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# On the Diophantine equation $L_n - L_m = 3 \cdot 2^a$

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**Abstract:** In this paper, we solve Diophantine equation in the tittle in positive integers $m, n$, and $a$. It is shown that solutions of the equation $L_n - L_m = 3 \cdot 2^a$ are given by $L_{11} - L_4 = 199 - 7 = 3 \cdot 2^6, L_4 - L_3 = 7 - 4 = 3 \cdot 2^0, L_4 - L_1 = 7 - 1 = 3 \cdot 2, L_3 - L_1 = 4 - 1 = 3 \cdot 2^0$. In order to prove our result, we use lower bounds for linear forms in logarithms and a version of the Baker–Davenport reduction method in Diophantine approximation.

**Keywords:** Fibonacci numbers, Lucas numbers, Exponential equations, Linear forms in logarithms, Baker's method.
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## 1 Introduction

The Fibonacci sequence $(F_n)$ is defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence $(L_n)$, which is similar to the Fibonacci sequence, is defined by the same recursive pattern with initial conditions $L_0 = 2,\ L_1 = 1$. These two sequences are the most important among the second order linear recursive sequences and have been investigated by many researchers. For a brief history of Fibonacci and Lucas sequences, one can consult reference [6]. Firstly, square terms and, later, perfect powers in the Fibonacci and Lucas sequences have attracted the attention of researchers. The problem of finding all perfect powers in these sequences remained an open problem and was finally resolved in 2006 by Bugeaud, Mignotte and Siksek in [5]. It is shown that the perfect powers in the Fibonacci and Lucas sequences are $F_0 = 0$, $F_1 = F_2 = 1, F_6 = 8 = 2^3, F_{12} = 144 = 12^2$, and $L_1 = 1, L_3 = 4 = 2^2$, respectively. In the last decade, some exponential Diophantine equations containing the terms of second order linear recursive sequences have been studied by the mathematicians. As an example, the Diophantine equation $L_n + L_m = 2^a$ has been tackled in [3] by Bravo and Luca. Two years later, the same authors solved Diophantine equation $F_n + F_m = 2^a$ in [4]. Meanwhile, the equation $F_n + F_m + F_l = 2^a$ has been solved by Erich F. Bravo and John J. Bravo [2]. Lastly, in [10], the authors dealt with the Diophantine equation $u_n + u_m = wp_1^{z_1}p_2^{z_2}\cdots p_s^{z_s}$ and they solved this equation in the case that $w = 1$, $p_1, ...p_{46}$ are all prime numbers, which is less than 200 and $u_n$ is the Fibonacci sequence or Lucas sequence. In [11] , we solved $F_n - F_m = 2^a$. In this paper, we consider the equation

$$L_n - L_m = 3 \cdot 2^a \tag{1}$$

and find all solutions $n, m$, and $a$ in positive integers. This study can be viewed as a continuation of the previous works on this subject. We follow the approach and the method presented in [3]. In section 2, we introduce necessary lemmas and theorems. Then in section 3, we prove our main theorem.

# 2 Auxiliary results

Lately, in many articles, to solve Diophantine equations such as the equation (1), authors have used Baker's theory lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving of Diophantine equations, we start with recalling some basic notions from algebraic number theory.

Let $\eta$ be an algebraic number of degree $d$ with minimal polynomial

$$a_0x^d + a_1x^{d-1} + ... + a_d = a_0\prod_{i=1}^{d}\left(X - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the $a_i$'s are relatively prime integers with $a_0 > 0$ and $\eta^{(i)}$'s are conjugates of $\eta$. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{2}$$

is called logarithmic height of $\eta$. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b > 1$, then $h(\eta) = \log\left(\max\{|a|, b\}\right)$.

The following properties of logarithmic height are found in many works stated in references:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{3}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{4}$$

$$h(\eta^s) = |s|h(\eta). \tag{5}$$

The following theorem is deduced from Corollary 2.3 of Matveev [9], provides a large upper bound for the subscript $n$ in the equation (1) (also see Theorem 9.4 in [5]).

