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On the prime factors of a quasiperfect number

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Abstract: A positive integer N is said to be *quasiperfect* if $\sigma(N) = 2N + 1$ where $\sigma(N)$ is the sum of the positive divisors of N. So far no quasiperfect number is known. If such N exists, let $\gamma(N)$ denote the product of the distinct primes dividing N. In this paper, we obtain a lower bound for $\gamma(N)$ in terms of $r = \omega(N)$, the number of distinct prime factors of N. Also, we show that every quasiperfect number N is divisible by a prime p with: (i) $p \equiv 1 \pmod 4$, (ii) $p \equiv 1 \pmod 5$ if $5 \nmid N$ and (iii) $p \equiv 1 \pmod 3$, if $3 \nmid N$.
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1 Introduction

For any natural number N let $\sigma(N)$ denote the sum of its positive divisors. W. Sierpinski [6] asked whether there is any natural number N satisfying

$$\sigma(N) = 2N + 1, \tag{1.1}$$

which is unanswered till date. Calling such N, if it exists, a *quasiperfect* number, Cattaneo [2] initiated the study of such numbers. H. L. Abbott et. al. [1] continued the investigations and proved the following:

If a quasiperfect number N exists and if $\omega(N)$ is the number of distinct prime factors of N then

$$\omega(N) \geq 5 \text{ and } N > 10^{20} \text{ ([1] , Theorem 2 and 4)} \tag{1.2}$$

and

$$\omega(N) \geq 15 \text{ and } N > 10^{57} \text{ if } (N, 15) = 1 \text{ ([3])} \tag{1.3}$$

In [4] M. Kishore improved (1.2) to $\omega(N) \geq 6$ and $N > 10^{30}$ while a further refinement of it to $\omega(N) \geq 7$ and $N > 10^{35}$ was obtained by G.L. Cohen and Peter Hagis Jr. [3].

For other details of research on quasiperfect numbers one can see the excellent book of J. Sandor and B. Crstici ([5], p. 38-39).

Recently the authors [7] have given a different proof for the first part of (1.3) for which Theorem 2.4 (given in Section 2 below) was used.

For any positive integer n let $\gamma(n)$ denote the product of its distinct prime factors ($\gamma(n)$ is called the *radical* of the integer n; and it is the maximal squarefree divisor of n, that is, the greatest divisor of n having no square factor > 1).

In this paper we obtain a lower bound for $\gamma(N)$ in terms of $r = \omega(N)$ for a quasiperfect number N. Also we prove that every quasiperfect number is divisible by a prime p with
(i) $p \equiv 1 \pmod 4$, (ii) $p \equiv 1 \pmod 5$ if $5 \nmid N$ and (iii) $p \equiv 1 \pmod 3$ if $3 \nmid N$.

2 Preliminaries

Throughout the rest of the paper N stands for a quasiperfect number. We first state a theorem due to Cattaneo [2] needed for our purpose:

Theorem 2.1.
(a) If N exists, then it is of the form

$$N = p_1^{2e_1} p_2^{2e_2} ... p_r^{2e_r}, \tag{2.2}$$

where $p_1, p_2, ..., p_r$ are distinct odd primes and $e_i \geq 1$ for $i = 1, 2, 3, ..., r$.
(b) If $p_i \equiv 1 \pmod 8$, then $e_i \equiv 0$ or $1 \pmod 4$; if $p_i \equiv 3 \pmod 8$, then $e_i \equiv 0 \pmod 2$ and if $p_i \equiv 5 \pmod 8$, then $e_i \equiv 0$ or $-1 \pmod 4$.
(c) If M is a natural number such that $\sigma(M) \geq 2M$, then no non-trivial multiple of M is quasiperfect.

Remark 2.3. It follows from Theorem 2.1 that every quasiperfect number is the square of an odd integer and that $\sigma(d) < 2d$ for every divisor d of N.

In [7] the authors have proved:

Theorem 2.4. If N exists and is of the form (2.2), then an odd number of $p_i^{2e_i}$ are such that either $p_i \equiv 1 \pmod 8$ and $e_i \equiv 1 \pmod 4$ or $p_i \equiv 5 \pmod 8$ and $e_i \equiv -1 \pmod 4$.
(Such $p_i^{2e_i}$ are called *special factors of N* in [7])

3 Lower bound for $\gamma(N)$

Suppose $A = \{a_1, a_2, ..., a_r\}$ is a set of positive real numbers and for any $k (1 \leq k \leq r)$ suppose $S_k(A)$ is the sum of the products of the elements in the k-element subsets of A. That is,

$$S_k(A) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq r} a_{i_1}.a_{i_2}...a_{i_k} \tag{3.1}$$

For example, $S_1(A) = \sum_{i=1}^{r} a_i$ and $S_2(A) = \sum_{1 \leq i_1 < i_2 \leq r} a_{i_1}.a_{i_2}$.

