Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 4, 120–127
DOI: 10.7546/nntdm.2018.24.4.120-127

On the convergence of second-order recurrence series

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Received: 21 October 2017 **Accepted:** 26 October 2018

Abstract: In this article, a generalized second-order linear recurrence sequence is considered and the range of the convergence of this sequence with power series is studied. An estimation for the speed of convergence of the second-order linear recurrence series is also given.

Keywords: Second-order recurrence relation, Power series, Range of convergence, Speed of convergence.
2010 Mathematics Subject Classification: 11B39, 11B83.

1 Introduction

For arbitrary integers a, b, p, q, a generalized second-order linear recurrence sequence $\{f_k\}$ denoted by $\{f_k\} = \{f_k(a, b;\ p, q)\}$ is defined by

$$f_0 = a, \quad f_1 = b, \quad f_k = pf_{k-1} + qf_{k-2}, \tag{1}$$

for $k \geq 2$. Particular cases of the sequence $\{f_k\}$ are the sequences

$$U_k(p, q) = U_k = f_k(0, 1;\ p, q);$$
$$V_k(p, q) = V_k = f_k(2, p;\ p, q);$$
$$W_k(p, q) = W_k = f_k(1, \frac{p}{2};\ p, q).$$

For different values of p and q the more special cases of the sequences are $U_k(1,1) = F_k$, where F_k denotes the k-th Fibonacci number, $U_k(2,1) = P_k$, where P_k denotes the k-th Pell number. The other sequences such as the sequence of balancing numbers, sequence of Lucas-balancing numbers, sequence of Pell–Lucas numbers are also obtained by assigning different values of p and q, i.e., $U_k(6,-1) = B_k$, $V_k(1,1) = L_k$, $W_k(2,1) = Q_k$ and $W_k(6,-1) = C_k$, where B_k, L_k, Q_k and C_k denote the k-th balancing number, k-th Lucas number, k-th Pell–Lucas number and k-th Lucas-balancing number respectively. It is worthy to define a balancing number and a Lucas-balancing number. A balancing number n is the solution of a simple Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$ with the balancer r [1]. A Lucas-balancing number C_n is defined as $C_n = \sqrt{8B_n^2+1}$ and both of their respective recurrences are $B_{n+1} = 6B_n - B_{n-1}$ and $C_{n+1} = 6C_n - C_{n-1}$ [1]. Some recent development of these number sequences are studied in [2,5,8–12].

The roots of the Eq. (1) are

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2} \quad \text{and} \quad \beta = \frac{p-\sqrt{p^2+4q}}{2},$$

with $\alpha+\beta = p$, $\alpha-\beta = \sqrt{p^2+4q}$ and $\alpha\beta = -q$. The Binet formula for the sequence $\{f_k\}$ is given by

$$f_k = \frac{A\alpha^k - B\beta^k}{\alpha-\beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$ [4].

The mathematical identity that connects three adjacent Fibonacci numbers is well-known under the name Cassini formula, and is used to establish many important identities involving Fibonacci numbers and their related sequences. The Cassini formula for the generalized second-order sequence $\{f_k\}$ is (see [4])

$$f_{k-1}f_{k+1} - f_k^2 = -AB(-q)^{k-1}. \tag{2}$$

For different values of a, b, p, q in the generalized Cassini formula (2), the Cassini formulas for the sequences such as Fibonacci sequence, Pell sequence, balancing sequence, Lucas sequence, Pell–Lucas sequence and the sequence of Lucas-balancing numbers are obtained and are respectively given by

$$F_{k+1}F_{k-1} - F_k^2 = (-1)^k;\ \ P_{k+1}P_{k-1} = P_k^2 = (-1)^k;\ \ B_{k+1}B_{k-1} - B_k^2 = -1;$$
$$L_{k+1}L_{k-1} - L_k^2 = 5(-1)^{k-1};\ \ Q_{k+1}Q_{k-1} - Q_k^2 = 2(-1)^{k-1};\ \ C_{k+1}C_{k-1} - C_k^2 = 8.$$

2 The range of convergence of a generalized second order recurrence sequence with power series

Glaister [3] has studied the Fibonacci power series and found many interesting relations on it. Subsequently, Koshy [7] has established the convergence of Pell and Pell–Lucas series and also studied the speed of convergence for these series. In this section, a special attempt is made to study the convergence of generalized second-order recurrence power series.

