Representation of higher even-dimensional rhotrix

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Abstract: The multiplication of higher even-dimensional rhotrices is presented and generalized. The concept of empty rhotrix, and the necessary and sufficient conditions for an even-dimensional rhotrix to be represented over a linear map, are investigated and presented.

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1 Introduction

A rhotrix is an arrangement of numbers in a rhomboid shape. This is similar to a matrix, which is an arrangement of numbers in a rectangular form. Rhotrix was first introduced by Ajibade [1], as an extension of the idea suggested by Atanassov and Shannon [4] in their work titled “matrix-tertions and matrix-noitrets”. A formal definition of a real rhotrix as presented in the maiden paper is given below:

**Definition 1.1.** [1] A real rhotrix set of dimension three, denoted as $\hat{R}_3(\mathbb{R})$ is defined as:

$$\hat{R}_3(\mathbb{R}) = \left\{ \begin{bmatrix} a & b & c & d \\ b & c & d & e \\ e & & & \end{bmatrix} : a, b, c, d, e \in \mathbb{R} \right\}$$

where $c = h(R)$ is called the heart of any rhotrix $R$ belonging to $\hat{R}_3(\mathbb{R})$ (a set of all real rhotrices of dimension 3) and $\mathbb{R}$ is the set of real numbers.
Examples showing extension of this set and analysis are copious in literature. A few are presented in these references [2, 3, 5–7, 10–14, 18, 19]. It has been noted that these heart-oriented rhotrices are always of odd dimension. Thus, a rhotrix with even dimension is recently being introduced by Isere [8, 9]. The algebra and analysis establishing this new structure as mathematically tractable were all presented in [9]. The heart-based rhotrices are classified as classical rhotrices, while even-dimensional rhotrices are classified as non-classical rhotrices [8].

Meanwhile, the addition and multiplication of heart-based rhotrices (\(h\)-rhotrices) were first presented in [1]. Thus, addition and multiplication of two heart-based rhotrices are defined as:

\[
R + Q = \begin{pmatrix} a & f \\ h(R) & j \end{pmatrix} + \begin{pmatrix} a + f \\ h(Q) & d + j \end{pmatrix}
\]

and

\[
R \circ Q = \begin{pmatrix} bh(Q) + gh(R) & ah(Q) + fh(R) \\ h(R)h(Q) & dh(Q) + jh(R) \\ eh(Q) + kh(R) \end{pmatrix},
\]

respectively. A generalization of this hearty multiplication is given in [14] and in [6]. A row-column multiplication of heart-based rhotrices was proposed by Sani [15] as:

\[
R \circ Q = \begin{pmatrix} a + f + dg \\ h(R)h(Q) & aj + dk \\ bj + ek \end{pmatrix}.
\]

A generalization of this row-column multiplication was also later given by Sani [16] as:

\[
R_n \circ Q_n = \langle a_{ij}, c_{ij} \rangle \circ \langle b_{ij}, d_{lk} \rangle = \begin{pmatrix} \sum_{i,j=1}^{t} (a_{ij}b_{ij}), \sum_{l,k=1}^{t-1} (c_{lk}d_{lk}) \end{pmatrix}, \quad t = (n + 1)/2,
\]

where \(R_n\) and \(Q_n\) are \(n\)-dimensional rhotrices (with \(n\) rows and \(n\) columns). These two methods of multiplication of rhotrices are very popular in literature. In both methods, the heart plays a significant role as shown above. A lot of work has been done on \(h\)-rhotrices. These works are also well known in literature, such as the conversion of a rhotrix into a coupled matrix by Sani [17]. A generalization of rhotrix was introduced as paraletrix by Aminu and Michael [3]. This concept shows more flexibility in mathematical arrays of numbers, where the number of rows and columns need not be the same. It was noted that not every paraletrix has a heart. Consequently, a rhotrix without a heart was introduced in [8, 9] as heartless rhotrices (\(hl\)-rhotrices). Such rhotrices were found to be even-dimensional. The simplest non-trivial even-dimensional rhotrix is of dimension two, and it is stated below:

