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The identities for generalized Fibonacci numbers via orthogonal projection

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Abstract: In this paper, we consider the space $R(p,1)$ of generalized Fibonacci sequences and orthogonal bases of this space. Using these orthogonal bases, we obtain the orthogonal projection onto a subspace $R(p,1)$ of $\mathbb{R}^n$. By using the orthogonal projection, we obtain the identities for the generalized Fibonacci numbers.
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1 Introduction

The generalized Fibonacci numbers G_n are defined by the following recurrence relation for $n \geq 1$

$$G_{n+1} = pG_n + G_{n-1} \tag{1}$$

with the initial conditions $G_0 = a,\ G_1 = b$. The characteristic equation of recurrence (1) is

$$\lambda^2 - p\lambda - 1 = 0. \tag{2}$$

The Binet's formula for the generalized Fibonacci numbers G_n is

$$G_n = A\alpha^n + B\beta^n,$$

where α and β are roots of the equation (2) and $A = \frac{(b-a\beta)}{\alpha-\beta}$, $B = \frac{a\alpha - b}{\alpha-\beta}$.

In particular, taking $a = 0$ in the initial conditions of the generalized Fibonacci number, we obtain the sequence $\{u_n\}$. Namely, the sequence $\{u_n\}$ is defined as

$$u_{n+1} = pu_n + u_{n-1}; \quad u_0 = 0, \quad u_1 = b.$$

Similarly, taking $2b$ and pb instead of a and b, we obtain the sequence $\{v_n\}$, which is defined by the following recurrence

$$v_{n+1} = pv_n + v_{n-1}; \quad v_0 = 2b, \quad v_1 = pb.$$

The Binet's formula for the numbers u_n and v_n is

$$u_n = b\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right); \quad v_n = b\left(\alpha^n + \beta^n\right).$$

Also, we can give some identities between u_n and v_n, as follows

$$pv_{n-2} = u_n - u_{n-4}, \tag{3}$$

$$u_n u_{n+k} = \frac{b}{p^2+4}\left(v_{2n+k} + (-1)^{n+1} v_k\right), \tag{4}$$

$$u_{2m} = \frac{1}{b} u_m v_m, \tag{5}$$

$$v_n = pu_n + 2u_{n-1}, \tag{6}$$

$$v_m v_{m+1} - u_m u_{m+1}\left(p^2+4\right) = 2\left(-1\right)^m b^2 p. \tag{7}$$

Taking $p = k$, $a = 0$ and $b = 1$ in (1), we have the k-Fibonacci numbers. Also, if we take $p = 2$, $a = b = 1$ in (1), we obtain the modified Pell numbers. We can give a few values of $\{G_n\}$, $\{u_n\}$ and $\{v_n\}$ sequences as the following table

n	G_n	u_n	v_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$b\left(p^2+1\right) - ap\left(p^2+2\right)$	$b\left(p^2+1\right)$	$-pb\left(p^2+3\right)$
-2	$a\left(p^2+1\right) - pb$	$-pb$	$b\left(p^2+2\right)$
-1	$b - pa$	b	$-pb$
0	a	0	$2b$
1	b	b	pb
2	$pb + a$	pb	$b\left(p^2+2\right)$
3	$b\left(p^2+1\right) + ap$	$b\left(p^2+1\right)$	$pb\left(p^2+3\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now, we give the theorems about orthogonal projection.

Theorem 1 ([2], p. 204, Theorem 5.1.4). *Consider a vector* $\overrightarrow{x} \in \mathbb{R}^n$ *and subspace* V *of* $\mathbb{R}^n$. *Then we can write* $\overrightarrow{x} = \overrightarrow{x}^{\shortparallel} + \overrightarrow{x}^{\perp}$ *where* $\overrightarrow{x}^{\shortparallel}$ *is in* V *and* $\overrightarrow{x}^{\perp}$ *is perpendicular to* V, *and this represantation is unique. The vector* $\overrightarrow{x}^{\shortparallel}$ *is called the orthogonal projection of* $\overrightarrow{x}$ *onto* V, *denoted by* $proj_V(\overrightarrow{x})$. *The transformation* $T(\overrightarrow{x}) = proj_V(\overrightarrow{x}) = \overrightarrow{x}^{\shortparallel}$ *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *is linear.*

Theorem 2 ([2], p. 206, Theorem 5.1.5). *If* V *is a subspace of* $\mathbb{R}^n$ *with orthonormal basis* $u_1, u_2, \ldots, u_m$, *then*

$$proj_V(x) = \langle u_1, x\rangle u_1 + \cdots + \langle u_m, x\rangle u_m \tag{8}$$

for all x *in* $\mathbb{R}^n$.

