mention that while preparing this manuscript we found that in paper [2] there are some mistakes. For example, the inequality in the last line of page 20 is false.

It is asserted that

$$\sum_{p|n} \ln\left(1 + \frac{1}{p}\right) > \left|\sum_{p|n} \ln\left(1 - \frac{1}{p}\right)\right|,$$

where p runs through the prime divisors of n.

This is not correct. Let for example, $n = 6$. Then the prime divisors of n are $p = 2$ and $p = 3$. We should have

$$\ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{3}\right) > \left|\ln\left(1 - \frac{1}{2}\right) + \ln\left(1 - \frac{1}{3}\right)\right|.$$

After simple computations, this reduces to $\ln 2 > \ln 3$, which is false.

References

[1] Atanassov, K. (2011). Note on φ, ψ and σ-functions. Part 3. *Notes on Number Theory and Discrete Mathematics*, 17 (3), 13–14.

[2] Kannan, V. & Srikanth, R. (2013). Note on φ and ψ functions, *Notes on Number Theory and Discrete Mathematics*, 19 (1), 19–21.

[3] Mitrinovich, D. (1970). *Analytic Inequalities*. Springer Verlag, Berlin.

[4] Atanassov, K., & Sándor, J. (2019). Inequalities between the arithmetic functions φ, ψ and σ. Part 1. *Notes on Number Theory and Discrete Mathematics*, 25 (1), 50–53.

[5] Sándor, J. (2018). *Theory of means and thier inequalities*. Available online: `http://www.math.ubbcluj.ro/%7Ejsandor/lapok/Sandor-Jozsef-Theory%20of%20Means%20and%20Their%20Inequalities.pdf`.

[6] Sándor, J. (2014). On certain inequalities for σ, φ, ψ and related functions, *Notes on Number Theory and Discrete Mathematics*, 20 (2), 52–60.