Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 4, 99–111
DOI: 10.7546/nntdm.2018.24.4.99-111

Explicit expression for symmetric identities of w-Catalan–Daehee polynomials

Taekyun Kim[1], Seog-Hoon Rim[2], Dmitry V. Dolgy[3] and Sung-Soo Pyo[4]

[1] Department of Mathematics, Tianjin Polytechnic University
Tianjin 300387, China, and
Department of Mathematics, Kwangwoon University
Seoul, Republic of Korea
e-mail: tkkim@kw.ac.kr

[2] Department of Mathematics Education, Kyungpook National University
Daegu, Republic of Korea
e-mail: shrim@knu.ac.kr

[3] Institute of Natural Sciences, Far Eastern Federal University
Vladivostok, 690950, Russia
e-mail: d_dol@mail.ru

[4] Department of Mathematics Education, Silla University
Busan, Republic of Korea
e-mail: ssoopyo@gmail.com

Received: 19 September 2018 **Accepted:** 26 October 2018

Abstract: Recently, Catalan–Daehee numbers are studied by several authors. In this paper, we consider the w-Catalan–Daehee polynomials and investigate some properties for those polynomials. In addition, we give explicit expression for the symmetric identities of the w- Catalan–Daehee polynomials which are derived from p-adic invariant integral on $\mathbb{Z}_p$.

Keywords: Catalan numbers, Daehee numbers, w-Catalan–Daehee numbers.
2010 Mathematics Subject Classification: 11B83, 11S80.

1 Introduction

In combinatorial mathematics, the Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively defined objects. The n-th Catalan numbers are defined in the terms of binomial coefficients which are given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n} \frac{n+k}{k} \quad (n \geq 0).$$

The generating function of Catalan number is given by

$$\frac{1-\sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} C_n t^n, \text{ (see [12, 15, 25])}. \tag{1}$$

In addition, the Catalan polynomials are also defined by the generating function to be

$$\frac{2}{1+\sqrt{1-4t}}(1-4t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!} \text{ (see [10, 14])}.$$

It is easy to show that the expression of $\sqrt{1+t}$ is given by

$$\sqrt{1+t} = \sum_{k=0}^{\infty}(-1)^{k-1}\binom{2k}{k}\frac{1}{4^k}\left(\frac{1}{2k-1}\right)t^k. \tag{2}$$

Replacing t by $-4t$ in (2), we have the generating function of Catalan numbers (1),

$$\begin{aligned}\sqrt{1-4t} &= 1 - 2\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{m+1}t^{m+1} \\ &= 1 - 2\sum_{m=0}^{\infty} C_m t^{m+1}.\end{aligned}$$

It is well known that the Daehee numbers, denoted by D_n, are defined by the generating function

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}. \tag{3}$$

Even though the Daehee numbers are easily calculated as $D_n = (-1)^n \frac{n!}{n+1}$, they play important roles in connecting relationships between special numbers (see [3, 5, 7, 9, 18, 20–25]).

The Catalan–Daehee numbers are defined by assigning $\sqrt{1-4t} - 1$ instead of t in the definition of Daehee numbers (3), as follows:

$$\frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1} = \sum_{n=0}^{\infty} d_n t^n, \text{(see [3, 14])}. \tag{4}$$

From (4), we note that

$$
\begin{aligned}
\frac{\frac{1}{2}\log(1-4t)}{\sqrt{1-4t}-1} &= \frac{1}{2}\sum_{l=0}^{\infty}\frac{4^l}{l+1}t^l\left(2-2\sum_{m=0}^{\infty}C_m t^{m+1}\right)\\
&= \sum_{l=0}^{\infty}\frac{4^l}{l+1}t^l\left(1-\sum_{m=0}^{\infty}C_m t^{m+1}\right)\\
&= \sum_{n=0}^{\infty}\frac{4^n}{n+1}t^n-\sum_{l=0}^{\infty}\frac{4^l}{l+1}t^l\sum_{m=0}^{\infty}C_m t^{m+1}\\
&= 1+\sum_{n=1}^{\infty}\left(\frac{4^n}{n+1}-\sum_{m=0}^{n-1}\frac{4^{n-m-1}}{n-m}C_m\right)t^n
\end{aligned}
\tag{5}
$$

From (4) and (5), we can derive the following equation (6)

$$
\begin{aligned}
d_n &= \begin{cases} 1, & \text{if } n=0,\\ \frac{4^n}{n+1}-\sum_{m=0}^{n-1}\frac{4^{n-m-1}}{n-m}C_m, & \text{if } n\geq 1,\end{cases}\\
&= -\sum_{m=0}^{n}\frac{4^{n-m}}{n-m+1}C_{m-1}^{*}, \text{ for all } n\geq 0,
\end{aligned}
\tag{6}
$$

where

$$
C_{m-1}^{*}=\begin{cases} -1 & \text{if } m=0,\\ C_{m-1} & \text{if } m\geq 1.\end{cases}
$$