**Theorem 2.1.** *Assume that $\gamma_1, \gamma_2, ..., \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree $D$, $b_1, b_2, ..., b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} ... \gamma_t^{b_t} - 1$$

*is not zero. Then*

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 A_2 ... A_t\right),$$

*where*

$$B \geq \max\{|b_1|, ..., |b_t|\},$$

*and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, ..., t$.*

The following lemma, proved by Dujella and Pethő [7], is a variation of a lemma of Baker and Davenport [1]. And this lemma will be used to reduce the upper bound for the subscript $n$ in the equation (1). In the following lemma, the function $|| \cdot ||$ denotes the distance from $x$ to the nearest integer, that is, $||x|| = \min\{|x - n| : n \in \mathbb{Z}\}$ for a real number $x$.

**Lemma 2.2.** *Let $M$ be a positive integer, let $p/q$ be a convergent of the continued fraction of the irrational number $\gamma$ such that $q > 6M$, and let $A, B, \mu$ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

*in positive integers $u, v$, and $w$ with*

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

It is well known that

$$L_n = \alpha^n + \beta^n, \tag{6}$$

where $\alpha = \dfrac{1 + \sqrt{5}}{2}$ and $\beta = \dfrac{1 - \sqrt{5}}{2}$, which are the roots of the characteristic equation $x^2 - x - 1 = 0$. The relation between Lucas number and $\alpha$ are given by

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n. \tag{7}$$

for $n \geq 0$. The inequality (7) can be proved by induction. It can be seen that $1 < \alpha < 2$ and $-1 < \beta < 0$.

The following theorems are given in [5] and [8], respectively.

**Theorem 2.3.** *If $L_n = 3x^2$, then $n = 2$.*

**Theorem 2.4.** *The equation $L_n = 6x^2$ has no solutions for $n \geq 1$.*

# 3 Main theorem

**Theorem 3.1.** *The only solutions of the Diophantine equation (1) in positive integers* $m, n$, *and* $a$ *with* $m < n$, *are given by*

$$(n, m, a) \in \{(11, 4, 6), (4, 3, 0), ((4, 1, 1), (3, 1, 0)\},$$

*namely*

$$L_{11} - L_4 = 199 - 7 = 3 \cdot 2^6, L_4 - L_3 = 7 - 4 = 3 \cdot 2^0$$

*and*

$$L_4 - L_1 = 7 - 1 = 3 \cdot 2, L_3 - L_1 = 4 - 1 = 3 \cdot 2^0.$$

*Proof.* Assume that the equation (1) holds. With the help of *Mathematica* program, we obtain the solutions in Theorem 3.1 for $1 \leq m < n \leq 200$. This takes a little time. From now on, assume that $n > 200$ and $n - m \geq 3$. Now, let us show that $a \leq n$. Using (7), we get the inequality

$$2^a < 3 \cdot 2^a = L_n - L_m < L_n < 2\alpha^n < 2^{n+1},$$

that is, $a \leq n$.

On the other hand, rearranging the equation (1) as $\alpha^n - 3 \cdot 2^a = L_m - \beta^n$ and taking absolute values, we obtain

$$|\alpha^n - 3 \cdot 2^a| = |L_m - \beta^n| \leq L_m + \frac{|\beta|^n}{\sqrt{5}} < 2\alpha^m + \frac{1}{2}$$

by (7). If we divide both sides of the above inequality by $\alpha^n$, we get

$$\left|1 - 3 \cdot 2^a \alpha^{-n}\right| < \frac{3}{\alpha^{n-m}}, \tag{8}$$

where we have used the facts that $\alpha^{-m} < 1$ and $n > m$. Now, let us apply Theorem 2.1 with $\gamma_1 := 3$, $\gamma_2 := \alpha$, $\gamma_3 := 2$ and $b_1 := 1$, $b_2 := -n$, $b_3 := a$. Note that the numbers $\gamma_i$ for $i = 1, 2, 3$ are positive real numbers and elements of the field $\mathbb{K} = Q(\sqrt{5})$, so $D = 2$. It can be shown that the number $\Lambda_1 := 3 \cdot 2^a \alpha^{-n} - 1$ is nonzero. For, if $\Lambda_1 = 0$, then we get

$$3 \cdot 2^a = \alpha^n = L_n - \beta^n > L_n - 1 > L_n - L_m = 3 \cdot 2^a,$$

which is impossible. Moreover, since $h(\gamma_1) = \log 3$, $h(\gamma_2) = \frac{\log \alpha}{2} = \frac{0.4812...}{2}$ and $h(\gamma_3) = \log 2$ by (2), we can take $A_1 := 2.2$, $A_2 := 0.5$ and $A_3 := 1.4$. Also, since $a \leq n$, we can take $B := \max\{|a|, |-n|, 1\} = n$. Thus, taking into account the inequality (8) and using Theorem 2.1, we obtain

$$\frac{3}{\alpha^{n-m}} > |\Lambda_1| > \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log n)\,(2.2)\,(0.5)\,(1.4)\right)$$

and so

$$(n - m)\log\alpha - \log 3 < 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log n)\,(2.2)\,(0.5)\,(1.4) \tag{9}$$