It is well-known that, the geometric progression $\sum_{k=0}^{\infty} t^k = 1 + t + t^2 + \dots$, which converges to $t/(1-t)$ if $|t| < 1$. Multiplying the terms t^k by f_k, we obtain the series $S = \sum_{k=0}^{\infty} f_k t^k$, which on further simplification gives

$$
\begin{aligned}
S &= a + bt + (bp + aq)t^2 + \sum_{k=3}^{\infty} p f_{k-1} t^k + \sum_{k=3}^{\infty} q f_{k-2} t^k \\
&= a + bt + (bp + aq)t^2 + pt \sum_{m=2}^{\infty} f_m t^m + qt^2 \sum_{n=1}^{\infty} f_n t^n \\
&= a + bt + (pb + aq)t^2 + pt(S - a - bt) + qt^2(S - a),
\end{aligned}
$$

which follows that

$$
S = \sum_{k=0}^{\infty} f_k t^k = \frac{a + (b - ap)t}{1 - pt - qt^2}. \tag{3}
$$

As the series $\sum_{k=0}^{\infty} c_k t^k$ for $1/(1 - ct)$ converges if and only if $|ct| < 1$, therefore we identify the radius of convergence for the series S as follows.

Further, factorizing the expression $1 - pt - qt^2$ into $(1 - \alpha t)(1 - \beta t)$, the right-hand side expression of (3) simplifies to $\frac{a + (b - ap)t}{(1 - \alpha t)(1 - \beta t)}$. Using partial fraction, (3) reduces to

$$
S = \sum_{k=0}^{\infty} f_k t^k = \frac{a(\alpha - p) + b/\alpha - \beta}{1 - \alpha t} + \frac{a(p - \beta) - b/\alpha - \beta}{1 - \beta t}. \tag{4}
$$

However, the series $S = \sum_{k=0}^{\infty} f_k t^k$ converges if and only if $|\alpha t| < 1$ and $|\beta t| < 1$. It follows that S converges if and only if $|t| < |\min(|\alpha^{-1}|, |\beta^{-1}|)|$. Since the product of α and β is $-q$, clearly $0 < \frac{-\beta}{q} < \alpha$. Therefore, the minimum value of $|\alpha^{-1}|$ and $|\beta^{-1}|$ is $\frac{-\beta}{q}$, which is nothing but the radius of convergence of S. Indeed, the range of t for which S converges is the inequality

$$
\frac{\beta}{q} < t < \frac{-\beta}{q}. \tag{5}
$$

The range of the sequences like Fibonacci sequence, Pell sequence, balancing sequence, Lucas sequence etc., can be obtained from (5) by assigning different values to a, b, p, q. Though, the values of a, b, p, q are used to represent the above mentioned sequences, but to find out the range of each sequence, it only depend on the values of p and q.

For $p = 1 = q$ so that $\beta = \frac{1 - \sqrt{5}}{2}$, by virtue of (5), the range of convergence of the Fibonacci sequence $U_k(1, 1) = F_k$ and the range of convergence of Lucas sequence $V_k(1, 1) = L_k$ are same and is given by $-0.618 < t < 0.618$. This result for Fibonacci and Lucas sequence has shown in [3, 6]. Koshy [7] has shown that the Pell series and the Pell–Lucas series both converge if and only if $-0.414 < t < 0.414$. It is observed that, this range can also be obtained from (5) by assigning the values $p = 2$, $q = 1$ for the Pell sequence $U_k(2, 1) = P_k$ and the Pell–Lucas sequence $W_k(2, 1) = Q_k$. For the balancing sequence $U_k(0, 1) = B_k$ and the Lucas-balancing sequence $W_k(1, 3) = C_k$, the range of convergence is $-0.172 < t < 0.172$.

3 Speed of the convergence of the series $\sum_{k=0}^{\infty} f_k t^k$

In the previous section, it is shown that the series $\sum_{k=0}^{\infty} f_k t^k$ converges if and only if $\frac{\beta}{q} < t < \frac{-\beta}{q}$. Assume that $t = \frac{f_{2n}}{f_{2n+1}}$. Then for $\frac{\beta}{q} < 0$ and by Binet formula,

$$\begin{aligned}\frac{f_{2n}}{f_{2n+1}} &= \frac{A\alpha^{2n} - B\beta^{2n}}{A\alpha^{2n+1} - B\beta^{2n+1}} \\ &< \frac{A\alpha^{2n}}{A\alpha^{2n+1}} \\ &= \frac{1}{\alpha} = \frac{-\beta}{q},\end{aligned}$$

which follows that the power series $\sum_{k=0}^{\infty} f_k t^k$ converges for $t = f_{2n}/f_{2n+1}$ when $\frac{\beta}{q} < 0$. However, the range $t = -f_{2n}/f_{2n+1}$ which is greater than $\frac{\beta}{q}$, is outside the interval of convergence.

