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Partitions generated by Mock Theta Functions $\rho(q)$, $\sigma(q)$ and $\nu(q)$ and relations with partitions into distinct parts

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Abstract: From two-line matrix interpretations of Mock Theta Functions $\rho(q)$, $\sigma(q)$ and $\nu(q)$ introduced in [5], we have obtained identities for the partitions generated by their respective general terms, whose proofs are done in a completely combinatorial way. We have also obtained relations between partitions into two colours generated by $\rho(q)$ and $\sigma(q)$, and also by $\nu(q)$.

Keywords: Mock Theta Function, Integer partition, Combinatorial interpretation, Partition enumeration.
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1 Introduction

A new combinatorial way to represent integer partitions as two-line matrices was introduced in [6]. The same idea was extended in [4] to specific types of partitions, allowing the authors to prove a great amount of new results. In a similar way, Brietzke et al. in [5] presented two-line matrix representations for the coefficients of some Mock Theta Functions. Without considering the signal when it appears, the general terms of these functions can be interpreted as generating functions for specific types of integer partitions.

Matrix representations for partitions have been useful to express other kinds of partitions. By considering the First Roger-Ramanujan's Identity, for example, partitions into 2-distinct parts are also represented as two-line matrices in [6]. Concerning the other side of the identity, partitions into parts congruent to $\pm 1 \pmod 5$ have their two-line matrix representation given in [7], where nice information was derived.

Considering the models given in [5], a study of Mock Theta Functions $\phi(q)$, $\psi(q)$, $f_0(q)$, $F_0(q)$, $f_1(q)$ and $F_1(q)$ as Generating Functions for partitions into one colour was done in [3]. In a similar way, the present work focus on Mock Theta Functions $\rho(q)$, $\sigma(q)$, $\nu(q)$ and $\lambda(q)$, the first two and the last one related to partitions into two colours. Although it is not apparent, we can associate some partitions generated by $\nu(q)$, which consider only one colour, to a specific type of partitions into two colours brought forth by $\lambda(q)$. As a consequence, we prove an identity for $\nu(q)$ in a combinatorial way.

The second line of matrices representing partitions generated by those Mock Theta Functions also describes properties of the related partition. By summing the elements of the second line, we can classify the partitions according to these sums and organize the data in a table, as done in [1] for two matrix representations of unrestricted partitions. It allows us to investigate and discover properties, unknown until then, that are suggested by the table.

2 Mock Theta Function $\rho(q)$

In this section, we study the Mock Theta Function of order 6

$$\rho(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}}, \tag{1}$$

and obtain results and patterns that are suggested by a table, whose construction is made according to the sum of the second line of its matrix representation, found in [5].

The general term

$$\frac{(1+q)(1+q^2)\cdots(1+q^s)q^{1+2+3+\cdots+s}}{(1-q)(1-q^3)\cdots(1-q^{2s+1})},$$

is the generating function for partitions of n into parts of two different colours:

- dark gray parts, ranging from 1 to s, with no gaps, and each part having multiplicity 1 or 2;
- any number of light gray odd parts less than or equal to $2s+1$.

In [5], we find the following combinatorial interpretation for this function in terms of two-line matrices.

Theorem 2.1. *The coefficient of q^n in the expansion of (1) is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{2}$$

with non-negative integer entries satisfying

$$\begin{aligned} &c_s = 0; \\ &c_t = i_t + c_{t+1} + 2d_{t+1}, \text{ with } i_t \in \{1,2\}, \forall t < s; \\ &n = \sum c_t + \sum d_t. \end{aligned}$$

Notation 1. From now on, in order to differentiate dark and light gray parts, we indicate the light ones by writing them inside a box. So, a partition in which $(5,3,1,1)$ are the light parts and $(2,1,1)$ are the dark gray ones will be expressed by

$$(\boxed{5}, \boxed{3}, 2, \boxed{1}, \boxed{1}, 1, 1).$$

Example 2.2. *Looking at the first few terms of the expansion*

$$\rho(q) = 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + 11q^6 + 14q^7 + 18q^8 + 24q^9 + \cdots,$$

we can see that there are 11 *partitions of* 6 *into parts we described above. Consequently, there are* 11 *matrices of type* (2) *whose sum of its terms is equal to* 6*. They are shown below.*

The entries in the second row of the matrices of type (2) describe the light gray odd parts of the partition associated to each matrix. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s+1$.

Definition 2.3. Let $p_\rho(n,k)$ be the number of partitions of n into parts of two different colours, counted by the general term of (1), having k light gray odd parts. We denote by $P_\rho(n,k)$ the set of partitions counted by $p_\rho(n,k)$.

Example 2.4. $p_\rho(10,3) = 5$ *and* $P_\rho(10,3) = \big\{(\boxed{3}, \boxed{3}, \boxed{3}, 1), (\boxed{3}, \boxed{3}, 2, \boxed{1}, 1),$ $(\boxed{5}, 2, \boxed{1}, \boxed{1}, 1, 1), (\boxed{3}, 2, 2, \boxed{1}, \boxed{1}, 1), (3, 2, \boxed{1}, \boxed{1}, \boxed{1}, 1, 1)\big\}.$

For a fixed n, we classify its partitions of type described in Definition 2.3 according to the sum on the second row of the matrix associated to each partition. By counting the appearance of each number in these sums, we can organize the data on a table, which is presented next. The entry in line n and column $n - j$ is the number of times j appears as sum of the entries of the second row in type (2) matrices.

By observing the table above, we see that the columns become constant below certain entries. This result is described as follows:

Partitions from $\rho(q)$	Matrices of type (2)
$(3,2,1)$	$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$(2,2,1,1)$	$\begin{pmatrix} 4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(2,2,\boxed{1},1)$	$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}$
$(\boxed{3},2,1)$	$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$(\boxed{3},\boxed{1},1,1)$	$\begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$
$(\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$
$(\boxed{3},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$
$(\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix}$
$(\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1})$	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$

Theorem 2.5. *For $n, k \geq 0$, we have*

(i) $p_\rho(3n+4+k, n+1+k) = p_\rho(3n+4, n+1)$;

(ii) $p_\rho(3n+5+k, n+1+k) = p_\rho(3n+5, n+1)$.

Proof. We prove only the first item by exhibiting a bijection between the two sets of partitions. The second one has a similar proof.

A map we build from $P_\rho(3n+4, n+1)$ to $P_\rho(3n+4+k, n+1+k)$ is simply described by adding i light gray parts 1 to partitions lying in the first set. Clearly, the resulting partition lies in the second set.