Note that

$$\prod_{i=1}^{r}(1 + a_i) = 1 + \sum_{k=1}^{r} S_k(A) \tag{3.2}$$

Observe that $S_k(A)$ has $\binom{r}{k}$ terms and that each $a_j \in A$ occurs exactly in $\binom{r-1}{k-1}$ terms of it. Therefore the product $P_k(A)$ of the terms in $S_k(A)$ is given by

$$P_k(A) = (a_1 a_2 ... a_r)^{\binom{r-1}{k-1}} \tag{3.3}$$

Therefore, the inequality between the arithmetic mean and the geometric mean gives

$$\frac{S_k(A)}{\binom{r}{k}} > (P_k(A))^{\frac{1}{\binom{r}{k}}}$$

(the strict inequality is due to the fact that a_j are distinct)
which, in view of (3.3), shows that

$$S_k(A) > \binom{r}{k} (a_1 a_2 ... a_r)^{\frac{k}{r}}. \tag{3.4}$$

Theorem 3.5. If N exists and is of the form (2.2), then

$$\gamma(N) > A_r,$$

where $A_r = \dfrac{1}{(2^{\frac{1}{r}} - 1)^r}$

Proof. Here $\gamma(N) = p_1 p_2 ... p_r$ is a divisor of N so that by Remark 2.3 and (3.2) we have

$$2 > \frac{\sigma(\gamma(N))}{\gamma(N)} = \prod_{i=1}^{r} \frac{\sigma(p_i)}{p_i} = \prod_{i=1}^{r}(1 + \frac{1}{p_i}) = 1 + \sum_{k=1}^{r} S_k(B),$$

where $B = \left\{\frac{1}{p_1}, \frac{1}{p_2}, ..., \frac{1}{p_r}\right\}$. Therefore, by (3.4), it follows that

$$\begin{aligned} 2 &> 1 + \sum_{k=1}^{r} \binom{r}{k} \left(\frac{1}{p_1 p_2 ... p_r}\right)^{\frac{k}{r}} \\ &= 1 + \sum_{k=1}^{r} \binom{r}{k} \{\gamma(N)^{-\frac{1}{r}}\}^k \\ &= \{1 + \gamma(N)^{-\frac{1}{r}}\}^r, \end{aligned}$$

which proves the theorem. $\square$

Remark 3.6. One of the reviewers has pointed out that a better lower bound for $\gamma(N)$ than A_r can be obtained by using known estimates for some functions over primes and this will be investigated later. Another reviewer has observed that the proof of Theorem 3.5 bears a close resemblance to the proof of a result of Anirudh Prabhu's paper available online via arXiv at `https://arxiv.org/pdf/1008.1114.pdf` and the authors were not aware of the paper earlier.

4 On prime factors of N

Theorem 4.1. If N is of the form (2.2), then $p_i \equiv 1 \pmod 4$ for some i.

Proof. If not, $p_i \equiv 3$ or $7 \pmod 8$ for each i, contradicting Theorem 2.4. $\square$

Theorem 4.2. If N is of the form (2.2) and $(N,5)=1$, then $p_i \equiv 1 \pmod 5$ for some i.

Proof. If $(N,5)=1$ then $p_i \equiv \pm 1$ or $\pm 2 \pmod 5$

First suppose $p_i \equiv \pm 1 \pmod 5$ so that $p_i^2 \equiv 1 \pmod 5$ and therefore

$$\begin{aligned}\sigma(p_i^{2e_i}) &= (1+p_i)(1+p_i^2+...+p_i^{2e_i-2})+p_i^{2e_i} \equiv (1+p_i)e_i+1 \pmod 5\\ &\equiv \begin{cases} 2e_i+1 \pmod 5 & \text{if } p_i \equiv 1 \pmod 5\\ 1 \pmod 5 & \text{if } p_i \equiv -1 \pmod 5\end{cases}\end{aligned} \tag{4.3}$$