Now, we try to apply Theorem 2.1 for the second time. Rearranging the equation (1) as $\alpha^n - \alpha^m - 2^a = -\beta^n + \beta^m$ and taking absolute values in here, we obtain

$$\left|\alpha^n(1-\alpha^{m-n}) - 3\cdot 2^a\right| = |-\beta^n + \beta^m| \leq |\beta|^n + |\beta|^m < 1,$$

where we used the fact that $|\beta|^n + |\beta|^m < 1$ for $n > 200$. Dividing both sides of the above inequality by $\alpha^n(1-\alpha^{n-m})$, we get

$$\left|1 - 3\cdot 2^a\alpha^{-n}(1-\alpha^{m-n})^{-1}\right| < \frac{1}{\alpha^n(1-\alpha^{m-n})}. \tag{10}$$

Since

$$\alpha^{m-n} = \frac{1}{\alpha^{n-m}} < \frac{1}{\alpha} < \frac{2}{3},$$

it is seen that

$$1 - \alpha^{m-n} > 1 - \frac{2}{3} = \frac{1}{3},$$

and therefore

$$\frac{1}{1-\alpha^{m-n}} < 3.$$

Then from (10), it follows that

$$\left|1 - 3\cdot 2^a\alpha^{-n}(1-\alpha^{m-n})^{-1}\right| < \frac{3}{\alpha^n}. \tag{11}$$

Thus, taking $\gamma_1 := \alpha,\ \gamma_2 := 2,\ \gamma_3 := 3(1-\alpha^{m-n})^{-1}$ and $b_1 := -n,\ b_2 := a,\ b_3 := 1$, we can apply Theorem 2.1. As one can see that, the numbers $\gamma_1, \gamma_2$, and $\gamma_3$ are positive real numbers and elements of the field $\mathbb{K} = Q(\sqrt{5})$, so $D = 2$. Since

$$\alpha^n - \alpha^m = L_n - \beta^n - F_m + \beta^m \neq 3\cdot 2^a$$

for $n > m$, the number $\Lambda_2 := 3\cdot 2^a\alpha^{-n}(1-\alpha^{m-n})^{-1} - 1$ is nonzero. Similarly, since $h(\gamma_1) = \dfrac{\log\alpha}{2} = \dfrac{0.4812...}{2}$ and $h(\ \gamma_2) = \log 2$ by (2), we can take $A_1 := 0.5$ and $A_2 = 1.4$. Besides, using (3), (4), and (5), we get that $h(\gamma_3) \leq \log 6 + (n-m)\dfrac{\log\alpha}{2}$, and so we can take $A_3 := \log 36 + (n-m)\log\alpha$. Also, since $a \leq n$, it follows that $B := \max\{|a|, |-n|, 1\} = n$. Thus, taking into account the inequality (11) and using Theorem 2.1, we obtain

$$\frac{3}{\alpha^n} > |\Lambda_2| > \exp(-C)(1+\log 2)(1+\log n)\,(0.5)\,(1.4)\,(\log 36 + (n-m)\log\alpha)$$

or

$$n\log\alpha - \log 3 < C(1+\log 2)(1+\log n)\,(0.5)\,(1.4)\,(\log 36 + (n-m)\log\alpha)\,, \tag{12}$$

where $C = 1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2$. Inserting the inequality (9) into the last inequality, a computer search with *Mathematica* gives us that $n < 9.1\cdot 10^{27}$.