In order to check that this map is in fact a bijection, we must ensure that a partition counted by $p_\rho(3n+4+k, n+1+k)$ always has k light gray parts of size 1. So, let us suppose that there

Table 1. Table from the characterization given by Theorem 2.1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	1																								
2	1	1	1																							
3	1	1	1	1																						
4	1	1	1	2	1																					
5	1	1	1	2	2	1																				
6	1	1	1	2	2	2	2																			
7	1	1	1	2	2	3	3	1																		
8	1	1	1	2	2	3	4	3	1																	
9	1	1	1	2	2	3	4	4	4	2																
10	1	1	1	2	2	3	4	5	5	4	2															
11	1	1	1	2	2	3	4	5	6	6	5	2														
12	1	1	1	2	2	3	4	5	6	7	8	5	2													
13	1	1	1	2	2	3	4	5	6	8	9	8	6	2												
14	1	1	1	2	2	3	4	5	6	8	10	10	10	7	2											
15	1	1	1	2	2	3	4	5	6	8	10	11	13	11	7	3										
16	1	1	1	2	2	3	4	5	6	8	10	12	14	14	13	9	3									
17	1	1	1	2	2	3	4	5	6	8	10	12	15	16	17	15	9	3								
18	1	1	1	2	2	3	4	5	6	8	10	12	15	17	20	20	16	10	3							
19	1	1	1	2	2	3	4	5	6	8	10	12	15	18	21	23	23	19	11	3						
20	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	25	27	26	21	12	4					
21	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	26	30	31	30	24	13	4				
22	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	31	34	37	34	26	15	4			
23	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	36	41	42	38	30	16	4		
24	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	37	44	47	48	45	32	17	5	
25	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	45	50	55	56	49	36	19	5

are x_{2j-1} light gray parts $2j-1$, with $x_1 < k$. As each dark gray part j has multiplicity $1+l_j$, with $l_j = 0$ or $l_j = 1$, then

$$\begin{aligned} 3n+4+k &= (1+l_1)\cdot 1 + (1+l_2)\cdot 2 + \cdots + (1+l_s)\cdot s + \sum_{j=1}^{s+1}(2j-1)\cdot x_{2j-1} \\ &= \sum_{j=1}^{s}(1+l_j)\cdot j + \sum_{j=2}^{s+1}(2j-1)\cdot x_{2j-1} + x_1 \\ &\geq \sum_{j=1}^{s}(1+l_j)\cdot j + \sum_{j=2}^{s+1} 3\cdot x_{2j-1} + x_1 \\ &\geq \sum_{j=1}^{s}(1+l_j)\cdot j + 3\cdot(n+1+k-x_1) + x_1. \end{aligned}$$

So,

$$1+k \geq \sum_{j=1}^{s}(1+l_j)\cdot j + 3k - 2x_1$$

$$1 + k > \sum_{j=1}^{s}(1 + l_j) \cdot j + 3k - 2k$$
$$1 > \sum_{j=1}^{s}(1 + l_j) \cdot j,$$

which is a contradiction. □

Since the columns become constant, we can see that this fixed values represent the same sequence as the one for the number of partitions of certain integers into distinct parts. This result is described next.

Theorem 2.6. *For $n \geq 0$, we have*

(i) $p_\rho(3n + 1, n) = p_d(2n + 1)$,

(ii) $p_\rho(3n + 2, n) = p_d(2n + 2)$.

Proof. We prove the first item by exhibiting a bijection between sets $P_\rho(3n+1, n)$ and $P_d(2n+1)$. The same map also holds for item *(ii)*.

We classify each partition lying in $P_\rho(3n + 1, n)$ according to its largest dark gray part, which is s, from the definition of the Mock Theta Function $\rho(q)$. As we also know, every positive part smaller than or equal to s must appear and there are n light gray odd parts smaller than $2s + 1$.

The following steps describe the bijection between both sets, each one of them illustrated by an example.

- Step 1: Given a partition in $P_\rho(3n + 1, n)$, consider its Young Diagram. Split it in two parts: the light gray and the dark gray one.

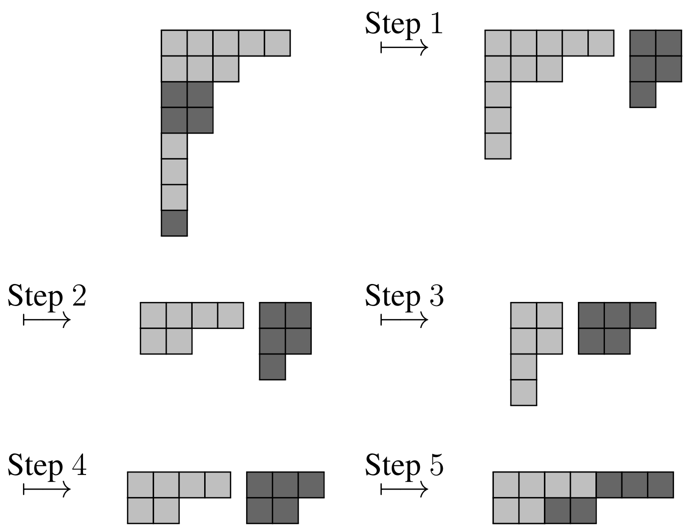
- Step 2: Decrease each light gray part by 1. Thus we get at most n light gray even parts smaller than or equal to $2s$.

- Step 3: Conjugate both partitions. In the dark gray one, as the largest part was s, now we have exactly s parts and in the light gray one, at most $2s$ parts.

- Step 4: Note that all dark gray parts are now distinct, because all numbers from 1 to s must appear. Since we have an even number of parts in the light gray partition, merge pairs of equal parts.

- Step 5: Finally, add both partitions side by side.

Example 2.7. *Taking $n = 5$, we start with* $(\boxed{5}, \boxed{3}, 2, 2, \boxed{1}, \boxed{1}, \boxed{1}, 1)$, *which lies in* $P_\rho(16, 5)$.



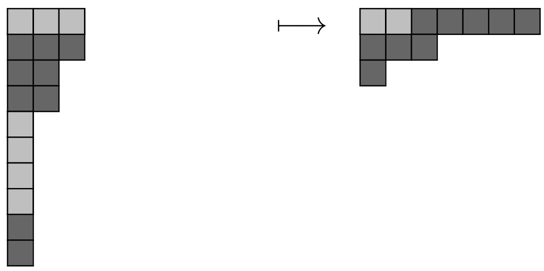
As image of $(\boxed{5}, \boxed{3}, 2, 2, \boxed{1}, \boxed{1}, \boxed{1}, 1)$ *we get* $(7, 4)$*, which lies in* $P_d(11)$.

Note that we always merge light gray parts of even size. Hence, if in the original dark gray partition the part s appears twice, the resulting smallest part will be even. In case it appears once, it will be odd.