Using (3), we have

$$\begin{aligned}\sum_{k=0}^{\infty} f_k \left(\frac{f_{2n}}{f_{2n+1}}\right)^k &= \frac{af_{2n+1}^2 + (b - ap)f_{2n}f_{2n+1}}{f_{2n+1}^2 - pf_{2n}f_{2n+1} - qf_{2n}^2} \\ &= \frac{af_{2n+1}^2 + (b - ap)f_{2n}f_{2n+1}}{q(f_{2n+1}f_{2n-1} - f_{2n}^2)}.\end{aligned}$$

By virtue of the Cassini formula (2), the above expression reduces to

$$\sum_{k=0}^{\infty} f_k \left(\frac{f_{2n}}{f_{2n+1}}\right)^k = \frac{af_{2n+1}^2 + (b - ap)f_{2n}f_{2n+1}}{-ABq^{2n}}. \tag{6}$$

For $a = 0, b = 1, p = 1$ and $q = 1$, the identity (6) leads to an expression for Fibonacci sequence as follows

$$\sum_{k=0}^{\infty} F_k \left(\frac{F_{2n}}{F_{2n+1}}\right)^k = F_{2n}F_{2n+1}.$$

For example, $\sum_{k=0}^{\infty} F_k \left(\frac{3}{5}\right)^k = 3 \times 5$ and $\sum_{k=0}^{\infty} F_k \left(\frac{8}{13}\right)^k = 8 \times 13$. By computing the sum, it can be observed that $\sum_{k=0}^{114} F_k \left(\frac{3}{5}\right)^k \approx 14.5$ and $\sum_{k=0}^{1243} F_k \left(\frac{8}{13}\right)^k \approx 103.5$, which indicates that the convergence of Fibonacci numbers is extremely slow.

The other examples for the speed of convergence for the sequence of Pell series and the sequence of balancing series are given in the following table:

Value of a,b,p,q	Series	Exact value of series	Approximate value of series	Speed of convergence
$0,1;2,1$	$\sum_{k=0}^{\infty} P_k \left(\frac{P_{2n}}{P_{2n+1}}\right)^k = P_{2n}P_{2n+1}$	$\sum_{k=0}^{\infty} P_k \left(\frac{12}{29}\right)^k = 12 \times 29$ $\sum_{k=0}^{\infty} P_k \left(\frac{70}{169}\right)^k = 70 \times 169$	$\sum_{k=0}^{6445} P_k \left(\frac{12}{29}\right)^k \approx 347.5$ $\sum_{k=0}^{337001} P_k \left(\frac{70}{169}\right)^k \approx 11829.5$	very slow
$0,1;6,-1$	$\sum_{k=0}^{\infty} B_k \left(\frac{B_{2n}}{B_{2n+1}}\right)^k = B_{2n}B_{2n+1}$	$\sum_{k=0}^{\infty} B_k \left(\frac{6}{35}\right)^k = 6 \times 35$ $\sum_{k=0}^{\infty} B_k \left(\frac{204}{1189}\right)^k = 204 \times 1189$	$\sum_{k=0}^{7179} B_k \left(\frac{6}{35}\right)^k \approx 209.5$ $\sum_{k=0}^{337001} B_k \left(\frac{204}{1189}\right)^k \approx 242555.5$	very slow

Table 1. Convergence of sequence of Pell series and sequence of balancing series

Further, assume $t = f_{2n-1}/f_{2n}$. Again by Cassini formula for generalized second-order recurrence sequence and for $n \geq 1$, we have

$$\sum_{k=0}^{\infty} f_k \left(\frac{f_{2n-1}}{f_{2n}}\right)^k = \frac{af_{2n}^2 + (b-ap)f_{2n}f_{2n-1}}{f_{2n}^2 - pf_{2n}f_{2n-1} - qf_{2n-1}^2} = \frac{af_{2n}^2 + (b-ap)f_{2n}f_{2n-1}}{f_{2n}^2 - f_{2n-1}f_{2n+1}},$$

which follows that

$$\sum_{k=0}^{\infty} f_k \left(\frac{f_{2n-1}}{f_{2n}}\right)^k = \frac{af_{2n}^2 + (b-ap)f_{2n}f_{2n-1}}{AB(-q)^{2n-1}}. \tag{7}$$

For Lucas sequence, take $a = 2, b = 1, p = 1$ and $q = 1$, for which

$$\sum_{k=0}^{\infty} L_k \left(\frac{L_{2n-1}}{L_{2n}}\right)^k = F_{2n-1}L_{2n},$$

where $n \geq 1$. For example,

$$\sum_{k=0}^{\infty} L_k \left(\frac{11}{18}\right)^k = 5 \times 18, \quad \sum_{k=0}^{\infty} L_k \left(\frac{29}{47}\right)^k = 13 \times 47.$$

Notice that

$$\sum_{k=0}^{1678} L_k \left(\frac{11}{18}\right)^k \approx 90, \quad \sum_{k=0}^{12756} L_k \left(\frac{29}{47}\right)^k \approx 611.$$

From the examples discussed above, it is clear that the speed of convergence of the Lucas series is very slow.

