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Generalized Fibonacci numbers and Bernoulli polynomials

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Abstract: Relationships, in terms of equations and congruences, are developed between the Bernoulli numbers and arbitrary order generalizations of the ordinary Fibonacci and Lucas numbers. Some of these are direct connections and others are analogous similarities.
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1 Introduction

A generating function for Fibonacci polynomials $\{u_n(x)\}$ [3, 4] was demonstrated in [3] to be

$$\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = \exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right) \tag{1.1}$$

in which

$$u_n = \begin{cases} \sum_{j=1}^{r} (-1)^{j+1} P_j u_{n-j}, & n > 0, \\ 1, & n = 0, \\ 0, & n < 0, \end{cases} \tag{1.2}$$

and

$$v_n = \begin{cases} \sum_{j=1}^{r} (-1)^{j+1} P_j v_{n-j}, & n \geq r, \\ \sum_{j=1}^{r} \alpha_j^n, & 0 \leq n < r, \\ 0, & n < 0, \end{cases} \tag{1.3}$$

where the P_j are arbitrary integers and the α_j are the distinct roots of the auxiliary equation

$$x^r = \sum_{j=1}^{r} (-1)^{j+1} P_j x^{r-j}. \tag{1.4}$$

From (1.1) it follows that

$$u_n = \frac{1}{n!} u_n(0),$$

and

$$\sum_{n=0}^{\infty} u_n(x) \frac{t^n}{n!} = e^{xt} \left(\sum_{n=0}^{\infty} u_n(0) \frac{t^n}{n!} \right),$$

which are comparable with similar results for the Bernoulli polynomials such as

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = e^{xt} \left(\sum_{n=0}^{\infty} B_n(0) \frac{t^n}{n!} \right).$$

Here we prove a more direct connection between these generalized Fibonacci numbers and the Bernoulli numbers, namely

$$n u_{n-1}(0) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(0) \Delta u_k(0), \tag{1.5}$$

in which the difference operator Δ [2] is defined as

$$\Delta u_n(x) = u_n(x+1) - u_n(x).$$

2 Proof of Fibonacci–Bernoulli connection

The proof of (1.5):

$$\begin{aligned} \sum_{n=0}^{\infty} \Delta u_n(x) \frac{t^n}{n!} &= e^{(x+1)t} \sum_{n=o}^{\infty} u_n(0) \frac{t^n}{n!} - e^{xt} \sum_{n=0}^{\infty} u_n(0) \frac{t^n}{n!} \\ &= \left(e^t - 1\right) e^{xt} \sum_{n=0}^{\infty} u_n(0) \frac{t^n}{n!} \end{aligned}$$

and so

$$\frac{t}{e^t - 1} \sum_{n=0}^{\infty} \Delta u_n(x) \frac{t^n}{n!} = t e^{xt} \sum_{n=0}^{\infty} u_n(0) \frac{t^n}{n!}.$$

But

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},$$

so

$$\frac{t}{e^t-1}\sum_{n=0}^{\infty}\Delta u_n(x)\frac{t^n}{n!}=\sum_{m=0}^{\infty}B_m\frac{t^m}{m!}\sum_{n=0}^{\infty}\Delta u_n(x)\frac{t^n}{n!}.$$
$$=\sum_{n=0}^{\infty}\sum_{n=0}^{n}\binom{n}{k}.B_{n-k}\Delta u_k(x)\frac{t^n}{n!}$$

and we also have that

$$te^{xt}\sum_{n=0}^{\infty}u_n(0)\frac{t^n}{n!}=t\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\sum_{n=0}^{\infty}u_n(0)\frac{t^n}{n!}$$
$$=t\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}u_{n-k}(0)x^k\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}u_{n-k}(0)x^k\frac{t^{n+1}}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k}u_{n-k-1}(0)x^k\frac{t^n}{(n-1)!}$$

from which we get the required result on equating coefficients of t^n when $x = 0$.

3 Fibonacci and Bernoulli congruences

The Appell set criterion was established in [6], namely,

$$\frac{d}{dx}u_n(x)=nu_{n-1}(x),\ n=1,2,3,...,$$

from which we can obtain

$$\int_x^{x+1}u_n(y)dy=\frac{\Delta u_{n+1}(x)}{n+1},$$

which parallels the known result [1]

$$\int_x^{x+1}B_n(y)dy=\frac{\Delta B_{n+1}(x)}{n+1}.$$

Just as there are many Bernoulli polynomial congruences [7], so there are too for the Fibonacci polynomials. For instance, we can also show that

$$u_{n+tm}(x)\equiv u_n(x)(u_m(x))^t\ (\mathrm{mod}\, m). \tag{3.1}$$

To prove this, we use induction on t and m.

When $t = 0$, it is obvious for all n.