By using the equation (8), we can give the matrix of orthogonal projection as the following theorem.

Theorem 3 ([2], p. 232, Theorem 5.3.10). *Consider a subspace* V *of* $\mathbb{R}^n$ *with orthonormal basis* $u_1, u_2, \ldots, u_m$. *The matrix* P *of the orthogonal projection onto* V *is*

$$P = u_1 u_1^T + \cdots + u_m u_m^T.$$

Theorem 4 ([4], p. 365, Theorem 6.12). *The projection matrix* P *for subspace* V *of* $\mathbb{R}^n$ *is both idempotent and symmetric. Conversely, every* $n \times n$ *matrix that is both idempotent and symmetric is a projection matrix.*

Many authors have investigated the second order recurrence sequences. In particular, we consider studies connected with the Generalized Fibonacci and Horadam numbers. In [6], the author defines certain sequences and gives the properties of the certain sequences. In [7], the authors consider Horadam numbers and Horadam polynomials. Also Horzum gives the properties of Horadam polynomials. In [9], the author investigates sums of Horadam sequence. In [8], the authors consider the general Fibonacci numbers and give some interesting properties. Dupree and Mathes investigate the singular values of Hankel matrices with k-Fibonacci and k-Lucas numbers. Also, they give the orthogonal projection onto the two dimensional space of k-Fibonacci and k-Lucas sequences in [3]. In [5], the authors consider the orthogonal projection onto the two dimensional space of k-Fibonacci and k-Lucas sequences in [3], and give a new proof of obtained results by Dupree and Mathes. In [1], the authors give some identities of the Pell, modified Pell and Pell–Lucas sequences via orthogonal projection.

Motivated by the above papers, we investigate the orthogonal projection onto the two dimensional space of Generalized Fibonacci sequences. We can see that the obtained orthogonal projection matrix is a Hankel matrix with the elements of $\{u_n\}$ sequence entries. Also, we give the identities for the Generalized Fibonacci numbers, the elements of $\{u_n\}$ and $\{v_n\}$ sequence by using the orthogonal projection matrix. Since k-Fibonacci and modified Pell numbers are special cases of the generalized Fibonacci number G_n, all results in this paper generalized results from [1, 3, 5].

2 Main results

Let $R\left(p,1\right)$ denote the subspace of $\mathbb{R}^n$, consisting of the $(G_i) \in \mathbb{R}^n$ which

$$G_{i+1} = pG_i + G_{i-1}$$

for $i = 1, 2, \ldots, n$. The elements of $R\left(p,1\right)$ whose first two coordinates a and b will be denoted (G_i) and is called the Generalized Fibonacci sequence.

In this paper, we obtain the matrix of orthogonal projection onto $R\left(p,1\right)$ as follows

$$\frac{p}{u_n}\begin{pmatrix} u_{-n+1} & u_{-n+2} & \ldots & u_0 \\ u_{-n+2} & u_{-n+3} & \ldots & u_1 \\ \vdots & \vdots & \ddots & \vdots \\ u_0 & u_1 & \ldots & u_{n-1} \end{pmatrix} \tag{9}$$

for an even n. Let us note that, the matrix

$$H_u = \begin{pmatrix} u_{-n+1} & u_{-n+2} & \ldots & u_0 \\ u_{-n+2} & u_{-n+3} & \ldots & u_1 \\ \vdots & \vdots & \ddots & \vdots \\ u_0 & u_1 & \ldots & u_{n-1} \end{pmatrix}$$

is called the central Hankel matrix with elements of the sequence $\{u_n\}$ entries. Replacing u_i with v_i yields H_v the central Hankel matrix. For an odd n, we obtain orthogonal projection matrix onto $R\left(p,1\right)$ which is connected with the central Hankel matrix H_v.