Moreover, if the dark gray part k from the original partition has multiplicity 2, the resulting λ_k and λ_{k+1} parts have the same parity. In case of multiplicity 1, they have distinct parity. We can see this in the following two cases for $n = 5$.

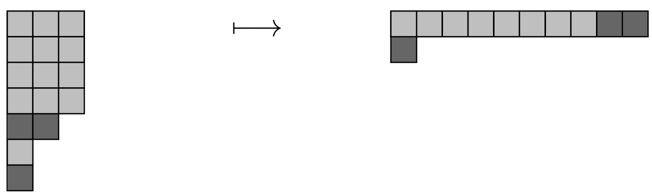
Example 2.8. *Case with dark gray parts with multiplicity* 2*:*

$$(\boxed{3}, 3, 2, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, 1, 1) \quad \longmapsto \quad (\boxed{2} + 5, 3, 1)$$



Case with dark gray parts with multiplicity 1*:*

$$(\boxed{3}, \boxed{3}, \boxed{3}, \boxed{3}, 2, \boxed{1}, 1) \quad \longmapsto \quad (\boxed{8} + 2, 1)$$



Furthermore, the number of parts in the resulting partition is the same as the number of largest dark gray parts of size s in the original partition.

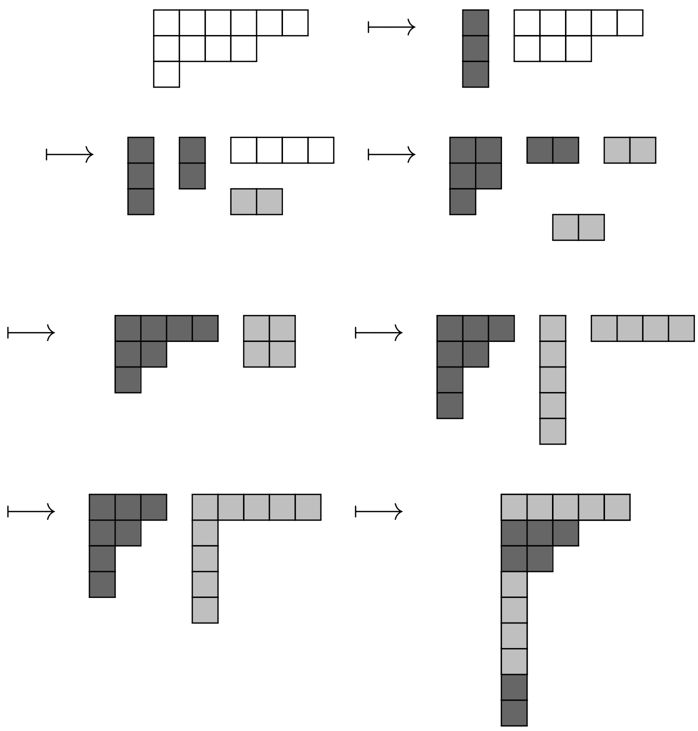
Based on the previous remarks, we are able to describe the inverse map. Taking a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of $2n+1$ into distinct parts, we need to know how many times each dark gray part from 1 to s will appear in the resulting partition.

We start by observing the smallest part λ_s. If it is odd, the part s in the resulting partition will have multiplicity 1. Otherwise, s will have multiplicity 2. According to this multiplicity, we remove the first or the first two columns of λ's Young Diagram, and also what is left from its s-th row, and save this parts we removed. We repeat the previous step for every remaining part in order to discover all dark gray parts of the resulting partition.

Consider the removed columns and rows from each of the s steps above. Organize the columns by joining them side by side, and the rows by joining them one above the other, in the same order they were removed in the previous steps. If these rows are of even size, split them into two equal parts. After all this adjustments, conjugate the remaining partitions and add the partition $(1, 1, \ldots, 1)$ with n parts to the light gray partition. Finally, join both partitions together.

In order to illustrate the inverse map, we describe it step-by-step in the following example.

Example 2.9. *Again with* $n = 5$*, consider* $(6, 4, 1)$ *which lies in* $P_d(11)$*. According to the inverse map, its image is* $(\boxed{5}, 3, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, 1, 1)$*, as we can see next.*



So, the map we described is a bijection and the equality in item (i) holds. □

3 Mock Theta Function $\sigma(q)$

In this section we consider the Mock Theta Function of order 6

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}. \tag{3}$$

Its general term
$$\frac{(1+q)(1+q^2)\cdots(1+q^s)q^{1+2+3+\cdots+(s+1)}}{(1-q)(1-q^3)(1-q^5)\cdots(1-q^{2s+1})},$$
generates the partitions of n into parts of two different colours:

- dark gray parts, ranging from 1 to $s+1$, with no gaps, the largest one with multiplicity exactly one and the others with multiplicity 1 or 2;
- any number of light gray odd parts less than or equal to $2s+1$.

Some partition identities we find in this section are similar to those we obtained from Table 1. In addition, the columns of $\sigma(q)$'s table, constructed similarly to the previous one, reveal a relation between partitions generated by $\sigma(q)$ and $\rho(q)$.

The Mock Theta Function $\sigma(q)$ may also be interpreted in a combinatorial way, as given by the next theorem, found in [5].

Theorem 3.1. *The coefficient of q^n in the expansion of (3) is equal to the number of elements in the set of matrices of the form*
$$A=\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{4}$$
with non-negative integer entries satisfying
$$\begin{aligned} c_s &= 1; \\ c_t &= i_t + c_{t+1} + 2d_{t+1}, \text{ with } i_t \in \{1,2\}, \forall t < s; \\ n &= \sum c_t + \sum d_t. \end{aligned}$$

Example 3.2. *Looking at the first few terms of the expansion*
$$\sigma(q) = q + q^2 + 2q^3 + 3q^4 + 3q^5 + 5q^6 + 7q^7 + 8q^8 + 11q^9 + \cdots,$$
we can see that there are 11 partitions of 9 into parts we described above. Consequently, there are 11 matrices of type (4) whose sum of the second line elements is equal to 9. They are shown below.

The entries in the second row of the matrices of type (4) describe the light gray odd parts of the partition associated to each matrix. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s+1$.

Definition 3.3. Let $p_\sigma(n,k)$ be the number of partitions of n into parts of two different colours, counted by the general term of (3), having k light gray odd parts. We denote by $P_\sigma(n,k)$ the set of partitions counted by $p_\sigma(n,k)$.

For a fixed n, we classify its partitions of type described in Definition 3.3 according to the sum on the second row of the matrix associated to each partition. Similar to the one in the previous section, we construct a table whose entry in line n and column $n-j$ is the number of times j appears as sum of the entries of the second row in type (4) matrices.