From the generating function (1), a kind of generalization of the Catalan numbers, the so called w-Catalan numbers are introduced in [10], as follows:

$$
\frac{2}{1+\sqrt{(1-4t)^w}}(1-4t)^{\frac{w}{2}x}=\sum_{n=0}^{\infty}C_{n,w}(x)t^n. \tag{7}
$$

Recently, a group of mathematicians studied the symmetric identities of special polynomials which are derived from the p-adic invariant integral on $\mathbb{Z}_p$ (see [1,2,4,6,8,11,17,20]).

In this paper, we define w-Catalan–Daehee polynomials and numbers. We give some identities for w-Catalan–Daehee polynomials and numbers. In addition, we give some new explicit expression for w-Catalan–Daehee numbers which are derived from p-adic integrals on $\mathbb{Z}_p$.

2 The w-Catalan–Daehee polynomials

For $w\in\mathbb{N}$, we define the *w-Catalan–Daehee polynomials*, $d_{n,w}(x)$, using the generating function, as follows:

$$
\frac{\frac{1}{2}\log(1-4t)}{\sqrt{(1-4t)^w}-1}(1-4t)^{\frac{wx}{2}}=\sum_{n=0}^{\infty}d_{n,w}(x)t^n. \tag{8}
$$

Note that $\lim_{w\to 1}d_{n,w}(x)=d_n(x),(n\geq 0)$, and we call $d_{n,w}=d_{n,w}(0)$ w-Catalan–Daehee numbers.

From the definition of w-Cataln-Daehee numbers and Daehee numbers,

$$\begin{aligned}
\frac{\frac{1}{2}\log(1-4t)}{\sqrt{(1-4t)^w}-1} &= \frac{\frac{1}{w}\log\left((1-4t)^{\frac{w}{2}}-1+1\right)}{(1-4t)^{\frac{w}{2}}-1} \\
&= \frac{1}{w}\sum_{l=0}^{\infty} D_n \frac{\left((1-4t)^{\frac{w}{2}}-1\right)^l}{l!} \\
&= \frac{1}{w}\sum_{l=0}^{\infty} \frac{D_n}{l!}\sum_{k=0}^{l}\binom{l}{k}(1-4t)^{\frac{w}{2}k}(-1)^{l-k} \\
&= \frac{1}{w}\sum_{l=0}^{\infty} \frac{D_n}{l!}\sum_{k=0}^{l}\sum_{i=0}^{\infty}\binom{l}{k}\binom{\frac{w}{2}k}{i}(-4t)^i(-1)^{l-k} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{w}\sum_{l=0}^{\infty}\sum_{k=0}^{l}(-1)^{n+l-k}\frac{D_n}{l!}\binom{l}{k}\binom{\frac{w}{2}k}{n}4^n\right)t^n.
\end{aligned} \tag{9}$$

From the equation (8) and (9), we have a relation between the w-Catalan–Daehee and Daehee numbers.

Proposition 1. *For any* $w, n \in \mathbb{N}$*,*

$$d_{n,w} = \frac{1}{w}\sum_{l=0}^{\infty}\sum_{k=0}^{l}(-1)^{n+l-k}\frac{D_n}{l!}\binom{l}{k}\binom{\frac{w}{2}k}{n}4^n.$$

The following can be obtained from the definition of the w-Catalan–Daehee polynomials.

$$\begin{aligned}
\frac{\frac{1}{2}\log(1-4t)}{\sqrt{(1-4t)^w}-1}(1-4t)^{\frac{wx}{2}} &= \left(\frac{1}{2}\sum_{l=1}^{\infty}\frac{4^l}{l}t^l\right)\left(\sum_{k=0}^{\infty}(1-4t)^{\frac{w}{2}k}\right)(1-4t)^{\frac{wx}{2}} \\
&= \left(\frac{1}{2}\sum_{l=1}^{\infty}\frac{4^l}{l}t^l\right)\left(\sum_{k=0}^{\infty}(1-4t)^{\frac{w}{2}(k+x)}\right) \\
&= \left(\frac{1}{2}\sum_{l=1}^{\infty}\frac{4^l}{l}t^l\right)\left(\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{\frac{w}{2}(k+x)}{i}(-4)^i t^i\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=1}^{n}\sum_{k=0}^{\infty}\frac{1}{2}\frac{4^l}{l}\binom{\frac{w}{2}(k+x)}{n-l}(-4)^{n-l}\right)t^n
\end{aligned} \tag{10}$$

The equation (10) gives us an explicit formula for the w-Catalan–Daehee polynomials.