Theorem 5. *For an even n, the spectral norm of $\frac{p}{u_n}$ H_u matrix is*

$$\left\| \frac{p}{u_n} H_u \right\|_2 = 1.$$

Proof. For an even n, the characteristic polynomial of $\dfrac{p}{u_n}$ H_u is

$$\left| \lambda I - \frac{p}{u_n} H_u \right| = \lambda^{n-2}(\lambda - 1)^2.$$

Hence, the roots of the characteristic equation $\left| \lambda I - \dfrac{p}{u_n} H_u \right| = 0$ are

$$\lambda_{1,2} = 1$$

and $\lambda_k = 0$ for $k = 3, 4, \ldots, n$. The spectral norm is the maximum eigenvalue of the matrix due to H_u symetric matrix. Clearly,

$$\left\| \frac{p}{u_n} H_u \right\|_2 = 1.$$

□

Theorem 6. *For odd* n $(n=2m+1)$ *, the eigenvalues of the Hankel matrix* H_v *are*

$$\lambda_1=\frac{v_m v_{m+1}}{pb}\ ,\ \lambda_2=\frac{(p^2+4)\,u_m u_{m+1}}{pb}$$

and $\lambda_k=0$ *for* $k=3,4,\ldots,n$. *Eigenvectors of corresponding eigenvalues* λ_1 *and* λ_2

$$u=(u_{-m},\ldots,u_0,\ldots,u_m)^T \text{ and } v=(v_{-m},\ldots,v_0,\ldots,v_m)^T .$$

Proof. For odd n $(n=2m+1)$, the characteristic polynomial of H_v is

$$|\lambda I-H_v|=\lambda^{n-2}(\lambda^2-\left(\frac{2v_{2m+1}}{p}\right)\lambda+\frac{(p^2+4)\,(u_{2m+1}^2-b^2)}{p^2})$$

The roots of the characteristic equation are

$$\lambda_1=\frac{v_m v_{m+1}}{pb},\ \lambda_2=\frac{(p^2+4)\,u_m u_{m+1}}{pb}$$

and $\lambda_k=0$ for $k=3,4,\ldots,n$. The eigenvectors corresponding eigenvalues λ_1 and λ_2 are

$$u=(u_{-m},\ldots,u_0,\ldots,u_m)^T$$

and

$$v=(v_{-m},\ldots,v_0,\ldots,v_m)^T .$$

□

Theorem 7. *For odd* n $(n=2m+1)$ *, the spectral norm of* H_v *matrix is*

$$\|H_v\|_2=\begin{cases}\dfrac{v_m v_{m+1}}{pb}, & m \text{ is even}\\[2ex] \dfrac{(p^2+4)\,u_m u_{m+1}}{pb}, & m \text{ is odd}\end{cases}.$$

Proof. H_v is a symmetric matrix, then the spectral norm of H_v matrix is the maximum eigenvalue of the matrix. Thus, the spectral norm of H_v matrix is

$$\|H_v\|_2=\begin{cases}\dfrac{v_m v_{m+1}}{pb}, & m \text{ is even}\\[2ex] \dfrac{(p^2+4)\,u_m u_{m+1}}{pb}, & m \text{ is odd}\end{cases}.$$

□

Now, we will prove the orthogonal projection matrix of the matrix $\frac{p}{u_n}H_u$ in (9). Therefore, we consider the matrix of order 2 as follows

$$E=\begin{pmatrix}0 & b\\ b & pb\end{pmatrix}.$$

It follows that

$$E^m = b^{m-1} \begin{pmatrix} u_{m-1} & u_m \\ u_m & u_{m+1} \end{pmatrix} \tag{10}$$

for all integers m.

Lemma 8. *For any integer t and any nonnegative integer s, we have*

$$\sum_{i=0}^{s} \frac{E^{t+4i}}{b^{t+4i-1}} = \frac{u_{2(s+1)}}{pb} \frac{E^{t+2s}}{b^{t+2s-1}}. \tag{11}$$

Proof. It suffices to prove the identity

$$\sum_{i=0}^{s} u_{t+4i} = \frac{u_{2(s+1)}}{pb} u_{t+2s}.$$

Using the Binet's formula for the $\{u_n\}$ sequence, we have

$$\begin{aligned}
\sum_{i=0}^{s} u_{t+4i} &= \sum_{i=0}^{s} b\left(\frac{\alpha^{t+4i} - \beta^{t+4i}}{\alpha - \beta}\right) \\
&= b\frac{\alpha^t}{\alpha - \beta}\sum_{i=0}^{s}\alpha^{4i} - b\frac{\beta^t}{\alpha - \beta}\sum_{i=0}^{s}\beta^{4i} \\
&= b\left[\frac{(\alpha^{4s+t} - \beta^{4s+t}) - (\alpha^{4s+t+4} - \beta^{4s+t+4}) - (\alpha^{t-4} - \beta^{t-4}) + (\alpha^t - \beta^t)}{(\alpha - \beta)(\alpha^4 - 1)(\beta^4 - 1)}\right] \\
&= -\frac{1}{p^4 + 4p^2}\left(u_{4s+t} - u_{4s+t+4} - u_{t-4} + u_t\right).
\end{aligned}$$