Now, let us try to reduce the upper bound on $n$ applying Lemma 2.2. Let

$$z_1 := a\log 2 - n\log\alpha + \log 3.$$

Then

$$|1 - e^{z_1}| < \frac{3}{\alpha^{n-m}}$$

by (8). The inequality

$$\alpha^n = L_n - \beta^n > L_n - 1 > L_n - L_m = 3 \cdot 2^a$$

implies that $z_1 < 0$. In that case, since $\frac{3}{\alpha^{n-m}} < 0.75$ for $n - m \geq 3$, it follows that $e^{|z_1|} < 4$. Hence

$$0 < |z_1| < e^{|z_1|} - 1 = e^{|z_1|} |1 - e^{z_1}| < \frac{12}{\alpha^{n-m}},$$

or

$$0 < |a \log 2 - n \log \alpha + \log 3| < \frac{12}{\alpha^{n-m}}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < |a \frac{\log 2}{\log \alpha} - n + \frac{\log 3}{\log \alpha}| < \frac{25}{\alpha^{n-m}}. \tag{13}$$

Putting $\gamma := \frac{\log 2}{\log \alpha}$ and taking $M := 8 \cdot 10^{145}$, we found that $q_{292}$, the denominator of the 292-nd convergent of $\gamma$ exceeds $6M$. Also let us take

$$\mu := \frac{\log 3}{\log \alpha}.$$

In this case, a quick computation with *Mathematica* gives us the inequality

$$\epsilon = ||\mu q_{64}|| - M||\gamma q_{64}|| \geq 0.348264.$$

Let $A := 25,\ B := \alpha$, and $w := n - m$ in Lemma 2.2. Thus, with the help of *Mathematica*, we can say that the inequality (13) has no solution for $n - m \geq 147.259$. So $n - m \leq 147$. Substituting this upper bound for $n - m$ into (12), we obtain $n < 3.87 \cdot 10^{15}$.

Now, let

$$z_2 := a \log 2 - n \log \alpha + \log \left(3(1 - \alpha^{m-n})^{-1}\right).$$

In this case,

$$|1 - e^{z_2}| < \frac{3}{\alpha^n}$$

by (10). It is seen that $\frac{3}{\alpha^n} < \frac{1}{2}$ for $n > 200$. If $z_2 > 0$, then $0 < z_2 < e^{z_2} - 1 < \frac{3}{\alpha^n}$. If $z_2 < 0$, then $|1 - e^{z_2}| = 1 - e^{z_2} < \frac{3}{\alpha^n} < \frac{1}{2}$. From this, we get $e^{|z_2|} < 2$ and therefore

$$0 < |z_2| < e^{|z_2|} - 1 = e^{|z_2|} |1 - e^{z_2}| < \frac{6}{\alpha^n}.$$

In any case, the inequality

$$0 < |z_2| < \frac{6}{\alpha^n}$$

is true. That is,

$$0 < \left|a \log 2 - n \log \alpha + \log \left(3(1 - \alpha^{m-n})^{-1}\right)\right| < \frac{6}{\alpha^n}.$$

Dividing both sides of the above inequality by $\log \alpha$, we get

$$0 < \left| a \left( \frac{\log 2}{\log \alpha} \right) - n + \frac{\log \left(3(1-\alpha^{m-n})^{-1}\right)}{\log \alpha} \right| < 13 \cdot \alpha^{-n}. \tag{14}$$

Putting $\gamma := \dfrac{\log 2}{\log \alpha}$ and taking $M := 3.87 \cdot 10^{15}$, we found that $q_{44}$, the denominator of the $44$-th convergent of $\gamma$ exceeds $6M$. Also, taking

$$\mu := \frac{\log \left(3(1-\alpha^{m-n})^{-1}\right)}{\log \alpha}$$

with $n - m \in [3, 719]$ , a quick computation using *Mathematica* gives us the inequality

$$\epsilon = ||\mu q_{44}|| - M||\gamma q_{44}|| \geq 0.499076.$$

Let $A := 13,\ B := \alpha$, and $w := n$ in Lemma 2.2. Thus, with the help of *Mathematica*, we can say that the inequality (14) has no solution for $n \geq 101.212$. In that case $n \leq 101$. This contradicts our assumption that $n > 200$. Thus, we have to consider the cases $n - m = 1$ and $2$ to complete the proof. By Theorems 2.3 and 2.4, if $n - m = 1$, then we have the equation $3 \cdot 2^a = L_{m+1} - L_m = L_{m-1}$, which implies that $(n, m, a) = (4, 3, 0)$ ; and if $n - m = 2$, then we have the equation $3 \cdot 2^a = L_{m+2} - L_m = L_{m+1}$, which implies that $(n, m, a) = (3, 1, 0)$ .

This completes the proof. □

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