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A note on the OEIS sequence A228059

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Abstract: The OEIS sequence A228059 lists odd numbers of the form $p^{1+4k}r^2$, where p is prime of the form $1+4m$, $r>1$, and $\gcd(p,r)=1$ that are closer to being perfect than previous terms. In this note, we present the prime factorizations of the first 37 terms of this sequence. We then predict appropriate conjectures for odd perfect numbers, in general.

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1 Introduction

Let x be a positive integer. Recall that we denote

$$\sum_{d|x} d = \sigma_1(x) = \sigma(x)$$

as the sum of divisors of x. Denote the abundancy index of x by $I(x)=\sigma(x)/x$, and the deficiency of x by $D(x)=2x-\sigma(x)$. Note that we have the identity

$$\frac{D(x)}{x}+\frac{\sigma(x)}{x}=\frac{D(x)}{x}+I(x)=2.$$

Note further that, if $y=\prod_{i=1}^{w} {z_i}^{s_i}$ is the prime factorization of y, then we have the formula

$$\sigma(y)=\sigma\left(\prod_{i=1}^{w} {z_i}^{s_i}\right)=\prod_{i=1}^{w}\left(\sigma({z_i}^{s_i})\right)=\prod_{i=1}^{w}\frac{{z_i}^{s_i+1}-1}{z_i-1},$$

where $w = \omega(y)$ is the number of distinct prime factors of y. This means that σ as a function satisfies $\sigma(ab) = \sigma(a)\sigma(b)$ if and only if $\gcd(a, b) = 1$, which means that σ is multiplicative.

Therefore, if $\gcd(a, b) = 1$, it follows from the above formula for σ that

$$I(ab) = \frac{\sigma(ab)}{ab} = \frac{\sigma(a)\sigma(b)}{ab} = \left(\frac{\sigma(a)}{a}\right) \cdot \left(\frac{\sigma(b)}{b}\right) = I(a)I(b)$$

which shows that the abundancy index I as a function is also multiplicative. Lastly, note that the deficiency D as a function is in general not multiplicative [9].

We say that a number N is perfect if $\sigma(N) = 2N$. The following result (due to Euclid and Euler) gives a necessary and sufficient condition for an even integer E to be perfect.

Theorem 1.1. *An even integer E is perfect if and only if $E = (2^p - 1)2^{p-1}$ for some integer p for which $2^p - 1$ is prime.*

Refer to Dickson [3] to see different proofs of Theorem 1.1. Prime numbers of the form $2^p - 1$ are called Mersenne primes, and if $2^p - 1$ is prime then p must be prime. (The converse of this last statement does not hold.) Observe that 6, 28, 496, and 8128 are examples of even perfect numbers, and these correspond to the Mersenne primes $2^p - 1$ with p given by 2, 3, 5, and 7, respectively. We still do not know if there are infinitely many even perfect numbers. Also, it is not known if there are odd perfect numbers. It is widely believed that no odd perfect numbers exist.

If there is an odd perfect number O, then Euler proved that it must have the form $O = q^s n^2$, where q is a prime satisfying $q \equiv s \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. We call q the special or Euler prime of O, q^s is the Euler factor, and n^2 is the non-Euler factor. (Notice that both E and O have the forms $N = Q^K M^2$ where Q is prime, $K \equiv 1 \pmod 4$, and $\gcd(Q, M) = 1$.) Descartes, Frenicle, and subsequently Sorli [10] predicted that $s = 1$ always holds. Sorli conjectured $s = 1$ after testing large odd numbers N' with $\omega(N') = 8$ for perfection. More recently, Beasley [2] reports that "Dickson has documented Descartes' conjecture as occurring in a letter to Marin Mersenne [on November 15,] 1638, with Frenicle's subsequent observation occurring in 1657".

Holdener [5] presented some conditions equivalent to the existence of odd perfect numbers. In [4], Dris gives some conditions equivalent to the Descartes–Frenicle–Sorli conjecture.