Table 2. Table of partitions from Example 6

Partitions from $\sigma(q)$	Matrices of type (4)
$(3,2,2,1,1)$	$\begin{pmatrix} 5 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(3,2,2,\boxed{1},1)$	$\begin{pmatrix} 4 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
$(3,2,\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$
$(3,2,\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix}$
$(\boxed{3},3,2,1)$	$\begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$(\boxed{3},\boxed{3},2,1)$	$\begin{pmatrix} 7 & 1 \\ 0 & 2 \end{pmatrix}$
$(\boxed{3},2,\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 3 & 1 \\ 5 & 0 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 2 & 1 \\ 6 & 0 \end{pmatrix}$
$(\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$
$(\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 1 \\ 8 \end{pmatrix}$

Remark 3.4. *From the definition of the Mock Theta Function of order* 5

$$1+2\Psi_0(q)=\sum_{n=0}^{\infty}(-1;q)_n q^{\binom{n+1}{2}},$$

which represents partitions whose parts appear at most twice, the largest part s *appears once and every integer smaller than* s *is also a part, it is easy to see that* $p_\sigma(n,0)$ *represents the coefficients of the Mock Theta Function* Ψ_0 *above.*

As it happens in the table of Mock Theta Function $\rho(q)$, the columns in the table of Mock Theta Function $\sigma(q)$ also become constant from certain values of n onwards, which leads us to the following theorem. Its proof is analogous to the one in Theorem 2.5.

Table 3. Table from the characterization given by Theorem 3.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																								
2	1	0																							
3	1	0	1																						
4	1	0	1	1																					
5	1	0	1	1	0																				
6	1	0	1	1	1	1																			
7	1	0	1	1	1	2	1																		
8	1	0	1	1	1	2	1	1																	
9	1	0	1	1	1	2	2	2	1																
10	1	0	1	1	1	2	2	3	2	1															
11	1	0	1	1	1	2	2	3	2	3	1														
12	1	0	1	1	1	2	2	3	3	4	3	1													
13	1	0	1	1	1	2	2	3	3	5	4	3	2												
14	1	0	1	1	1	2	2	3	3	5	4	5	4	1											
15	1	0	1	1	1	2	2	3	3	5	5	6	6	3	2										
16	1	0	1	1	1	2	2	3	3	5	5	7	7	6	5	2									
17	1	0	1	1	1	2	2	3	3	5	5	7	7	8	8	5	1								
18	1	0	1	1	1	2	2	3	3	5	5	7	8	9	10	8	6	2							
19	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	11	10	6	2						
20	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	13	13	10	7	2					
21	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	14	15	14	13	7	3				
22	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	17	18	13	8	3			
23	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	19	21	18	16	9	2		
24	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	20	23	22	23	17	9	3	
25	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	24	25	28	24	19	11	3

Theorem 3.5. *For $n, k \geq 0$, we have*

(i) $p_\sigma(3n+3+k, n+1+k) = p_\sigma(3n+3, n+1)$;

(ii) $p_\sigma(3n+1+k, n+k) = p_\sigma(3n+4, n+1)$.

The fixed values in σ's table columns are also related to partitions of certain integers into distinct parts, greater than or equal to 2. This result is described next.

Theorem 3.6. *Denoting by $p_d(n, \geq k)$ the number of partitions into distinct parts greater than or equal to k, for $n \geq 1$ we have*

(i) $p_\sigma(3n, n-1) = p_d(2n, \geq 2)$;

(ii) $p_\sigma(3n+1, n-1) = p_d(2n+1, \geq 2)$.

Proof. Again, this theorem is proved by setting a bijection between two different sets of partitions. In order to avoid extensive details, we only highlight some adjustments we make in the proof of Theorem 2.6, so that we have an analogous proof for Theorem 3.6.

When in Theorem 2.6 we decreased by one each part from the light gray partition, now we must also decrease the largest dark parts by one. It is possible since we always have one dark part. The remaining steps are equal to the original map.

In the inverse map, we set how many dark gray parts we would have according to the parity of the parts. Unlike there, now if λ_i is odd, there will be two parts of size i in the resulting partition. Otherwise there will be just one part of size i. This occurs because now we have the dark part $s+1$, which does not appear in $\rho(q)$. □

By observing the constant values in the columns of Tables 1 and 3, we can observe that the sum of two certain numbers in Table 3 also appears in Table 1. This property is proved next.

Corollary 3.7. *For all* $n \geq 1$*, we have:*

(i) $p_\sigma(3n, n-1) + p_\sigma(3n+1, n-1) = p_\rho(3n+1, n)$;

(ii) $p_\sigma(3n+1, n-1) + p_\sigma(3n+3, n) = p_\rho(3n+2, n)$.

Proof. Both items have analogous proofs, so we present only the proof of item (i).

By Theorems 2.6 and 3.6, statement (i) is equivalent to

$$p_d(2n+1, \geq 2) + p_d(2n, \geq 2) = p_d(2n+1).$$

This identity can be easily proved by splitting the set whose cardinality is counted by $p_d(2n+1)$ into two new ones: one of partitions of $2n+1$ into distinct parts with no part of size 1, and the other of partitions of $2n+1$ into distinct parts having 1 as a part. Noting that the last set has cardinality equal to $p_d(2n, \geq 2)$, the corollary follows. □

Combining Corollary 3.7 and Theorems 2.5 and 3.5, we get another relation between Mock Theta Functions $\rho(q)$ and $\sigma(q)$.

Corollary 3.8. *For all* $n \geq 1$*, we have*

$$p_\sigma(2n-1, n-1) + p_\sigma(2n+1, n) = p_\rho(2n-1, n-1).$$

Although Corollary 3.8 can be obtained from previous results, alternatively we can demonstrate it by a bijective proof, which is described next.

Alternative proof for Corollary 3.8. Let $P_\rho(2n-1, n-1)$ be the set of all partitions counted by $p_\rho(2n-1, n-1)$, and define its two disjoint subsets as follows.