From the identity (3), we obtain

$$\sum_{i=0}^{s} u_{t+4i} = \frac{1}{p^3 + 4p}\left(v_{4s+t+2} - v_{t-2}\right).$$

Taking $n = 2s + 2$ and $k = t - 2$ in the identity (4), we have

$$\sum_{i=0}^{s} u_{t+4i} = \frac{u_{2(s+1)}}{pb} u_{t+2s}.$$

□

Theorem 9. *For an even n, the matrix $\frac{p}{u_n} H_u$ is orthogonal projection matrix onto $R(p, 1)$.*

Proof. The orthogonal projection matrix is both symmetric and idempotent. The matrix $\frac{p}{u_n} H_u$ is clearly symmetric. Therefore, we need to show only that $\frac{p}{u_n} H_u$ is idempotent. Namely, we will prove that

$$H_u^2 = \frac{u_n}{p} H_u.$$

We can express the matrix H_u as

$$H_u = \begin{pmatrix} \frac{E^{-n+2}}{b^{-n+1}} & \frac{E^{-n+4}}{b^{-n+3}} & \cdots & \frac{E^{-2}}{b^{-3}} & \frac{E^{0}}{b^{-1}} \\ \frac{E^{-n+4}}{b^{-n+3}} & \frac{E^{-n+6}}{b^{-n+5}} & \cdots & \frac{E^{0}}{b^{-1}} & \frac{E^{2}}{b} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{E^{-2}}{b^{-3}} & \frac{E^{0}}{b^{-1}} & \cdots & \frac{E^{n-6}}{b^{n-7}} & \frac{E^{n-4}}{b^{n-5}} \\ \frac{E^{0}}{b^{-1}} & \frac{E^{2}}{b} & \cdots & \frac{E^{n-4}}{b^{n-5}} & \frac{E^{n-2}}{b^{n-3}} \end{pmatrix}$$

with the matrix E^m. Taking $n = 2m$, we have

$$H_u = \left[\frac{E^{2(i+j)-2(m+1)}}{b^{2(i+j)-2(m+1)-1}} \right]_{i,j=1}^{m}.$$

Using the Lemma 6, we obtain

$$\begin{aligned} H_u^2 &= \left[\sum_{r=1}^{m} \frac{E^{2(i+r)-2(m+1)}}{b^{2(i+r)-2(m+1)-1}} \frac{E^{2(r+j)-2(m+1)}}{b^{2(r+j)-2(m+1)-1}} \right]_{i,j=1}^{m} \\ &= \left[\sum_{r=1}^{m} \frac{E^{2(i+j)-4m+4(r-1)}}{b^{2(i+j)-4m+4(r-1)-2}} \right]_{i,j=1}^{m} \\ &= \frac{u_{2m}}{p} \left[\frac{E^{2(i+j)-4m+2(m-1)}}{b^{2(i+j)-4m+2(m-1)-1}} \right]_{i,j=1}^{m} \\ &= \frac{u_n}{p} \left[\frac{E^{2(i+j)-2(m+1)}}{b^{2(i+j)-2(m+1)-1}} \right]_{i,j=1}^{m} \\ &= \frac{u_n}{p} H_u. \end{aligned}$$

□

Using the idempotency of the matrix $\frac{p}{u_n} H_u$, we obtain the identity for the elements of $\{u_n\}$ sequence as

$$\frac{p}{u_n} u_{i+j-n+1} = \frac{p^2}{u_n^2} \sum_{k=0}^{n-1} u_{i+k-n+1} u_{j+k-n+1}$$

for all even n and $-n+1 \leq i, j \leq n-1$. Thus, we give the following corollary.

Corollary 10. *For an even n and $-n+1 \leq i, j \leq n-1$, we have*

$$u_n u_{i+j-n+1} = p \sum_{m=0}^{n-1} u_{i-m} u_{j-m}.$$

Now, we give the matrix of orthogonal projection onto $R(p,1)$ for the even values of n. Hence, we consider the different orthogonal bases of the space $R(p,1)$.