The OEIS sequence A228059 [6] lists odd numbers of the form $p^{1+4k} r^2$, where p is prime of the form $1 + 4m$, $r > 1$, and $\gcd(p, r) = 1$ that are closer to being perfect than previous terms. In this note, we present the prime factorizations of the first 37 terms of this sequence. We then predict appropriate conjectures for odd perfect numbers, in general.

2 Some observations on the first 37 terms of the OEIS sequence A228059

Tony D. Noe computed the first 10 terms of the OEIS sequence A228059, using *Mathematica*. Giovanni Resta computed the 11-th till 37-th terms, using another language and a special algorithm.

Here are the first 37 terms of the OEIS sequence A228059, together with their prime factorizations:

$$45 = 5 \cdot 3^2$$
$$405 = 5 \cdot 3^4$$
$$2205 = 5 \cdot (3 \cdot 7)^2$$
$$26325 = 13 \cdot (3^2 \cdot 5)^2$$
$$236925 = 13 \cdot (3^3 \cdot 5)^2$$
$$1380825 = 17 \cdot (3 \cdot 5 \cdot 19)^2$$
$$1660725 = 61 \cdot (3 \cdot 5 \cdot 11)^2$$
$$35698725 = 61 \cdot (3^2 \cdot 5 \cdot 17)^2$$
$$3138290325 = 53 \cdot (3^4 \cdot 5 \cdot 19)^2$$
$$29891138805 = 5 \cdot (3^2 \cdot 11^2 \cdot 71)^2$$
$$73846750725 = 509 \cdot (3 \cdot 5 \cdot 11 \cdot 73)^2$$
$$194401220013 = 21557 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$194509436121 = 21569 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$194581580193 = 21577 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$194689796301 = 21589 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$194798012409 = 21601 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$194906228517 = 21613 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$194942300553 = 21617 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$195230876841 = 21649 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$195339092949 = 21661 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$195447309057 = 21673 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$195699813309 = 21701 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$195808029417 = 21713 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$196024461633 = 21737 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$196204821813 = 21757 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$196349109957 = 21773 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$196745902353 = 21817 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$196781974389 = 21821 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$196962334569 = 21841 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$

$$197323054929 = 21881 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$197431271037 = 21893 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$197755919361 = 21929 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$197828063433 = 21937 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$198044495649 = 21961 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$198188783793 = 21977 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$198369143973 = 21997 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$
$$198513432117 = 22013 \cdot (3 \cdot 7 \cdot 11 \cdot 13)^2$$

The authors used *WolframAlpha* to compute these prime factorizations one by one. Note that each of the first 37 terms of the OEIS sequence A228059 have a p with exponent 1.

Furthermore, note that the non-Euler factor value (n^2) of

$$(3 \cdot 7 \cdot 11 \cdot 13)^2$$

is deficient-perfect [1], with deficiency

$$D((3 \cdot 7 \cdot 11 \cdot 13)^2) = D(9018009) = 819,$$

and that this condition is known to be equivalent [4] to the Descartes–Frenicle–Sorli conjecture that $s = 1$, if $O = q^s n^2$ is an odd perfect number with special/Euler prime q. Lastly, note that the Descartes–Frenicle–Sorli conjecture is true for $O = q^s n^2$ an odd perfect number with special/Euler prime q if and only if

$$\frac{n^2}{D(n^2)} = \frac{q+1}{2}.$$

Substituting $n^2 = 9018009$ and $D(n^2) = 819$, we obtain

$$\frac{q+1}{2} = \frac{9018009}{819} = 11011,$$

from which we finally get

$$q = 22022 - 1 = 22021.$$

Notice that 22021 is not prime, as it can be factorized as

$$22021 = 19^2 \cdot 61.$$

This then finally gives the Descartes spoof

$$\mathscr{D} = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021 = 198585576189.$$