- $P_\rho(2n-1, n-1)^*$: the set of partitions such that, if s is the largest dark part, it appears twice, or the largest light part is $2s+1$.
- $P_\rho(2n-1, n-1)^\#$: the set of partitions such that the largest dark part s appears once and the largest light part is smaller than or equal to $2s-1$. So we have

$$P_\rho(2n-1, n-1) = P_\rho(2n-1, n-1)^\# \cup P_\rho(2n-1, n-1)^*$$

Now, separate all the partitions according to the set they belong. As an example, take $n = 9$:

	$P_\rho(17,8)$
$P_\rho(17,8)^{\#}$	$(\boxed{3},\boxed{3},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
	$(\boxed{3},3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$
	$(3,2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$
$P_\rho(17,8)^{*}$	$(\boxed{3},\boxed{3},\boxed{3},\boxed{3},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
	$(\boxed{5},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
	$(\boxed{3},\boxed{3},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
	$(\boxed{5},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
	$(3,3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$

It is easy to see that $P_\rho(2n-1,n-1)^{\#} = P_\sigma(2n-1,n-1)$. So, it remains to be proved that $|P_\rho(2n-1,n-1)^{*}| = |P_\sigma(2n+1,n)|$, which is done by exhibiting a bijection between both sets. Let λ be a partition lying in $P_\rho(2n-1,n-1)^{*}$. We define the map according to the following steps:

- If the largest dark part s of λ appears twice, increase one of them by 1 and add $\boxed{1}$ as a light gray part. Thus we have a partition of $2n+1$ into n light gray parts and the dark ones ranging from 1 to $s+1$, with possible repetition of parts between 1 and $s-1$. Hence, a partition lying in $P_\sigma(2n+1,n)$.
- In case s appears once and the largest light gray part of λ is $\boxed{2s+1}$, split it into two new dark parts s and $s+1$. Then add two light parts $\boxed{1}$. Once again, we have a partition lying in $P_\sigma(2n+1,n)$ with dark parts $1,\ldots,s,s,s+1$.

For example, for $n = 9$ it works as follows.

$$(\boxed{3},\boxed{3},\boxed{3},\boxed{3},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longmapsto (\boxed{3},\boxed{3},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$$

$$(\boxed{5},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longmapsto (\boxed{3},3,2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$$

$$(\boxed{3},\boxed{3},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longmapsto (\boxed{3},\boxed{3},3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$$

$$(\boxed{5},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longmapsto (\boxed{5},3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$$

$$(3,3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longmapsto (4,3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1).$$

The inverse map depends on how many dark parts of size s the partition lying in $P_\sigma(2n+1,n)$ has. In case it has two, merge one part of size s and one of size $s+1$, getting a light gray part $\boxed{2s+1}$, and remove two parts $\boxed{1}$. Otherwise, decrease the part $s+1$ by 1 and remove one part $\boxed{1}$. □

4 Mock Theta Function $\nu(q)$

Overlooking signs in coefficients of Mock Theta Function of order 5

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}},$$

we see that they represent some kind of partitions, as before, related to certain partitions into distinct parts. In this section, we present these relations and also an identity that easily follows from a formula found in [2].

Consider the unsigned version of function $\nu(q)$,

$$\nu(-q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q^2)_{n+1}}. \tag{5}$$

Its general term

$$\frac{q^{2+4+6+\cdots+2s}}{(1-q)(1-q^3)\cdots(1-q^{2s+1})}$$

generates the partitions of n containing exactly one copy of each of the even parts $2, 4, \ldots, 2s$ and any number of the odd parts less than or equal to $2s+1$. Again, we can find in [5] an interpretation in terms of two-line matrices for this partitions.

Theorem 4.1. *The coefficient of q^n in the expansion of (5) is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s+1} \\ d_1 & d_2 & \cdots & d_{s+1} \end{pmatrix}, \tag{6}$$

with non-negative integer entries satisfying

$$\begin{aligned} c_{s+1} &= 0; \\ c_t &= 2 + c_{t+1} + 2d_{t+1},\ \forall t \leq s; \\ n &= \sum c_t + \sum d_t. \end{aligned}$$

The entries in the second row of the matrices of type (6) describe the odd parts from 1 to $2s+1$ of the partition associated to each matrix. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s$.

Definition 4.2. Let $p_\nu(n,k)$ be the number of partitions of n counted by the general term of (5), having k odd parts between 1 to $2s+1$. We denote by $P_\nu(n,k)$ the set of partitions counted by $p_\nu(n,k)$.

For a fixed n, we classify its partitions of type described in Definition 4.2 according to the sum on the second row of the matrix associated to each partition. We construct a table (see Table 4) in the same way we did for functions $\rho(q)$ and $\sigma(q)$, now for Mock Theta Function $\nu(q)$.

By observing the table above we get some interesting results.

Table 4. Table from the characterization given by Theorem 4.1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	0																								
2	1	0	1																							
3	1	0	1	0																						
4	1	0	1	0	0																					
5	1	0	1	0	1	0																				
6	1	0	1	0	1	0	1																			
7	1	0	1	0	1	0	1	0																		
8	1	0	1	0	1	0	2	0	0																	
9	1	0	1	0	1	0	2	0	1	0																
10	1	0	1	0	1	0	2	0	1	0	0															
11	1	0	1	0	1	0	2	0	2	0	1	0														
12	1	0	1	0	1	0	2	0	2	0	2	0	1													
13	1	0	1	0	1	0	2	0	2	0	2	0	1	0												
14	1	0	1	0	1	0	2	0	2	0	3	0	2	0	0											
15	1	0	1	0	1	0	2	0	2	0	3	0	3	0	1	0										
16	1	0	1	0	1	0	2	0	2	0	3	0	3	0	2	0	0									
17	1	0	1	0	1	0	2	0	2	0	3	0	4	0	3	0	1	0								
18	1	0	1	0	1	0	2	0	2	0	3	0	4	0	4	0	2	0	0							
19	1	0	1	0	1	0	2	0	2	0	3	0	4	0	4	0	3	0	1	0						
20	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	4	0	2	0	1					
21	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	5	0	4	0	1	0				
22	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	5	0	5	0	3	0	0			
23	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	6	0	4	0	1	0		
24	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	7	0	6	0	2	0	0	
25	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	7	0	7	0	4	0	1	0

Remark 4.3. *It is clear to see that, for all $n \geq 1$, we have*

$$p_\nu(n,0) = \begin{cases} 1, & \text{if } n = s^2 + s \\ 0, & \text{otherwise.} \end{cases}$$

and

$$p_\nu(n,1) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 4.4. *For all $n \geq 1$ and $i = 0, 2, 4$, we have*

$$p_\nu(4n^2 - 2n + i, 2) = n.$$

Proof. If $i = 0$, the largest even part of any partition counted by $p_\nu(4n^2 - 2n, 2)$ has to be $4n - 4$. So, we have to write

$$4n^2 - 2n = 2 + 4 + \cdots + 4n - 4 + r + s,$$

with odd r and s and $1 \leq s \leq r \leq 4n - 3$, which implies

$$r + s = 4n - 2.$$

Equivalently, writing $r = 2p - 1$ and $s = 2q - 1$, with $1 \leq q \leq p \leq 2n - 1$, we have to determine the number of solutions of equation $p + q = 2n$, with $1 \leq q \leq p \leq 2n - 1$.