**Definition 1.2.** A real rhotrix of dimension two is given as

\[
A = \left\{ \begin{pmatrix} a \\ b \\ e \\ d \end{pmatrix} : a, b, d, e \in \mathbb{R} \right\}.
\]
It is to be noted that an $n$-dimensional rhotrix with $n$ being even has its cardinality as $|R_n| = \frac{1}{2}(n^2 + 2n)$ $\forall$ $n \in 2N$. The multiplication of $h$-rhotrices, as remarked in [1], can be done in many ways. This is also true with even-dimensional rhotrices. In this work, we define multiplication of two even-dimensional rhotrices elementwise as follows:

$$A \circ B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \circ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

Moreover, we shall be looking at multiplication of higher even-dimensional rhotrices, the concept of empty rhotrix and the representation of an even-dimensional rhotrix over a linear map. The concept of rhotrix linear transformation was first investigated by Mohammed et al [13]. The necessary and sufficient conditions for a rhotrix to be represented by a linear map were given in [13]. It is to be noted that the rhotrix investigated was an $h$-rhotrix. These conditions will be stated in the next section. However, an extension of these conditions will be considered in this work, and the necessary and sufficient conditions for an even-dimensional rhotrix to be represented by a linear map will be presented.

2 Preliminaries

Some definitions will be considered in this section that will be useful in achieving the results anticipated in this work.

**Definition 2.1.** [13] A rhotrix $R$ of dimension $n$ is given as:

$$R_n = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{t1} & a_{tt} \end{pmatrix}.$$  

The element $a_{ij}(i,j = 1,2,...,t)$ and $c_{kl}(k,l = 1,2,...,t-1)$ are called the major and minor entries of $R$, respectively. This is usually denoted as $R_n = \langle a_{ij}, c_{kl} \rangle$.

**Definition 2.2.** [13] Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an $n$ dimensional rhotrix. Then, $a_{ij}$ is the $(i,j)$-entries called the major entries of $R_n$ and $c_{kl}$ is the $(k,l)$-entries called the minor entries of $R_n$.

**Definition 2.3.** [16] A rhotrix $R_n = \langle a_{ij}, c_{kl} \rangle$ of $n$ dimension is a couple of two matrices $(a_{ij})$ and $(c_{kl})$ consisting of its major and minor matrices of $R_n$.

**Definition 2.4.** [13] Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an $n$ dimensional rhotrix. Then, rows and columns of $a_{ij}(c_{kl})$ will be called the major (minor) rows and columns of $R_n$, respectively.
Definition 2.5. [13] For any odd integer \( n \), a \( n \times n \) matrix \((a_{ij})\) is called a filled coupled matrix if \( a_{ij} = 0 \) for all \( i, j \) whose sum \( i + j \) is odd. We shall refer to these entries as the null entries of the filled coupled matrix.

Remark 2.1. (i) \( R_n = \langle a_{ij}, c_{kl} \rangle \) is a representation of any rhotrix. (ii) Moreover, an even-dimensional rhotrix can also be represented as \( R_n = \langle a_{ij}, c_{kl} \rangle \) or simply as \( R_n = \langle a_i, \rangle \). (iii) a \((n \times n)\) filled coupled matrix has \( n^2 \) entries.

Definition 2.6. For any odd integer \( n \), a \((n \times n)\) matrix \((a_{ij})\) is called a completely filled coupled matrix if \( a_{ij} = 0 \) for all \( i, j \) whose sum \( i + j \) is odd and for all \( i = j = \frac{n+1}{2} \). The entry corresponding to \( a_{ij} = 0, i = j = \frac{n+1}{2} \) is a special null-entry called the null entry of the completely filled coupled matrix.

Definition 2.7. The entries \( a_{ij} \) whose sum \( i + j \) is even, except when \( i = j = \frac{n+1}{2} \) \( \forall n \in 2Z^+ + 1 \), are called the real entries of the completely filled coupled matrix.