Lastly, note that the Descartes spoof

$$\mathscr{D} = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021 = 198585576189$$

is not a member of the OEIS sequence A228059, because of the following reasoning:

The abundancy index $I(x) = \sigma(x)/x$ (where $\sigma(x)$ is the sum of the divisors of $x \in \mathbb{N}$) of the first 9 terms of the OEIS sequence A228059 are:

$$\begin{aligned}
I(45) &= \frac{26}{15} \approx 1.73333 \\
I(405) &= \frac{242}{135} \approx 1.79259 \\
I(2205) &= \frac{494}{245} \approx 2.01633 \\
I(26325) &= \frac{52514}{26325} \approx 1.99483 \\
I(236925) &= \frac{474362}{236925} \approx 2.00216 \\
I(1380825) &= \frac{307086}{153425} \approx 2.00154 \\
I(1660725) &= \frac{3323138}{1660725} \approx 2.00102 \\
I(35698725) &= \frac{71396534}{35698725} \approx 1.99997 \\
I(3138290325) &= \frac{77488034}{38744325} \approx 1.99998
\end{aligned}$$

Notice that, by the definition of the OEIS sequence A228059, $|I(x_i) - 2|$ must be a strictly decreasing sequence.

Therefore, since

$$I(198585576189) = \frac{23622}{11011} \approx 2.14531,$$

it follows that the Descartes spoof

$$\mathscr{D} = 198585576189$$

is not a member of the OEIS sequence A228059.

We state this result as our lone lemma for this section.

Lemma 2.1. *The Descartes spoof*

$$\mathscr{D} = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021 = 198585576189$$

is not a member of the OEIS sequence A228059.

3 Conjectures

Using the data in the previous section, we predict the truth of the following conjectures:

Conjecture 3.1. *If $O = q^s n^2$ is an odd perfect number with special/Euler prime q, then $s = 1$.*

Conjecture 3.2. *If $O = q^s n^2$ is an odd perfect number with special/Euler prime q, then $3 \mid O$.*

Conjecture 3.3. *If $O = q^s n^2$ is an odd perfect number with special/Euler prime q, then $5 \nmid O$.*

Remark 3.1. *Conjectures 3.2 and 3.3 are consistent with the observation that $105 \nmid O$, if $O = q^s n^2$ is an odd perfect number with special/Euler prime q.*

4 Further research

The terms of the OEIS sequence A228059 are determined by an inequality rather than an equation. This directly contributes to the overall difficulty of the problem. Perhaps investigating a few more terms of the OEIS sequence A228059 will shed light into the problem of existence of odd perfect numbers. Per an e-mail response from Giovanni Resta [7]: "Given the very small abundance of 198513432117 (i.e., $\sigma(198513432117) - 2(198513432117) = 6552$), (he) suspect(s) the next number is quite large."

Resta also indicated that we need to use the right language (which is not *Mathematica*) and the right algorithm (which is not computing $\sigma(x)$ by means of prime factorization for all the odd numbers), in order to use less computing memory and therefore speed up the computation for the succeeding terms of the sequence.

Remark 4.1. *After this paper was initially submitted to NNTDM, Giovanni Resta [8] communicated a term past the* 37*-th. Here is his e-mail message: "After your email I got curious. I knew the approach I used to search up to* 10^{12} *was not suited to extend the search much further, so I wrote another little program using a different method. I do think that the next term is*

$$283665529390725 = 15349 \cdot (3^3 \cdot 5 \cdot 19 \cdot 53)^2,$$

but since the method is not exhaustive I'm not 100% *sure."*

That 283665529390725 *is a member of the OEIS sequence A228059, can be proved using a method similar to that used to prove Lemma 2.1.*

Remark 4.2. *Note that*

$$3 \mid 283665529390725$$

and

$$5 \mid 283665529390725.$$

This additional datum, in a certain sense, lends further support to Conjecture 3.2 and "disproves" Conjecture 3.3. However, even with further data points, we cannot really be 100% *sure.*

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