Assume that n is even and $s = (G_0, G_1, \ldots, G_{n-2}, G_{n-1}) \in R(p,1)$, which is a column vector. We define

$$s^{\perp} = (-G_{n-1}, G_{n-2}, \ldots, -G_1, G_0)^T.$$

It is clear that $\left\{s, s^{\perp}\right\}$ is an orthogonal basis for the space $R\left(p,1\right)$. Normalizing s and $s^{\perp}$, we consider the t and w vectors as

$$t=\frac{s}{\|s\|},\ w=\frac{s^{\perp}}{\|s^{\perp}\|},$$

where

$$\|s\|^2=\left\|s^{\perp}\right\|^2=\frac{G_nG_{n-1}+(pa^2-ab)}{p}.$$

From Theorem 3, the matrix of orthogonal projection onto $R\left(p,1\right)$ is

$$P=tt^T+ww^T, \tag{12}$$

which is the Hankel matrix in (9).

Theorem 11. *For an even n and $0\leq i,j\leq n-1$, we have*

$$u_{i+j-n+1}\left(G_nG_{n-1}+\left(pa^2-ab\right)\right)=u_n\left(G_iG_j+(-1)^{i+j}G_{n-i-1}G_{n-j-1}\right). \tag{13}$$

Proof. By equalizing the ij-th entries of the matrices in (9) and (12), we have

$$\frac{p}{G_nG_{n-1}-G_0G_{-1}}\left(G_iG_j+(-1)^{i+j}G_{n-i-1}G_{n-j-1}\right)=\frac{p}{u_n}u_{i+j-n+1}.$$

From $G_0=a$ and $G_{-1}=b-pa$, we obtain

$$u_{i+j-n+1}\left(G_nG_{n-1}+\left(pa^2-ab\right)\right)=u_n\left(G_iG_j+(-1)^{i+j}G_{n-i-1}G_{n-j-1}\right).$$

□

By using the above theorem, we have the identities for the elements of $\{G_n\}$ and $\{u_n\}$ sequences as follows.

For $i=j=0$ and $i=j=n-1$ in (13), we obtain

$$\frac{u_n}{u_{-n+1}}=\frac{G_nG_{n-1}+(pa^2-ab)}{a^2+G_{n-1}^2}.$$

Taking $i=j=\dfrac{n}{2}$ in (13), we have

$$\frac{(G_nG_{n-1}+(pa^2-ab))\,b}{u_n}=G_{\frac{n}{2}}^2+\ G_{\frac{n}{2}-1}^2.$$

Let $i=n-1,\ j=1$ in (13), then

$$b\left(G_nG_{n-1}+\left(pa^2-ab\right)\right)=u_n\left(bG_{n-1}+aG_{n-2}\right).$$

Let α and β be roots of the characteristic equation (2). Now, we consider the another orthogonal basis of $R(p,1)$, which is $\{s,t\}$, where

$$s=\left(1,\alpha,\alpha^2,\ldots,\alpha^{n-1}\right)^T \text{ and } t=\left(1,\beta,\beta^2,\ldots,\beta^{n-1}\right)^T$$

The orthogonal projection matrix onto $R(p,1)$ is

$$P=\frac{1}{\|s\|^2}ss^T+\frac{1}{\|t\|^2}tt^T. \tag{14}$$

This matrix is a Hankel matrix with the elements of the sequence $\{u_n\}$ in (9).

Theorem 12. *For n is even, we have*

$$\frac{\alpha^{i+j+1}}{\alpha^{2n}-1}+\frac{\beta^{i+j+1}}{\beta^{2n}-1}=\frac{1}{u_n}u_{i+j-n+1}. \tag{15}$$

for all $0\leq i,j\leq n-1$.

Proof. By equalizing the ij-th entries of the matrices in (9) and (14), we have

$$\frac{\alpha^2-1}{\alpha^{2n}-1}\alpha^{i+j}+\frac{\beta^2-1}{\beta^{2n}-1}\beta^{i+j}=\frac{p}{u_n}u_{i+j-n+1}.$$

For $\alpha^2-1=p\alpha$ and $\beta^2-1=p\beta$, the result is clear. □

In particular, taking $i=1$ and $j=n-1$ in (15), we have

$$\frac{\alpha^{n+1}}{\alpha^{2n}-1}+\frac{\beta^{n+1}}{\beta^{2n}-1}=\frac{1}{u_n}u_1.$$