If $p_i \equiv \pm 2 \pmod 5$, then $p_i^2 \equiv -1 \pmod 5$ and therefore

$$\begin{aligned}\sigma(p_i^{2e_i}) &= (1+p_i)(1+p_i^2+...+p_i^{2e_i-2})+p_i^{2e_i}\\ &\equiv (1+p_i)\{1+(-1)+(-1)^2+...+(-1)^{e_i-1}\}+(-1)^{e_i} \pmod 5\\ &\equiv \begin{cases} 1 \pmod 5 & \text{if } e_i \text{ is even}\\ 2 \pmod 5 & \text{if } e_i \text{ is odd, } p_i \equiv 2 \pmod 5\\ -2 \pmod 5 & \text{if } e_i \text{ is odd, } p_i \equiv -2 \pmod 5\end{cases}\end{aligned} \tag{4.4}$$

If possible, suppose no $p_i \equiv 1 \pmod 5$, then either $p_i \equiv -1 \pmod 5$ or $p_i \equiv \pm 2 \pmod 5$. Therefore, by (4.3) and (4.4), we get

$$\begin{aligned}\sigma(N) &\equiv \prod_{\substack{p_i \equiv 2 \pmod 5\\ e_i \text{ is odd}}} (2) \times \prod_{\substack{p_i \equiv -2 \pmod 5\\ e_i \text{ is odd}}} (-2) \cdot \pmod 5\\ &\equiv 2^{k+k'} \cdot (-1)^{k'} \pmod 5,\end{aligned} \tag{4.5}$$

where $k = \#\{p_i^{2e_i} : p_i \equiv 2 \pmod 5, e_i\ \textit{odd}\}$ and $k' = \#\{p_i^{2e_i} : p_i \equiv -2 \pmod 5, e_i\ \textit{odd}\}$. Also

$$2N+1 \equiv 2.\prod_{i=1}^{r}(p_i^2)^{e_i}+1 \equiv 2.(-1)^{k+k'}+1 \pmod 5. \tag{4.6}$$

Now (4.5) and (4.6) imply that

$$2.(-1)^{k+k'} + 1 \equiv 2^{k+k'}.(-1)^{k'} \pmod 5, \tag{4.7}$$

which reduces to

$$2.(-1)^{k} + (-1)^{k'} \equiv 2^{k+k'} \pmod 5, \tag{4.8}$$

and this congruence is impossible for all choices of integers k and k', a contradiction, proving the theorem. □

Theorem 4.9. If N is of the form (2.2) and $(N, 3) = 1$, then $p_i \equiv 1 \pmod 3$ for some i.

Proof. If $(N, 3) = 1$ then $p_i \equiv \pm 1 \pmod 3$ for each i and since each p_i is odd it follows $p_i \equiv \pm 1 \pmod 6$ for each i so that $p_i^2 \equiv 1 \pmod 6$. Therefore,

$$2N + 1 \equiv 2.\prod_{i=1}^{r}(p_i^2)^{e_i} + 1 \equiv 3 \pmod 6 \tag{4.10}$$

and for each i,

$$\begin{aligned}\sigma(p_i^{2e_i}) &= (1+p_i)(1+p_i^2+...+p_i^{2e_i-2}) + p_i^{2e_i} \\ &\equiv (1+p_i)e_i + 1 \pmod 6 \\ &\equiv \begin{cases} 2e_i + 1 \pmod 6 & \text{if } p_i \equiv 1 \pmod 6 \\ 1 \pmod 6 & \text{if } p_i \equiv -1 \pmod 6 \end{cases}\end{aligned}$$

If possible, suppose no $p_i \equiv 1 \pmod 6$. Then

$$\sigma(N) = \prod_{i=1}^{r}\sigma(p_i^{2e_i}) \equiv 1 \pmod 6 \tag{4.11}$$

Now, by (4.10) and (4.11), we have

$$1 \equiv 3 \pmod 6 \tag{4.12}$$

a contradiction. This proves the theorem. □

Under certain stronger conditions we have a more general result given below:

Theorem 4.13. If N is of the form (2.2) and $(N, m) = 1$ for some odd $m > 2$ and if $p_i \equiv \pm 1 \pmod m$ for all i, then $p_j \equiv 1 \pmod m$ for some j $(1 \le j \le r)$. Also if there is exactly one j with this property then $e_j \equiv 1 \pmod m$.

Proof. Similar to the proof of Theorem 4.9 for the first part. If there is exactly one j with $p_j \equiv 1 \pmod m$ then $2e_j + 1 \equiv 3 \pmod m$ giving $e_j \equiv 1 \pmod m$ since m is odd. □

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