First of all, the number of positive solutions of equation $p + q = 2n$ with no limitation for p and $q \leq p$ is $\lfloor \frac{2n}{2} \rfloor = n$. The solutions we do not want are those where $p > 2n - 1$. Although, note that $p > 2n - 1$ implies $p = 2n$, and so $q = 0$, which never occurs. So, there is no solution to eliminate and the number we are looking for is just n.

If $i = 2$ or 4, there is only one partition counted by $p_\nu(4n^2 - 2n + i, 2)$ with largest even part $4n - 2$. Indeed, $2 + 4 + \cdots + 4n - 2 + r + s = 4n^2 - 2n + i$, with odd r and s and $1 \leq s \leq r \leq 4n - 1$, implies $r + s = i$. As in those conditions 2 and 4 can only be written as $1 + 1$ and $3 + 1$, respectively, there is only one partition counted by $p_\nu(4n^2 - 2n + i, 2)$ with the largest part $4n - 2$. The other partitions have the largest even part equal to $4n - 4$. So,

$$4n^2 - 2n + i = 2 + 4 + \cdots + 4n - 4 + r + s,$$

with odd r and s and $1 \leq s \leq r \leq 4n - 3$, which implies

$$r + s = 4n - 2 + i.$$

Again writing $r = 2p-1$ and $s = 2q-1$, with $1 \leq q \leq p \leq 2n-1$, and $i = 2j$, it is equivalent to determine the number of solutions of equation $p + q = 2n + j$, with $1 \leq q \leq p \leq 2n - 1$.

The number of positive solutions of equation $p + q = 2n + j$ with no limitation for p and $q \leq p$ is $\lfloor \frac{2n+j}{2} \rfloor$. The solutions we do not want are those where $p > 2n - 1$. As q has to be at least 1 and $p > 2n - 1$, we can write

$$p = 2n - 1 + k \quad \text{with } 1 \leq k \leq j,$$

and so, for each value of k we get one value of p.

Then, the number of solutions we want is

$$\left\lfloor \frac{2n + j}{2} \right\rfloor - j = n - 1.$$

Adding these to the other solution we already have, we get $p_\nu(4n^2 - 2n + i, 2) = n$, for $i = 0, 2, 4$. □

As it happens in Tables 1 and 3, the columns in Table 4 become constant from certain values of n onwards. This fact can be proved analogously to those in Theorems 2.5 and 3.5. Hence, we omit the proof of the next theorem.

Theorem 4.5. *For all $n \geq 0$ and $i \geq 0$, we have*

$$p_\nu(3n + 2 + i, n + i) = p_\nu(3n + 2, n).$$

The constant values in the columns of the table are also related to certain partitions into distinct parts.

Theorem 4.6. *For all $n \geq 0$ and $i \geq 0$, we have*

$$p_\nu(3n + 2, n) = p_d(n + 1).$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_d(n+1)$. We describe how to associate λ to a partition $\mu \in P_\nu(3n+2, n)$ step-by-step and illustrate it with an example.

- Step 1: As λ has k distinct parts, subtract and save $k, k-1, k-2, \ldots, 1$ from $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively.
- Step 2: Conjugate the remaining partition and call the new parts $r_1, r_2, \ldots, r_{\lambda_1 - k}$.
- Step 3: Double the partitions $(k, k-1, k-2, \ldots, 1)$ and $(r_1, r_2, \ldots, r_{\lambda_1-k})$ and add 1 to each part r_i.
- Step 4: Join the partitions.
- Step 5: Add $n - (\lambda_1 - k)$ parts of size 1.

Overlooking the order of the parts, the partition μ obtained with the steps above is the partition

$$\mu = (2k, 2(k-1), 2(k-2), \ldots, 2, (2r_1+1), \ldots, (2r_{\lambda_1-k}+1), \underbrace{1, 1, \ldots, 1}_{n-(\lambda_1-k)}).$$

The partition μ really lies in $P_\nu(3n+2, n)$, because every even part from 2 to $2k$ appears exactly once, $2r_i + 1 \leq 2k+1$ for all $r_i \leq k$, the odd parts are in number of n and

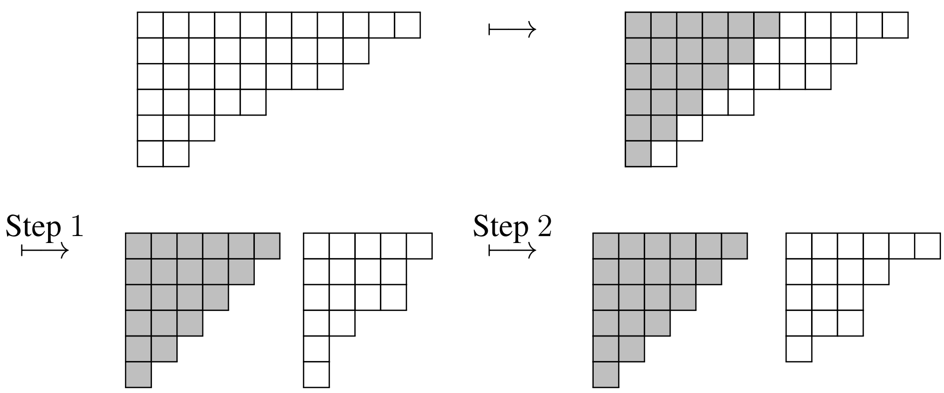
$$\begin{aligned} & 2k + 2(k-1) + \cdots + 2 + (2r_1+1) + \cdots + (2r_{\lambda_1-k}+1) + \underbrace{1 + 1 + \cdots + 1}_{n-(\lambda_1-k)} \\ &= k(k+1) + 2(r_1 + \cdots + r_{\lambda_1-k}) + \lambda_1 - k + n - (\lambda_1 - k) \\ &= k(k+1) + 2(n + 1 - \frac{k(k+1)}{2}) + n \\ &= 3n+2. \end{aligned}$$