When $t = 1$ and $n = 1$, we note that

$$u_1(x) = x + v_1$$

and

$$u_{m+1}(x) = (x + v_1)u_m(x) + \sum_{j=1}^{m} v_{j+1} m^{\underline{j}} u_{m-j}(x),$$

in which we have used the falling factorial coefficient $m^{\underline{j}}$ [5], and

$$\sum_{j=1}^{m+s} (m+s)^{\underline{j}} \equiv \sum_{j=1}^{m+s} s^{\underline{j}} \pmod{m} \equiv \sum_{j=1}^{s} s^{\underline{j}},$$

which we use below. Thus,

$$\begin{aligned} u_{m+1}(x) &\equiv (x + v_1)u_m(x) \pmod{m} \\ &\equiv u_1(x)u_m(x) \pmod{m}. \end{aligned}$$

Assume the result is true for $n = 2, 3, \ldots, s$; that is

$$u_{m+n}(x) \equiv u_m(x)u_n(x) \pmod{m}.$$

Then

$$\begin{aligned} u_{m+s+1}(x) &= (x + v_1)u_{m+s}(x) + \sum_{j=1}^{m+s} v_{j+1}(m+s)^{\underline{j}} u_{m+s-j} \\ &\equiv (x + v_1)u_{m+s}(x) + \sum_{j=1}^{s} v_{j+1} s^{\underline{j}} u_{m+s-j} \pmod{m} \\ &\equiv (x + v_1)u_{m+s}(x) + \sum_{j=1}^{s} v_{j+1} s^{\underline{j}} u_{s-j}\, u_m(x) \pmod{m} \\ &\equiv (x + v_1)u_m(x)u_s(x) + \sum_{j=1}^{s} v_{j+1} s^{\underline{j}} u_{s-j}(x) \pmod{m} \\ &\equiv u_m(x)\left((x + v_1)u_s(x) + \sum_{j=1}^{s} v_{j+1} s^{\underline{j}} u_{s-j}(x) \right) \pmod{m} \\ &\equiv u_m(x)u_{s+1}(x) \pmod{m} \end{aligned}$$

so when $t = 1$, for all n

$$u_{n+m}(x) \equiv u_n(x)u_m(x) \pmod{m}$$

and when $t = 2$, for all n,

$$u_{n+2m}(x) \equiv u_n(x)(u_m(x))^2 \pmod{m}.$$

Assume the result holds for $t = 3, 4, \ldots, k$

$$\begin{aligned} u_{n+(k+1)m}(x) &\equiv u_{n+km}(x)u_m(x) \pmod{m} \\ &\equiv u_n(x)(u_m(x))^{k+1} \pmod{m}. \end{aligned}$$

□

To continue the Bernoulli connections, it follows that for $n = 0, 1, 2, \ldots$

$$u_n(x) - (u_m(x))^t - u_{n+tm}(x) = \sum_{j=-n}^{tm} B_j(n)u_{n+j}(x) \tag{3.2}$$

in which the $B_j(n)$ are also polynomials depending on n with integral coefficients modulo m.

We then have the following result. Let the monic polynomial elements of the set

$$\{u_0(x), u_1(x), ..., u_n(x)\}$$

with coefficients modulo m such that

$$u_s(x) = \sum_{j=0}^{s} a_{s,j} x^j \ \ (a_{s,s} = 1, 0 \le s \le n)$$

and

$$\sum_{s=0}^{n} A_s u_s(x) \equiv 0 \ (\mathrm{mod}\ m),$$

where A_s ($s = 0, 1, 2, \ldots, n$) are integral modulo m. Then,

$$A_s \equiv 0 \ (\mathrm{mod}\, m) \ (0 \le s \le n). \tag{3.3}$$

Proof:

$$\begin{aligned}\sum_{s=0}^{n} A_s u_s(x) &= \sum_{s=0}^{n}\sum_{j=0}^{s} A_s a_{s,j} x^j \\ &\equiv 0 \ (\mathrm{mod}\, m).\end{aligned}$$

Now

$$\sum_{s=0}^{n}\sum_{j=0}^{s} A_s a_{s,j} x^j = [A_0, A_1, A_2, ..., A_n] \begin{bmatrix} a_{0,0} & 0 & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & 0 & \cdots & 0 \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & 0 \\ & & & \cdots & \\ a_{n,0} & a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \cdots \\ x^n \end{bmatrix}, \tag{3.4}$$

so that

$$\sum_{j=0}^{n} x^j \sum_{s=j}^{n} A_s a_{s,j} \equiv 0 \ (\mathrm{mod}\, m),$$

which implies that

$$\sum_{s=j}^{n} A_s a_{s,j} \equiv 0 \ (\mathrm{mod}\, m),$$

since if $u_s(x)$ is a polynomial with integral coefficients, the statement $u_s(x) \equiv 0 \ (\mathrm{mod}\, m)$ means that each coefficient of $u_s(x)$ is divisible by m. Thus, $m \mid A_s$, $s = 0, 1, 2, \ldots, n$, because of the triangularity of the matrix in (3.4). This completes the proof of (3.3). ☐

Concluding comment

These ideas can be extended to results which further connect the Fibonacci and Bernoulli recurrence relations.

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