Theorem 2.1. [13] Let \( n \in 2Z^+ + 1 \) and \( F \) be a field. Then, a linear map \( T : F^n \mapsto F^n \) can be represented by a rhotrix with respect to the standard basis if and only if \( T \) is defined as:

\[
T(x_1, y_1, x_2, y_2, \ldots, y_{t-1}, x_t) = (\alpha_1(x_1, x_2, \ldots, x_t), \beta_1(y_1, y_2, \ldots, y_{t-1}), \alpha_2(x_1, x_2, \ldots, x_t), \beta_2(y_1, y_2, \ldots, y_{t-1}), \ldots, \beta_{t-1}(y_1, y_2, \ldots, y_{t-1}), \alpha_t(x_1, x_2, \ldots, x_t))
\]

where \( t = \frac{n+1}{2} \), \( \alpha_1, \alpha_2, \ldots, \alpha_t \) and \( \beta_1, \beta_2, \ldots, \beta_{t-1} \) are any linear maps on \( F^t \) and \( F^{t-1} \), respectively.

Lemma 2.2. Let \([a_{ij}]_n\) be a \((n \times n)\) filled coupled matrix, then:

(a) The number of all the real entries is given as

\[
\Pi_n = \frac{1}{2}(n^2 + 1) \quad \forall n \in 2Z^+ + 1
\]

(b) The number of all the null entries is given as

\[
\emptyset_n = \frac{1}{2}(n^2 - 1) \quad \forall n \in 2Z^+ + 1
\]

Proof. Since a \((n \times n)\) filled coupled matrix has \( n^2 \) entries, then \( (a) + (b) = n^2 \). Consider:

\[
\frac{1}{2}(n^2 + 1) + \frac{1}{2}(n^2 - 1) = n^2
\]

Then, \( (a) \) and \( (b) \) are true.

Remark 2.2. \( \Pi_n + \emptyset_n \) as in Lemma 2.2 is an odd-dimensional rhotrix, i.e., the real entries are odd.

Lemma 2.3. Let \([a_{ij}]_n\) be a completely filled coupled matrix, then:
(a) The number of all the real entries is given as
\[ \Pi_n = \frac{1}{2}(n^2 - 1) \quad \forall n \in 2\mathbb{Z}^+ + 1 \]

(b) The number of all the null entries is given as
\[ \emptyset_n = \frac{1}{2}(n^2 + 1) \quad \forall n \in 2\mathbb{Z}^+ + 1. \]

Proof. The proof is similar to the proof of Lemma 2.2 above.

Remark 2.3. \( \Pi_n + \emptyset_n \) as in Lemma 2.3 is an even-dimensional rhotorix, i.e., the real entries are even.

Theorem 2.4. There is a one-to-one correspondence between the set of all \( n \)-dimensional rhotorices over a field \( F \) and the set of all \( n \times n \) completely filled coupled matrices over \( F \).

Proof. The proof follows from Lemma 2.3 and the fact that any \( n \)-dimensional rhotorix is \( n^2 \) entries.

Remark 2.4. (i) The set of all real entries \( \Pi_n \) of the completely filled coupled matrix corresponds to the entries of an even-dimensional rhotorix \( R_n = \langle a_{ij}, c_{kl} \rangle \) or simply as \( R_n = \langle a_i \rangle \).

(ii) A filled coupled matrix and a completely filled coupled matrix comprise of both real and null entries.

(iii) All heart-based rhotorices result in a filled coupled matrix while all even-dimensional rhotorices result in a completely filled coupled matrix.

3 Main Results

This section presents the main results starting with the concept of empty rhotorix, then some examples of filled and completely filled coupled matrices and multiplication of higher even-dimensional rhotorices.

3.1 The concept of empty rhotorix

Definition 3.1. A rhotorix that has no entry is an empty rhotorix, e.g., \( A = \langle \rangle \).

Lemma 3.1. An empty rhotorix \( A \) of \( n \)-th dimension contains null-entry of a completely-filled matrix as its only entry.

Proof. Recall that for an even-dimensional rhotorix \( |R_n| = \frac{1}{2}(n^2 + 2n) \quad \forall n \in 2\mathbb{N} \). Since \( n \in 2\mathbb{N} \) implies that \( 0 \in 2\mathbb{N} \) and \( R_0 = \langle \rangle \). The proof follows.