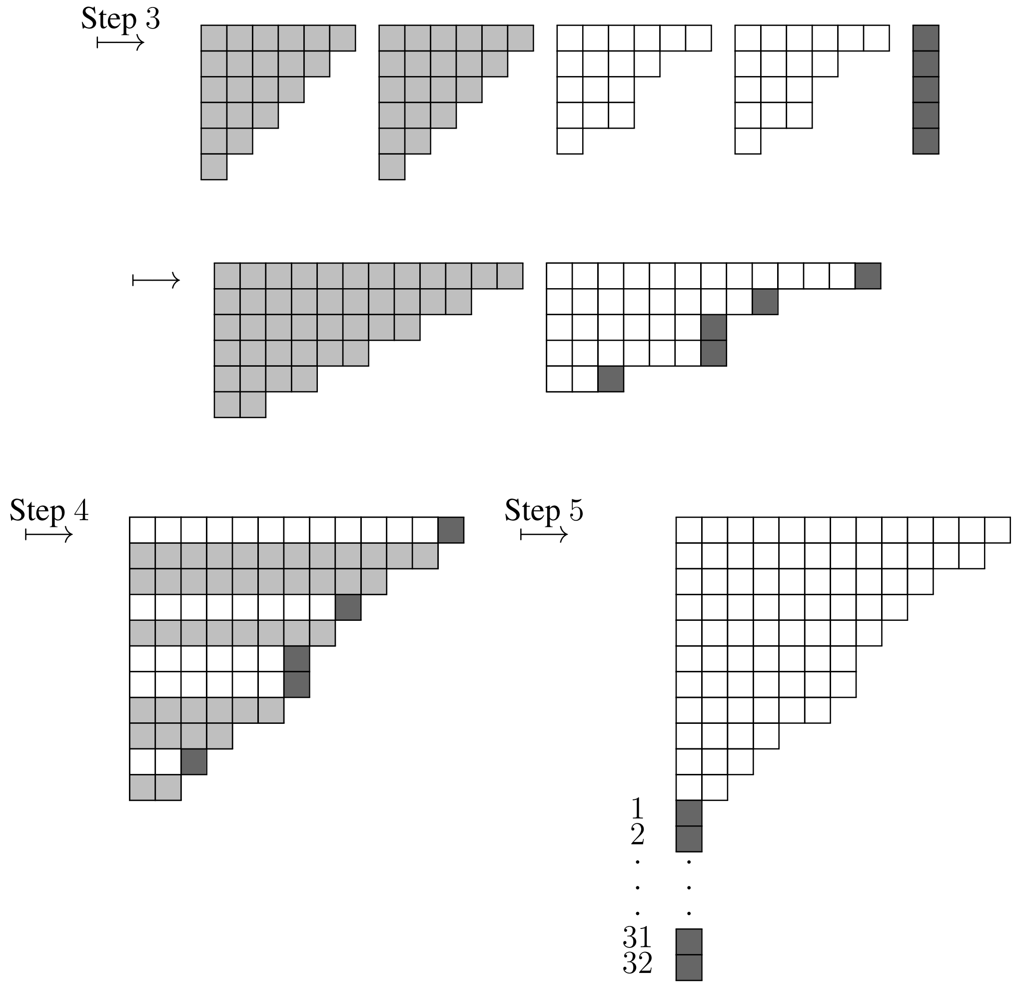
The inverse map is easy to build. □

Example 4.7. *For $n = 37$, consider*

$$\lambda = (11, 9, 8, 5, 3, 2) \textit{ and } \mu = (13, 12, 10, 9, 8, 7, 7, 6, 4, 3, 2\underbrace{1, 1, \ldots, 1}_{32 \textit{ times}}).$$

The following diagram illustrates how to get from λ to μ.





By replacing x and y by q in Equation 6 at page 29 of [2], we get the next identity. Although it is already proved, we present a new combinatorial way to demonstrate it.

Theorem 4.8. *We have the identity*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} (-q;q^2)_n q^n. \tag{7}$$

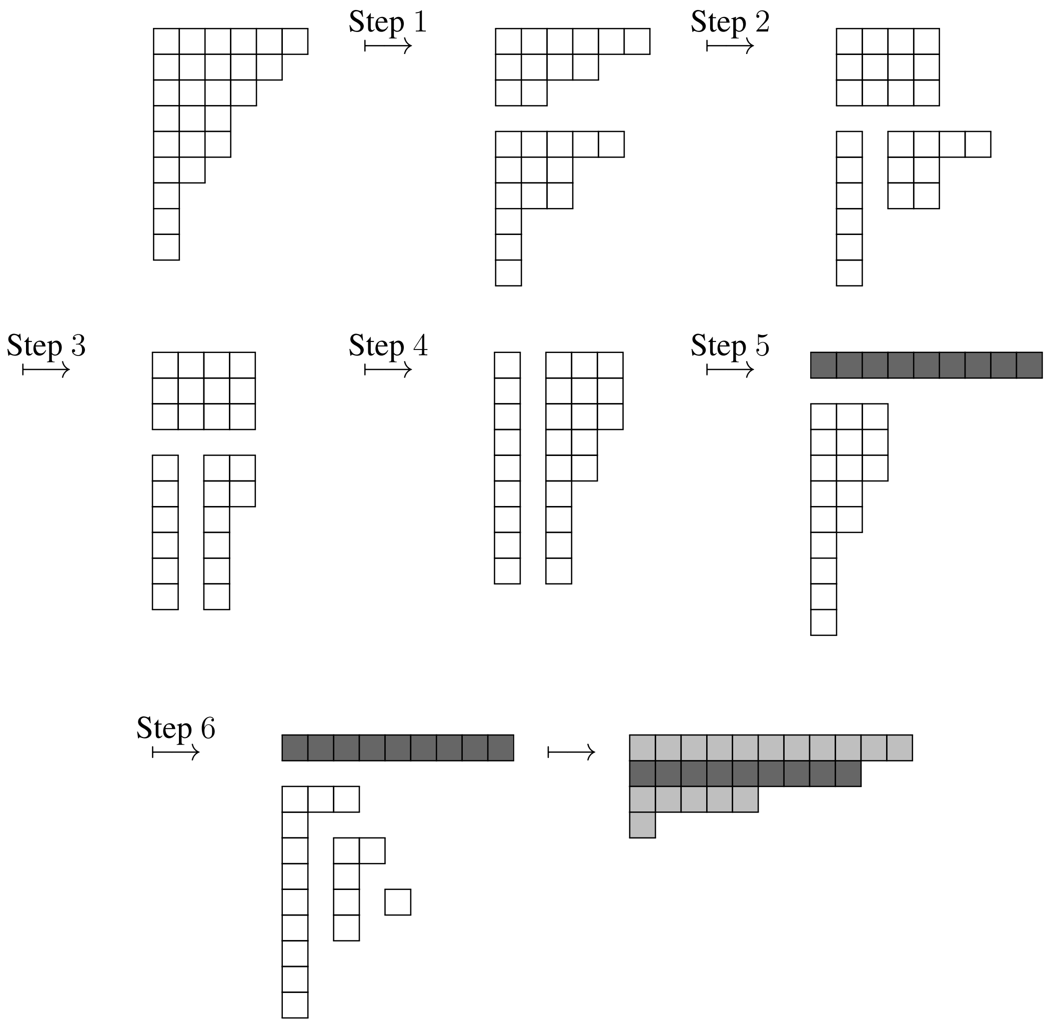
Proof. Let us consider the two following sets of partitions:

- $N(n)$: The set of partitions of n such that, if $2s$ is the largest even part, then all parts $2, 4, \ldots, 2s$ must appear exactly once, any odd part must be smaller than $2s+1$ and may appear in any number.
- $\Lambda(n)$: The set of partitions of n into dark gray and light gray parts such that there is only one dark part s and the light parts must be distinct odd parts, smaller than $2s-1$.