Using $\alpha\beta=-1$ and an even n,

$$\left(\frac{\alpha^{n+1}}{\alpha^{2n}-1}+\frac{\beta^{n+1}}{\beta^{2n}-1}\right)(\alpha^n-\beta^n)=(\alpha-\beta).$$

Hence, the Binet's formula for the elements of $\{u_n\}$ sequence appears as follows

$$u_n=b\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right).$$

Now, we give the matrix of orthogonal projection onto $R(p,1)$ for the odd values of n, $n=2m+1$. Therefore, we consider the eigenvectors u and v of the matrix H_v, which are

$$u=(u_{-m},\ldots,u_0,\ldots,u_m)^T \text{ and } v=(v_{-m},\ldots,v_0,\ldots,v_m)^T,$$

respectively. We have the norms of u and v

$$\|u\|^2=\frac{2u_mu_{m+1}}{p} \text{ and } \|v\|^2=\frac{2v_mv_{m+1}}{p}.$$

Also, using the fact that $u_{-i} = (-1)^{i+1} u_i$ and $v_{-i} = (-1)^i v_i$, we have

$$\sum_{i=-m}^{m} u_i v_i = -\sum_{i=1}^{m} u_i v_i + u_0 v_0 + \sum_{i=1}^{m} u_i v_i = 0.$$

Namely $\{u, v\}$ is an orthogonal basis of $R(p, 1)$. The orthogonal projection matrix onto $R(p, 1)$ is given by

$$P = \frac{p}{2u_m u_{m+1}} uu^T + \frac{p}{2v_m v_{m+1}} vv^T. \tag{16}$$

The following theorem gives the second expression for this projection.

Theorem 13. *The orthogonal projection matrix in (16) is*

$$\frac{p^2 (-1)^{m+1}}{(p^2+4) u_{2m} u_{2m+2}} vv^T + \frac{pb}{u_m u_{m+1} (p^2+4)} H_v. \tag{17}$$

Proof. Let us consider the Hankel matrix with elements of $\{v_n\}$ sequence entries, H_v. The eigenvalues of H_v are $\dfrac{u_m u_{m+1}(p^2+4)}{pb}$ and $\dfrac{v_m v_{m+1}}{pb}$. The eigenvectors u and v are connected with these eigenvalues. Thus

$$\begin{aligned} H_v &= \frac{u_m u_{m+1}(p^2+4)}{pb} \frac{p}{2u_m u_{m+1}} uu^T + \frac{v_m v_{m+1}}{pb} \frac{p}{2v_m v_{m+1}} vv^T \\ &= \frac{p^2+4}{2b} uu^T + \frac{1}{2b} vv^T. \end{aligned}$$

The projection matrix in (16) is

$$\begin{aligned} P &= \frac{p}{2u_m u_{m+1}} uu^T + \frac{p}{2v_m v_{m+1}} vv^T \\ &= \frac{2b}{p^2+4} \frac{p}{2u_m u_{m+1}} \left(H_v - \frac{1}{2b} vv^T \right) + \frac{p}{2v_m v_{m+1}} vv^T \\ &= \left(\frac{p}{2v_m v_{m+1}} - \frac{bp}{2b(p^2+4) u_m u_{m+1}} \right) vv^T + \frac{pb}{(p^2+4) u_m u_{m+1}} H_v. \end{aligned}$$

Using the identities (5) and (7), we simplify the above equation. Namely, we have

$$P = \frac{p^2 (-1)^{m+1}}{(p^2+4) u_{2m} u_{2m+2}} vv^T + \frac{pb}{u_m u_{m+1} (p^2+4)} H_v.$$

□

Corollary 14. *If n is odd, then*

$$\frac{p}{2u_m u_{m+1}} u_i u_j + \frac{p}{2v_m v_{m+1}} v_i v_j = \frac{p^2 (-1)^{m+1}}{(p^2+4) u_{2m} u_{2m+2}} v_i v_j + \frac{pb}{u_m u_{m+1}(p^2+4)} v_{i+j} \tag{18}$$

for $-m \leq i, j \leq m$.

Proof. By equalizing the ij-th entries of the matrices in (16) and (17), the result is clear. □

Taking $i = j = m$ in (18), we have the identity for the elements of $\{u_n\}$ and $\{v_n\}$ sequences as follows

$$bu_{4m+2} = v_{2m}u_{2m+2} - pb^2.$$

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