Corollary 3.1.1. An empty real rhotorix is even-dimensional.
Proof. We prove by contradiction. Let \( R_n \) be any \( n \)-dimensional real rhotrix. Suppose, \( n \) is odd, then, its cardinality can be represented as
\[
|R_n| = \frac{1}{2} (n^2 + 1) \quad n \in 2\mathbb{Z}^+ + 1.
\]
Since, an empty rhotrix has no entry, its cardinality is zero. That is
\[
0 = \frac{1}{2} (n^2 + 1)
\]
implies that \( n = \pm i \). Then, we have a contradiction. Now, suppose that \( n \) is even, then
\[
|R_n| = \frac{1}{2} (n^2 + 2n) \quad n \in 2\mathbb{N}
\]
implies that \( n = 0 \in 2\mathbb{N} \). Thus, an empty rhotrix is even-dimensional. 

Remark 3.1. \( \mathbb{N} \) is a set of non-negative integers

3.2 Some examples of filled and completely filled coupled matrices

Example 3.1. A rhotrix of dimension five \((R_5)\) is given by:
\[
R_5 = \begin{pmatrix}
a_{11} & a_{12} & c_{11} & a_{13} \\
a_{21} & c_{11} & a_{12} & c_{13} \\
a_{31} & c_{21} & a_{22} & c_{23} \\
a_{32} & c_{22} & a_{23} & c_{23} \\
a_{33} & & & \\
\end{pmatrix}
\]

Then its corresponding filled coupled matrix is presented below:
\[
M(R_5) = \begin{bmatrix}
a_{11} & 0 & a_{12} & 0 & a_{13} \\
0 & c_{11} & 0 & c_{12} & 0 \\
a_{21} & 0 & a_{22} & 0 & a_{23} \\
0 & c_{21} & 0 & c_{22} & 0 \\
a_{31} & 0 & a_{32} & 0 & a_{33} \\
\end{bmatrix}
\]

Example 3.2. A rhotrix of dimension seven \((R_7)\) is given by:
\[
R_7 = \begin{pmatrix}
a_{11} & a_{12} & c_{11} & a_{13} \\
a_{21} & c_{11} & a_{12} & c_{13} \\
0 & c_{21} & a_{22} & c_{23} \\
a_{31} & c_{31} & a_{32} & c_{33} \\
a_{41} & c_{41} & a_{42} & c_{43} \\
a_{42} & c_{42} & a_{43} & c_{44} \\
a_{43} & c_{43} & a_{44} & a_{44} \\
\end{pmatrix}
\]

Then its corresponding filled coupled matrix will be presented below:
Example 3.3. A rhotrix of dimension four \((R_4)\) is given by:

\[
R_4 = \begin{pmatrix}
  a_{11} & a_{21} & a_{31} & a_{41} \\
  a_{12} & c_{11} & a_{22} & a_{32} \\
  a_{13} & a_{23} & c_{22} & a_{33} \\
  a_{14} & a_{24} & a_{34} & c_{33} \\
\end{pmatrix}
\]

Then its corresponding completely filled coupled matrix is presented below:

\[
C(R_4) = \begin{pmatrix}
  a_{11} & 0 & 0 & a_{14} \\
  0 & c_{11} & 0 & c_{13} \\
  a_{21} & 0 & 0 & a_{24} \\
  0 & c_{21} & 0 & c_{23} \\
  a_{31} & 0 & a_{32} & a_{34} \\
  0 & c_{31} & 0 & c_{33} \\
  a_{41} & 0 & a_{42} & a_{44} \\
\end{pmatrix}
\]

Example 3.4. A rhotrix of dimension six \((R_6)\) is given by:

\[
R_6 = \begin{pmatrix}
  a_{11} & a_{21} & a_{31} & a_{41} \\
  a_{12} & c_{11} & a_{22} & a_{32} \\
  a_{13} & a_{23} & c_{22} & a_{33} \\
  a_{14} & a_{24} & a_{34} & c_{33} \\
\end{pmatrix}
\]

Then its corresponding completely filled coupled matrix is:

\[
C(R_6) = \begin{pmatrix}
  a_{11} & 0 & 0 & a_{14} \\
  0 & c_{11} & 0 & c_{13} \\
  a_{21} & 0 & 0 & a_{24} \\
  0 & c_{21} & 0 & c_{23} \\
  a_{31} & 0 & a_{32} & a_{34} \\
  0 & c_{31} & 0 & c_{33} \\
  a_{41} & 0 & a_{42} & a_{44} \\
\end{pmatrix}
\]