Note that the left and right hand sides of equality (7) are the generating functions for partitions lying in $N(n)$ and $\Lambda(n)$, respectively. If we prove that $|N(n)| = |\Lambda(n)|$, the theorem follows. So, we describe step-by-step a map from $N(n)$ to $\Lambda(n)$.

- Step 1: Given a partition counted by $N(n)$, consider its corresponding Young Diagram. Separate even and odd parts, getting two new diagrams.
- Step 2: As each even part appears exactly once and the greatest one is $2s$, change the s even parts for s parts of size $s+1$. Then, subtract and save one unit from each odd part.
- Step 3: As the partition into odd parts has turned into one with even parts, split each of these parts into two parts of equal size.
- Step 4: Join all the partitions together (for more details see Example 4.9) and then subtract and save one unit from each part of the new diagram.
- Step 5: Turn the removed units into a dark part.
- Step 6: The light gray parts are obtained by making hooks with the remaining rows and columns: for the first part, take the first row together with the first column; for the second part, take what is left from the second row together with what is left from the second column. Keep doing this process until there are no more squares left in the old diagram.
- Step 7: Finally, get the parts together.

Example 4.9. *We illustrate how to get partition* $(\boxed{11}, 9, \boxed{5}, \boxed{1}) \in \Lambda(26)$ *from partition* $(6,5,4,3,3,2,1,1,1) \in N(26)$.

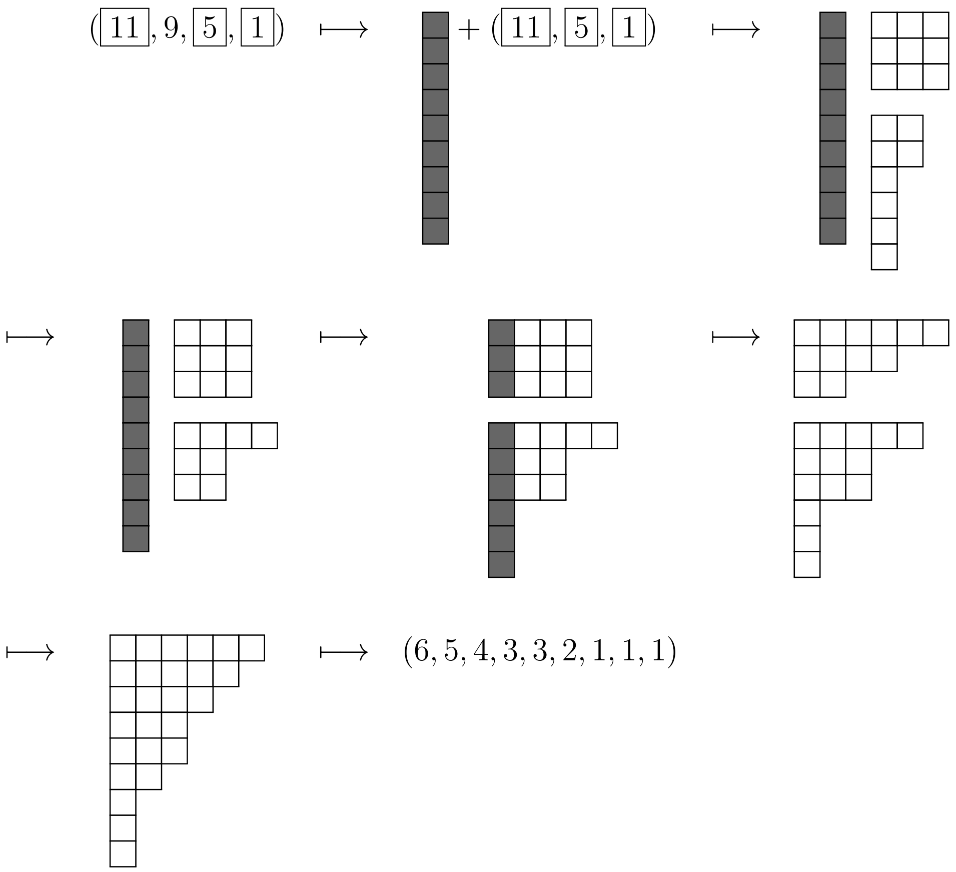


The resulting light gray parts are odd because the parts below the Durfee Square of the partition after Step 5 appear in pairs. This also proves that these parts are distinct.

Note that the unique dark part of the resulting partition is equal to the number of parts of the original partition (s plus the number of odd parts). The largest light part is, at most, 2 times the number of odd parts plus $2s - 1$. So, the resulting partition lies in $\Lambda(n)$.

To describe the inverse map, take the dark part and draw it as a column. Besides it, build a square with side equal to the number of light gray parts of the partition. Below this square, display the remaining parts as hooks to complete the Young Diagram. The steps that complete the inverse map are easy to see and we illustrate them in the next example.

Example 4.10. *Considering the partition* $(\boxed{11}, 9, \boxed{5}, \boxed{1}) \in \Lambda(26)$, *we set the way back to* $(6, 5, 4, 3, 3, 2, 1, 1, 1) \in N(26)$.



Thus, with the maps we described above we have set a bijection between $N(n)$ and $\Lambda(n)$ and, therefore, equality (7) holds. $\square$

From Theorem 4.8, changing q by $-q$, we have the following corollary.

Corollary 4.11. *We have the identity*

$$\nu(q) = \sum_{n=0}^{\infty} (-1)^n (q; q^2)_n q^n.$$

Remark 4.12. *Knowing the Mock Theta Function*

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^n}{(-q;q)_n},$$

we set the two-variable generating function for $\lambda(q)$

$$\lambda(q,z) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^n}{(-zq;q)_n}.$$

Considering the coefficient of z^0*, we get the generating function* $\sum_{n=0}^{\infty}(-1)^n(q;q^2)_n q^n$*, which is equal to* $\nu(q)$.

Having analogous representation for other Mock Theta Functions, we hope to obtain more relations and identities for different types of partitions.

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