Inequalities between arithmetic functions

\( \varphi, \psi \) and \( \sigma \). Part 1

Krassimir T. Atanassov\(^1\) and József Sándor\(^2\)

\(^1\) Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8010, Bulgaria
e-mail: krat@bas.bg

\(^2\) Department of Mathematics, Babeș–Bolyai University
Str. Kogalniceanu 1, 400084 Cluj-Napoca, Romania
e-mail: jjsandor@hotmail.com

Abstract: For three of the basic arithmetic functions \( \varphi, \psi \) and \( \sigma \) are proved the inequalities
\( \psi(n)^n > \sigma(n)^{\varphi(n)} \) and \( \sigma(n)^n < \psi(n)^{\sigma(n)} \) for each natural number \( n \geq 2 \).

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1 Introduction

One of the most interesting areas of the number theory is related to the arithmetic functions. Some properties of them are discussed in a series of papers of the authors [1–4, 7]. In the paper, two new inequalities will be formulated and proved.

For the natural number

\[ n = \prod_{i=1}^{k} p_i^{\alpha_i}, \]

(1)

where \( k, \alpha_1, ..., \alpha_k, k \geq 1 \) are natural numbers and \( p_1, ..., p_k \) are different primes, the following
arithmetic functions are defined by:

\[ \varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1), \varphi(1) = 1, \]
\[ \psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1), \psi(1) = 1, \]
\[ \sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}, \sigma(1) = 1 \]

(see, e.g. [5, 6]).

Also, we use the following notation for the above \( n \):

\[ \text{set}(1) = \emptyset, \text{set}(n) = \{p_1, \ldots, p_k\}. \]

One of the interesting inequalities containing the arithmetic functions \( \varphi \) and \( \psi \) is:

\[ \psi(n)^\varphi(n) < n^n < \varphi(n)^\psi(n) \tag{2} \]

(see [2]).

It can be easily seen that the following inequalities

\[ \psi(n)^\varphi(n) < \sigma(n)^\varphi(n) < n^n < \varphi(n)^\psi(n) < \varphi(n)^\sigma(n) \tag{3} \]

are valid, too.

In the present paper, two new inequalities related to (2) and (3), will be defined and proved, using different methods.

## 2 Main result

**Theorem 1.** For each natural number \( n \geq 2 \):

\[ \psi(n)^n > \sigma(n)^\varphi(n) \tag{4} \]

**Proof:** Let \( n \) be a prime number. Then from (4) we obtain:

\[ \psi(n)^n - \sigma(n)^\varphi(n) = (n + 1)^n - (n + 1)^{n-1} > 0. \]

Let for the natural number \( n \geq 2 \) of the form (1), the inequality (4) be valid. Let \( p \) be a prime number. For it there are two possibilities.

**Case 1.** Let \( p \not\in \text{set}(n) \). Then,

\[ \psi(np)^np - \sigma(np)^\varphi(np) = (\psi(n)(p + 1))^{np} - (\sigma(n)(p + 1))^{\varphi(n)(p-1)} \]
\[ = \psi(n)^np.(p + 1)^{np} - \sigma(n)^\varphi(n)(p-1).(p + 1)^{\varphi(n)(p-1)} \]
\[ = (\psi(n)^n)^p.(p + 1)^{np} - (\sigma(n)^\varphi(n))^{p-1}.(p + 1)^{\varphi(n)(p-1)} \]

51
(by the induction assumption)

\[ = (\psi(n)^{p})(p + 1)^{np} - (\psi(n)^{n})^{p^{p} - 1}(p + 1)^{\varphi(n)(p - 1)} > 0. \]

**Case 2.** Let \( p \in \text{set}(n) \). Then \( n = p^a m \) for the natural numbers \( a, m \geq 1 \), where \( (m, p) = 1 \).

First, obviously, for \( q \geq 3 \)

\[ q > (1 + \frac{1}{q})^{q - 1} > (1 + \frac{1}{q^2})^{q - 1}. \] (5)

Therefore, for each prime number \( q \geq 3 \):

\[ q^{2q - 1} = q.q^{2q - 2} > q^{2q - 2}(1 + \frac{1}{q})^{q - 1} = (q^2(1 + \frac{1}{q}))^{q - 1}. \]

Second, we see that for \( q \geq 3 \) and for \( a \geq 1 \):

\[ q + \frac{1}{q} - \frac{q^{a+2} - 1}{q^{a+1} - 1} = \frac{q^2 + 1}{q} - \frac{q^{a+2} - 1}{q^{a+1} - 1} \]

\[ = \frac{1}{q(q^{a+1} - 1)}((q^{a+3} + q^{a+1} - q^2 - 1) - (q^{a+3} + q)) \]

\[ = \frac{1}{q(q^{a+1} - 1)}(q^{a+1} - q^2 + q - 1) > 0. \] (6)

Third,

\[ \sigma(np) = \sigma(mp^{a+1}) = \sigma(m)p^{a+2} - \frac{1}{p - 1} = \sigma(n)p^{a+2} - \frac{1}{p^{a+1} - 1}. \]

Now, we obtain sequentially:

\[ \psi(np)^{np} - \sigma(np)^{\varphi(np)} = (\psi(n)p)^{np} - \left( \sigma(n)p^{a+2} - \frac{1}{p^{a+1} - 1} \right)^{\varphi(n)p} \]

(by the induction assumption)

\[ > \psi(n)^{np}p^{np} - \psi(n)^{np}\left( \frac{p^{a+2} - 1}{p^{a+1} - 1} \right)^{\varphi(n)p} \]

\[ = \psi(n)^{np}\left( p^{np} - \left( \frac{p^{a+2} - 1}{p^{a+1} - 1} \right)^{\varphi(n)p} \right) \]

\[ \geq \psi(n)^{np}\left( p^{np} - \left( p + \frac{1}{p} \right)^{(n-1)p} \right) \]

(from (6))

\[ \geq \psi(n)^{np}\left( p^{np} - \left( p + \frac{1}{p} \right)^{(n-1)p} \right) \]

52
ψ(n)p(n−1)p(n−1)p

> 0.

This completes the proof. □

**Theorem 2.** For each natural number \( n \geq 2 \):

\[
\sigma(n)^{\frac{1}{n}} < \psi(n)^{\sigma(n)}.
\]  

(7)

**Proof:** It is well-known that the function \( f(x) = x^{\frac{1}{x}} \) is strictly decreasing for \( x \geq e \) – Euler’s number. As \( 3 > e \), particularly we get that

\[
\sigma(n)^{\frac{1}{\sigma(n)}} \leq \psi(n)^{\frac{1}{\psi(n)}},
\]

as \( \sigma(n) \geq \psi(n) \geq 3 \) for \( n \geq 2 \). Now, as \( \psi(n) > n \) for \( n \geq 2 \), we get that

\[
\sigma(n)^{\frac{1}{\sigma(n)}} < \psi(n)^{\frac{1}{n}},
\]

which implies (7). This completes the proof. □

In near future, other inequalities related to \( \varphi, \psi, \sigma \) and other arithmetic functions will be discussed.

**References**