Remark 3.2. A completely filled coupled matrix is obtained from even-dimensional rhotrix, and contains the null-entry of the completely filled coupled matrix denoted as \(0^*\), while a filled coupled matrix is obtained from odd-dimensional rhotrices.
3.3 Multiplication of higher even-dimensional rhotrices

Multiplication of higher even-dimensional rhotrices whether even or odd dimensional can be defined in many ways. In this work, elementwise multiplication method is presented for higher even-dimensional rhotrices. Examples of rhotrices of dimension four are presented for the purpose of demonstration. Let

\[ A = \left\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12} \right\rangle, \quad B = \left\langle b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12} \right\rangle \]

then

\[ A \odot B = \left\langle a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4, a_5 b_5, a_6 b_6, a_7 b_7, a_8 b_8, a_9 b_9, a_{10} b_{10}, a_{11} b_{11}, a_{12} b_{12} \right\rangle \]

Example 3.5. Let

\[ A = \left\langle 2, 3, 1, 4, 5, 6, 7, 8, 9, 10, 5, 3 \right\rangle, \quad B = \left\langle 2, 4, 1, 7, 8, 9, 5, 6, 8, 3, 10, 6 \right\rangle \]

then

\[ A \odot B = \left\langle 2 \odot 3, 3 \odot 2, 1 \odot 4, 4 \odot 1, 6 \odot 3, 6 \odot 2, 10 \odot 8, 8 \odot 5, 3 \odot 10, 30 \right\rangle = \left\langle 35, 48, 63, 40, 54, 80, 15, 30 \right\rangle \]

Generally, a rhotrix $R$ of dimension $n$ ($n$ being even) can be written as:

\[ R_n = \left\langle \begin{array}{cccccccccc}
    & & & & & & & & & & \\
    r_1 & & & & & & & & & \\
    & r_2 & & & & & & & & \\
    & & r_3 & & & & & & & \\
    & & & r_4 \\
    & & & & - & - & - & - & - & - \\
    & & & & - & - & - & - & - & - \\
    & & & & - & - & - & - & - & - \\
    & & & & - & - & - & - & - & - \\
    & & & & - & - & - & - & - & - \\
    r_{\frac{n^2+2n-6}{2}} & r_{\frac{n^2+2n-4}{2}} & r_{\frac{n^2+2n-2}{2}} & & & & & & & \\
    & & & & & & & & & & \\
\end{array} \right\rangle \]
A generalization of the elementwise multiplication of even-dimensional rhotorices is as follows. Let $R_n = \langle a_i \rangle$ and $Q_n = \langle b_j \rangle$, be two even-dimensional rhotorices, then their multiplication is as follows

$$R_n \circ Q_n = \langle a_i \rangle \circ \langle b_j \rangle = \left( \sum_{i=1}^{t} a_i \right) \circ \left( \sum_{j=1}^{t} b_j \right) = \left( \sum_{k=1}^{t} (a_k \cdot b_k) \right), \; t = (n^2 + 2n)/2, \; n \in 2\mathbb{N},$$

where the product $(a_{ij}b_{ij})$ is empty whenever $i = j = \frac{t+1}{2}$ $\forall$ $t \in 2\mathbb{Z}^+ + 1$.

# 4 Linear maps on an even-dimensional rhotorix

The concept of representation by a linear map helps to establish the existence of a linear structure. In this section, we investigate the representation of an even-dimensional rhotorix over a linear map.

**Theorem 4.1.** Let $n \in 2\mathbb{Z}^+ + 1$ and $F$ be a field. Then, a linear map $\tau : F^n \mapsto F^n$ can be represented by an even-dimensional rhotorix with respect to the standard basis if and only if $\tau$ is defined as:

$$\tau(x_1, y_1, x_2, y_2, \ldots, y_{t-1}, x_t) = (\alpha_1(x_1, x_2, \ldots, x_t), \beta_1(y_1, y_2, \ldots, y_{t-1}),$$