The example of the speed of convergence for the sequence of Pell–Lucas series is given in the following table:

Indeed, for the Lucas-balancing series, the values of $t = f_{2n-1}/f_{2n}$ or f_{2n}/f_{2n+1} lie outside the range of convergence $\left(\sqrt{8}-3, 3-\sqrt{8}\right)$.

Value of a,b,p,q	Series	Exact value of series	Approximate value of series	Speed of convergence
$1,1;2,1$	$\sum_{k=0}^{\infty} Q_k \left(\frac{Q_{2n-1}}{Q_{2n}}\right)^k = P_{2n-1}Q_{2n}$	$\sum_{k=0}^{\infty} Q_k \left(\frac{41}{99}\right)^k = 29 \times 99$ $\sum_{k=0}^{\infty} Q_k \left(\frac{239}{338}\right)^k = 169 \times 338$	$\sum_{k=0}^{6445} Q_k \left(\frac{12}{29}\right)^k \approx 347.5$ $\sum_{k=0}^{337001} Q_k \left(\frac{239}{338}\right)^k \approx 57112$	very slow

Table 2. Convergence of Pell–Lucas series

The convergence of the series involving mixed types sequences such as Fibonacci and Lucas sequences, Pell and Pell–Lucas sequences and balancing and Lucas-balancing sequences are found by using (6) and (7) as discuss below.

Suppose that, $t = L_{2n-1}/L_{2n}$. Then, $\frac{\sqrt{5}-1}{2} < t < \frac{1-\sqrt{5}}{2}$, it follows that $\sum_{k=0}^{\infty} F_k t^k$ converges for $t = L_{2n-1}/L_{2n}$. Consequently, it follows by Cassini formula for L_m that

$$\sum_{k=0}^{\infty} F_k \left(\frac{L_{2n-1}}{L_{2n}}\right)^k = \frac{L_{2n-1}L_{2n}}{5}.$$

Likewise, let $t = F_{2n}/F_{2n+1}$, then

$$\sum_{k=0}^{\infty} L_k \left(\frac{F_{2n}}{F_{2n+1}}\right)^k = L_{2n}F_{2n+1}.$$

Further, suppose that $t = Q_{2n-1}/Q_{2n}$, then $1-\sqrt{2} < t < \sqrt{2}-1$. Therefore, $\sum_{k=0}^{\infty} P_k t^k$ converges for $t = \frac{Q_{2n-1}}{Q_{2n}}$. By Cassini formula for Q_m,

$$\sum_{k=0}^{\infty} P_k \left(\frac{Q_{2n-1}}{Q_{2n}}\right)^k = \frac{Q_{2n-1}Q_{2n}}{2}.$$

Likewise, let $t = \frac{P_{2n}}{P_{2n+1}}$, then

$$\sum_{k=0}^{\infty} Q_k \left(\frac{P_{2n}}{P_{2n+1}}\right)^k = Q_{2n}P_{2n+1}.$$

Again, taking $t = B_{2n+1}/B_{2n}$, then $\sqrt{8}-3 < t < 3-\sqrt{8}$. So, $\sum_{k=0}^{\infty} C_k t^k$ converges for $t = B_{2n}/B_{2n+1}$. Using Cassini formula for B_m,

$$\sum_{k=0}^{\infty} C_k \left(\frac{B_{2n}}{B_{2n+1}}\right)^k = B_{2n+1}C_{2n}.$$

The examples for the speed of convergence for the above mentioned series are given in the following table:

Series	Exact value of series	Approximate value of series	Speed of convergence
$\sum_{k=0}^{\infty} F_k \left(\frac{L_{2n-1}}{L_{2n}}\right)^k = \frac{L_{2n-1}L_{2n}}{5}$	$\sum_{k=0}^{\infty} F_k \left(\frac{1}{3}\right)^k = \frac{1\times 3}{5}$ $\sum_{k=0}^{\infty} F_k \left(\frac{4}{7}\right)^k = \frac{4\times 7}{5}$	$\sum_{k=0}^{23} F_k \left(\frac{1}{3}\right)^k \approx 0.6$ $\sum_{k=0}^{207} F_k \left(\frac{4}{7}\right)^k \approx 5.6$	extremely slow
$\sum_{k=0}^{\infty} L_k \left(\frac{F_{2n}}{F_{2n+1}}\right)^k = L_{2n}F_{2n+1}$	$\sum_{k=0}^{\infty} L_k \left(\frac{1}{2}\right)^k = 3\times 2$ $\sum_{k=0}^{\infty} L_k \left(\frac{3}{5}\right)^k = 7\times 5$	$\sum_{k=0}^{76} L_k \left(\frac{1}{2}\right)^k \approx 6$ $\sum_{k=0}^{609} L_k \left(\frac{3}{5}\right)^k \approx 35$	extremely slow
$\sum_{k=0}^{\infty} P_k \left(\frac{Q_{2n-1}}{Q_{2n}}\right)^k = \frac{Q_{2n-1}Q_{2n}}{2}$	$\sum_{k=0}^{\infty} P_k \left(\frac{1}{3}\right)^k = \frac{1\times 3}{2}$ $\sum_{k=0}^{\infty} P_k \left(\frac{7}{17}\right)^k = \frac{7\times 17}{2}$	$\sum_{k=0}^{69} P_k \left(\frac{1}{3}\right)^k \approx 1.5$ $\sum_{k=0}^{3136} P_k \left(\frac{7}{17}\right)^k \approx 59.5$	extremely slow
$\sum_{k=0}^{\infty} Q_k \left(\frac{P_{2n}}{P_{2n+1}}\right)^k = Q_{2n}P_{2n+1}$	$\sum_{k=0}^{\infty} Q_k \left(\frac{2}{5}\right)^k = 3\times 5$ $\sum_{k=0}^{\infty} Q_k \left(\frac{12}{29}\right)^k = 17\times 29$	$\sum_{k=0}^{493} Q_k \left(\frac{2}{5}\right)^k \approx 15$ $\sum_{k=0}^{20390} Q_k \left(\frac{12}{29}\right)^k \approx 493$	extremely slow
$\sum_{k=0}^{\infty} C_k \left(\frac{B_{2n}}{B_{2n+1}}\right)^k = C_{2n}B_{2n+1}$	$\sum_{k=0}^{\infty} C_k \left(\frac{6}{35}\right)^k = 17\times 35$ $\sum_{k=0}^{\infty} C_k \left(\frac{204}{1189}\right)^k = 577\times 1189$	$\sum_{k=0}^{24834} C_k \left(\frac{6}{35}\right)^k \approx 595$ $\sum_{k=0}^{98985539} C_k \left(\frac{204}{1189}\right)^k \approx 686053$	slow

Table 3. Speed of convergence of the series involving mixed sequences

References

[1] Behera, A., & Panda, G. K. (1999) On the square roots of triangular numbers, *Fibonacci Quart.*, 37, 98–105.

[2] Catarino, P., Campos, H., & Vasco, P. (2015) On some identities for balancing and cobalancing numbers, *Ann. Math. Inform.*, 45, 11–24.

[3] Glaister, P. (1995) Fibonacci power series, *Math. Gaz.*, 79, 521–525.

[4] Horadam, A. F. (1965) Basic properties of a certain generalized sequence of numbers, *Fibonacci Quart.*, 3, 161–176.

[5] Komatsu, T., & Szalay, L. (2014) Balancing with binomial coefficients, *Intern. J. Number Theory*, 10, 1729–1742.

[6] Koshy, T. (1999) The Convergence of a Lucas Series, *Math. Gaz.*, 83, 272–274.

[7] Koshy, T. (2012) Convergence of Pell and Pell–Lucas Series, *Math. Spectr.*, 45, 10–13.

[8] Patel, B. K. & Ray, P. K. (2016) The period, rank and order of the sequence of balancing numbers modulo m, *Math. Rep. (Bucur.)*, 18, 395–401.

[9] Ray, P. K. (2014) Some congruences for balancing and Lucas-balancing numbers and their applications, *Integers*, 14, #A8.

[10] Ray, P. K. (2015) Balancing and Lucas-balancing sums by matrix methods, *Math. Rep. (Bucur.)*, 17, 225–233.

[11] Ray, P. K., & Patel, B. K. (2016) Uniform distribution of the sequence of balancing numbers modulo m, *Unif. Distrib. Theory*, 11, 15–21.

[12] Ray, P. K. (2018) On the properties of k-balancing numbers, *Ain. Shams Eng. J.*, 9, 395–402.