Proposition 2. *For any* $w, n \in \mathbb{N}$*,*

$$d_{n,w}(x) = \sum_{l=1}^{n}\sum_{k=0}^{\infty}\frac{1}{2}\frac{4^l}{l}\binom{\frac{w}{2}(k+x)}{n-l}(-4)^{n-l}.$$

It is natural to look for for a relationship between Daehee numbers and w-Catalan–Daehee numbers. For this, substituting $\frac{1-(1+t_0)^{\frac{2}{w}}}{4}$ instead of t in the definition of w-Catalan–Daehee numbers (8), the left side becomes

$$\frac{\frac{1}{2}\log\left(1-4\left(\frac{1-(1+t)^{\frac{2}{w}}}{4}\right)\right)}{\sqrt{\left(1-4\left(\frac{1-(1+t)^{\frac{2}{w}}}{4}\right)\right)^{w}}-1}=\frac{\frac{1}{w}\log(1+t)}{t}=\sum_{n=0}^{\infty}\frac{D_n}{w}\frac{t^n}{n!}, \tag{11}$$

and the right side becomes

$$\begin{aligned}\sum_{l=0}^{\infty}d_{l,w}\left(\frac{1-(1+t_0)^{\frac{2}{w}}}{4}\right)^{l}&=\sum_{l=0}^{\infty}\frac{d_{l,w}}{4^l}\sum_{k=0}^{l}(-1)^k(1+t)^{\frac{2}{w}k}\\&=\sum_{l=0}^{\infty}\frac{d_{l,w}}{4^l}\sum_{k=0}^{l}(-1)^k\sum_{i=0}^{\infty}\binom{\frac{2}{w}k}{i}t^i\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{d_{l,w}}{4^l}(-1)^k\binom{\frac{2}{w}k}{n}\right)t^n.\end{aligned} \tag{12}$$

From (11) and (12), we get the following

Proposition 3. *For any $w, n \in \mathbb{N}$,*

$$D_n = w\sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{d_{l,w}}{4^l}(-1)^k\binom{\frac{2}{w}k}{n}.$$

To observe relations between Catalan numbers and w-Catalan–Daehee numbers, substitute $\frac{1-(2+\sqrt{1-4t})^{\frac{2}{w}}}{4}$ for t in the definition of w-Catalan–Daehee numbers.

$$\begin{aligned}&\frac{\frac{1}{2}\log\left(1-4\left(\frac{1-(2+\sqrt{1-4t})^{\frac{2}{w}}}{4}\right)\right)}{\sqrt{\left(1-4\left(\frac{1-(2+\sqrt{1-4t})^{\frac{2}{w}}}{4}\right)\right)^{w}}-1}\\&\quad=\frac{\frac{1}{w}\log\left(2+\sqrt{1-4t}\right)}{1+\sqrt{1-4t}}\\&\quad=-\frac{1}{2w}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}(1+\sqrt{1-4t})^k\sum_{m=0}^{\infty}C_mt^m\\&\quad=-\frac{1}{2w}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sum_{l=0}^{k}\binom{k}{l}(1-4t)^{\frac{l}{2}}\sum_{m=0}^{\infty}C_mt^m\\&\quad=-\frac{1}{2w}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{i=0}^{\infty}\binom{\frac{l}{2}}{i}(-4t)^i\sum_{m=0}^{\infty}C_mt^m\\&\quad=-\frac{1}{2w}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{\frac{l}{2}}{i}(-4)^iC_{j-i}t^j\\&\quad=\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\sum_{l=0}^{k}\sum_{i=0}^{n}\frac{1}{2w}\frac{(-1)^{i+k+1}}{k}\binom{k}{l}\binom{\frac{l}{2}}{i}(-4)^iC_{n-i}\right)t^n.\end{aligned} \tag{13}$$