$$\alpha_2(x_2, x_3, \ldots, x_t), \beta_2(y_2, y_3, \ldots, y_{t-1}), \ldots,$$

$$\beta_{\frac{t}{2}}(y_1, y_2, \ldots, 0(y_{\frac{t}{2}}), \ldots, y_{t-1}) \; \forall \; t - 1 \in 2\mathbb{Z}^+ + 1,$$

$$\alpha_{\frac{t-1}{2}}(x_1, x_2, \ldots, 0(x_{\frac{t-1}{2}}), \ldots, x_t) \; \forall \; t \in 2\mathbb{Z}^+ + 1, \ldots,$$

$$\beta_{t-1}(y_1, y_2, \ldots, y_{t-1}), \alpha_t(x_1, x_2, \ldots, x_t),$$

where $t = \frac{n+2}{2}$, $\alpha_1, \alpha_2, \ldots, \alpha_{\frac{t-1}{2}}, \ldots, \alpha_t$ and $\beta_1, \beta_2, \ldots, \beta_{\frac{t}{2}}, \ldots, \beta_{t-1}$ are any linear maps on $F^t$ and $F^{t-1}$, respectively.

**Proof.** **Case 1** (when $t \in 2\mathbb{Z}^+ + 1$).

Given that

$$\tau(x_1, y_1, x_2, y_2, \ldots, y_{t-1}, x_t) = (\alpha_1(x_1, x_2, \ldots, x_t), \beta_1(y_1, y_2, \ldots, y_{t-1}),$$

$$\alpha_2(x_2, x_3, \ldots, x_t), \beta_2(y_2, y_3, \ldots, y_{t-1}), \ldots,$$

$$\beta_{\frac{t}{2}}(y_1, y_2, \ldots, 0(y_{\frac{t}{2}}), \ldots, y_{t-1}) \; \forall \; t - 1 \in 2\mathbb{Z}^+ + 1,$$

$$\alpha_{\frac{t-1}{2}}(x_1, x_2, \ldots, 0(x_{\frac{t-1}{2}}), \ldots, x_t) \; \forall \; t \in 2\mathbb{Z}^+ + 1, \ldots,$$

$$\beta_{t-1}(y_1, y_2, \ldots, y_{t-1}), \alpha_t(x_1, x_2, \ldots, x_t))$$

where $t = \frac{n+2}{2}$, $\alpha_1, \alpha_2, \ldots, \alpha_{\frac{t-1}{2}}, \ldots, \alpha_t$ and $\beta_1, \beta_2, \ldots, \beta_{\frac{t}{2}}, \ldots, \beta_{t-1}$ are any linear maps on $F^t$ and $F^{t-1}$, respectively.
Now let us consider the standard basis:

\[
\begin{align*}
\tau(e_1) &= \begin{bmatrix} \alpha_1(1,0,\ldots,0), & 0, & \ldots, & \alpha_t(1,0,\ldots,0) \end{bmatrix} \\
\tau(e_1) &= \begin{bmatrix} 0, & \beta_1(1,0,\ldots,0), & 0, & \ldots, & \beta_{t-1}(1,0,\ldots,0) \end{bmatrix} \\
\vdots \\
\tau(e_t) &= \begin{bmatrix} \alpha_1(1,0,\ldots,0), & \ldots, & \alpha_{t+1}(0,\ldots,0(x_{t+1}),\ldots,0), & 0, & \ldots, & \alpha_t(1,0,\ldots,0) \end{bmatrix}
\end{align*}
\]

Putting the above linear equations into a matrix, we have

\[
\begin{bmatrix}
\alpha_{11} & 0 & \alpha_{12} & \ldots & \alpha_{1t-1} & 0 & \alpha_{1t} \\
0 & \beta_{11} & 0 & \ldots & 0 & \beta_{1t-1} & o \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{t+1}{\frac{1}{2}} & 0 & \alpha_{t+1}{\frac{1}{2}} & \ldots & 0 & \ldots & \alpha_{t+1}{\frac{1}{2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \beta_{t-11} & 0 & \ldots & 0 & \beta_{t-1t-1} & 0 \\
\alpha_{t1} & 0 & \alpha_{t2} & \ldots & \alpha_{t-1} & 0 & \alpha_{tt} \\
\end{bmatrix}
\]

The transpose of the above matrix is the matrix of transformation denoted as

\[
m(\tau) = \begin{bmatrix}
\alpha_{11} & 0 & \alpha_{12} & \ldots & \alpha_{1t-1} & 0 & \alpha_{1t} \\
0 & \beta_{11} & 0 & \ldots & 0 & \beta_{1t-1} & o \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{t+1}{\frac{1}{2}} & 0 & \alpha_{t+1}{\frac{1}{2}} & \ldots & 0 & \ldots & \alpha_{t+1}{\frac{1}{2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \beta_{t-11} & 0 & \ldots & 0 & \beta_{t-1t-1} & 0 \\
\alpha_{t1} & 0 & \alpha_{t2} & \ldots & \alpha_{t-1} & 0 & \alpha_{tt} \\
\end{bmatrix}^T
\]

The result is a completely filled coupled matrix from which we have the even-dimensional rhotrix representation.

Conversely, suppose that \(\tau : F^n \rightarrow F^n\) has an even-dimensional rhotrix representation \(\langle \alpha_{ij}, \beta_{kl} \rangle\) in the standard basis. Then, the corresponding matrix representation of \(\tau\) is the com-
pletely filled coupled matrix given above. From this, we obtain the linear system below:

\[
\begin{align*}
\tau(e_1) &= [\alpha_1(1, 0, \ldots, 0), 0, \ldots, \alpha_t(1, 0, \ldots, 0)] \\
\tau(e_1) &= [0, \beta_1(1, 0, \ldots, 0), 0, \ldots, \beta_{t-1}(1, 0, \ldots, 0)] \\
\vdots \\
\tau(e_1) &= [\alpha_1(1, 0, \ldots, 0), 0, \ldots, \alpha_{t+1}(0, \ldots, 0)(x_{t+1}), \ldots, 0), \ldots, \alpha_t(1, 0, \ldots, 0)] \\
\vdots \\
\tau(e_t) &= [\alpha_1(0, \ldots, t), 0, \ldots, \alpha_t(0, \ldots, 1)]
\end{align*}
\]

Case 1 (when \( t - 1 \in 2\mathbb{Z}^+ + 1 \)).
The proof follows similarly.

**Remark 4.1.** The above theorem is seeing our even-dimensional rhotrix as a completely filled couple matrix.

**Example 4.1.** Consider the linear mapping \( \tau : \mathbb{R} \mapsto \mathbb{R} \) defined by \( \tau(x, y, z) = (ax + dz, 0, bx + ez) \). Find the hl-rhotrix represented by the linear transformation(linear map) \( \tau \) with respect to the standard basis.

**Solution:**

\[
\begin{align*}
\tau(1, 0, 0) &= (a, 0, b) \\
\tau(0, 1, 0) &= (0, 0, 0) \\
\tau(0, 0, 1) &= (d, 0, e)
\end{align*}
\]

Then, putting this into matrix gives

\[
\begin{pmatrix}
a & 0 & b \\
0 & 0 & 0 \\
d & 0 & e
\end{pmatrix}
\]

Thus, the matrix of representation is the transpose of the above matrix

\[
m(\tau)^T =
\begin{pmatrix}
a & 0 & d \\
0 & 0 & 0 \\
b & 0 & e
\end{pmatrix}
\]

which is a completely filled coupled matrix. Then the even-dimensional rhotrix by \( \tau \) is

\[
R(\tau) =
\begin{pmatrix}
a \\
b & d \\
e
\end{pmatrix}
\]

**Example 4.2.** Consider the linear mapping \( \tau : \mathbb{R} \mapsto \mathbb{R} \) defined by \( \tau(a, b, c, d, e) = (a + 2c - 5e, 3b + 6d, 4a + 10c, 8b - 11d, 9a + 12c + 13e) \). Find the even-dimensional rhotrix represented by the linear transformation(linear map) \( \tau \) with respect to the standard basis.
Thus, the matrix of representation is given below:

\[
m(\tau) = \begin{bmatrix}
1 & 0 & 4 & 0 & 9 \\
0 & 3 & 0 & 8 & 0 \\
2 & 0 & 0 & 0 & 12 \\
0 & 6 & 0 & -11 & 0 \\
-5 & 0 & 10 & 0 & 13
\end{bmatrix}^T = \begin{bmatrix}
1 & 0 & 2 & 0 & -5 \\
0 & 3 & 0 & 6 & 0 \\
4 & 0 & 0 & 0 & 10 \\
0 & 8 & 0 & -11 & 0 \\
9 & 0 & 12 & 0 & 13
\end{bmatrix}
\]

which is a completely filled coupled matrix. Then the even-dimensional rhotrix by \( \tau \) is

\[
R(\tau) = \begin{bmatrix}
1 \\
4 & 3 & 2 \\
9 & 8 & 6 & -5 \\
12 & -11 & 10 \\
13
\end{bmatrix}
\]

This is an even-dimensional rhotrix of dimension 4.

**Example 4.3.** Consider the linear mapping \( \tau : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( \tau(a, b, c, d, e, f, g) = (3a + 2c - 4g - 2e, 5b + 4d + 3f, 5a - 7c + 3e - g, 8b - 5f, 7a + 12c - 3e + 5g, -4b + 2d + f, a + 14c - 7e + 10g) \).

Find the even-dimensional rhotrix represented by the linear transformation (linear map) \( \tau \) with respect to the standard basis.

**Solution:**

\[
\begin{align*}
\tau(1, 0, 0, 0, 0, 0, 0) &= (3, 0, 5, 0, 7, 0, 1) \\
\tau(0, 1, 0, 0, 0, 0, 0) &= (0, 5, 0, 8, 0, -4, 0) \\
\tau(0, 0, 1, 0, 0, 0, 0) &= (2, 0, -7, 0, 12, 0, 14) \\
\tau(0, 0, 0, 1, 0, 0) &= (0, 4, 0, 0, 0, 0, 2, 0) \\
\tau(0, 0, 0, 0, 1, 0, 0) &= (-2, 0, 3, 0, -3, 0, -7) \\
\tau(0, 0, 0, 0, 0, 1, 0, 0) &= (0, 3, 0, -5, 0, 1, 0) \\
\tau(0, 0, 0, 0, 0, 0, 1, 0, 0) &= (-4, 0, -1, 0, 5, 0, 10)
\end{align*}
\]

Thus, the matrix of representation is given below:

\[
m(\tau) = \begin{bmatrix}
3 & 0 & 5 & 0 & 7 & 0 & 1 \\
0 & 5 & 0 & 8 & 0 & -4 & 0 \\
2 & 0 & -7 & 0 & 12 & 0 & 14 \\
0 & 4 & 0 & 0 & 0 & 2 & 0 \\
-2 & 0 & 3 & 0 & -3 & 0 & -7 \\
0 & 3 & 0 & -5 & 0 & 1 & 0 \\
-4 & 0 & -1 & 0 & 5 & 0 & 10
\end{bmatrix}^T = \begin{bmatrix}
3 & 0 & 2 & 0 & -2 & 0 & -4 \\
0 & 5 & 0 & 4 & 0 & 3 & 0 \\
5 & 0 & -7 & 0 & 3 & 0 & -1 \\
0 & 8 & 0 & 0 & 0 & -5 & 0 \\
7 & 0 & 12 & 0 & -3 & 0 & 5 \\
0 & -4 & 0 & 2 & 0 & 1 & 0 \\
1 & 0 & 14 & 0 & -7 & 0 & 10
\end{bmatrix}
\]
which is a completely filled coupled matrix. Then the even-dimensional rhotrix by \( \tau \) is

\[
R(\tau) = \begin{pmatrix}
3 & 5 & 5 & 2 \\
7 & 8 & -7 & 4 & -2 \\
1 & -4 & 12 & 3 & 3 & -4 \\
14 & 2 & -3 & -5 & -1 \\
-7 & 1 & 5 \\
10
\end{pmatrix}
\]

This is an even-dimensional rhotrix of dimension 6.

5 Conclusion

A strenuous effort was made to represent an even-dimensional rhotrix over a linear map. This representation showed that an even-dimensional rhotrix is a linear structure, and that it is a special type of rhotrix. All even-dimensional rhotrices are rhotrices except for the converse. Representing a rhotrix this way enables us to have by definition, even-dimensional rhotrices. Therefore, this work is an expansion and a contribution to rhotrix algebra.

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References