And the right side becomes

$$\begin{aligned}
&\sum_{k=0}^{\infty} d_{n,w}\left(\frac{1-(2+\sqrt{1-4t})^{\frac{2}{w}}}{4}\right)^k \\
&\quad= \sum_{k=0}^{\infty}\frac{d_{n,w}}{4^k}\sum_{l=0}^{k}(-1)^l(1+\sqrt{1-4t})^{\frac{2}{w}l} \\
&\quad= \sum_{k=0}^{\infty}\frac{d_{n,w}}{4^k}\sum_{l=0}^{k}(-1)^l\sum_{i=0}^{\infty}\binom{\frac{2}{w}l}{i}(1-4t)^{\frac{i}{2}} \\
&\quad= \sum_{k=0}^{\infty}\frac{d_{k,w}}{4^k}\sum_{l=0}^{k}(-1)^l\sum_{i=0}^{\infty}\binom{\frac{2}{w}l}{i}\sum_{j=0}^{\infty}\binom{\frac{i}{2}}{j}(-4t)^j \\
&\quad= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{d_{k,w}}{4^k}\sum_{l=0}^{k}\sum_{i=0}^{\infty}(-1)^{n+l}4^n\binom{\frac{2}{w}l}{i}\binom{\frac{i}{2}}{j}\right)t^n
\end{aligned} \tag{14}$$

From the equation (13) and (14), we have the following identity between the Catalan and w-Catalan–Daehee numbers.

Proposition 4. *For any $w, n \in \mathbb{N}$,*

$$\sum_{k=1}^{\infty}\sum_{l=0}^{k}\sum_{i=0}^{n}\frac{1}{2w}\frac{(-1)^{i+k+1}}{k}\binom{k}{l}\binom{\frac{l}{2}}{i}(-4)^iC_{n-i} = \sum_{k=0}^{\infty}\frac{d_{k,w}}{4^k}\sum_{l=0}^{k}\sum_{i=0}^{\infty}(-1)^{n+l}4^n\binom{\frac{2}{w}l}{i}\binom{\frac{i}{2}}{j}.$$

3 Symmetric identities of Catalan–Daehee numbers

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $f(x)$ be a uniformly differential function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic invariant integral $I_1(f)$ is given by

$$\begin{aligned}
I_1(f) &= \int_{\mathbb{Z}_p} f(x)d\mu(x) \\
&= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x),
\end{aligned}$$

(see [1, 2, 4, 6, 8, 11, 17, 20].

By taking $f_1(x) = f(x+1)$, the following integral equation is well-known

$$I_1(f_1) = I_1(f) + f'(0),$$

where $f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}$.

Note that

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{w(x+y)}{2}}d\mu(y)=\frac{\frac{1}{2}\log(1-4t)}{\sqrt{(1-4t)^w}-1}(1-4t)^{\frac{wx}{2}}$$
$$=\sum_{n=0}^{\infty}d_{n,w}(x)t^n$$

and since

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{w(x+y)}{2}}d\mu(y)=\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{\frac{w(x+y)}{2}}{n}(-4t)^n d\mu(y)$$
$$=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{w(x+y)}{2}}{n}d\mu(y)(-4t)^n,$$

we have

$$\int_{\mathbb{Z}_p}\binom{\frac{w(x+y)}{2}}{n}d\mu(y)=\frac{(-1)^n}{4^n}d_{n,w}(x).$$

Let us observe that

$$\begin{aligned}
&\frac{2}{w\log(1-4t)}\left(\int_{\mathbb{Z}_p}(1-4t)^{\frac{w(x+n)}{2}}d\mu(y)-\int_{\mathbb{Z}_p}(1-4t)^{\frac{wx}{2}}d\mu(y)\right)\\
&=\sum_{i=0}^{n-1}(1-4t)^{\frac{wi}{2}}\\
&=\sum_{i=0}^{n-1}\sum_{k=0}^{\infty}\left(\frac{wi}{2}\right)^k\frac{1}{k!}\left(\log(1-4t)\right)^k\\
&=\sum_{i=0}^{n-1}\sum_{k=0}^{\infty}\left(\frac{wi}{2}\right)^k\sum_{m=k}^{\infty}S_1(m,k)(-4)^m\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}(-1)^m\sum_{k=0}^{m}w^k\sum_{i=0}^{n-1}i^k 2^{2m-k}S_1(m,k)\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}w^k\sum_{i=0}^{n-1}i^k 2^{2m-k}\left|S_1(m,k)\right|\right)\frac{t^m}{m!},
\end{aligned}\tag{15}$$

where $S_1(m,k)$ denote the Stirling numbers of the first kind, so $|S_1(m,k)|$ mean the unsigned Stirling numbers of the first kind.

For simplicity, from now on we use $S_k(n)$ to denote $\sum_{i=0}^{n-1}i^k$ and

$$T_m(n,w)=\sum_{k=0}^{m}w^k S_k(n)2^{2m-k}\left|S_1(m,k)\right|.$$

Then the equation (15) becomes

$$\begin{aligned}
&\frac{2}{w\log(1-4t)}\left(\int_{\mathbb{Z}_p}(1-4t)^{\frac{w(x+y)}{2}}d\mu(y)-\int_{\mathbb{Z}_p}(1-4t)^{\frac{wx)}{2}}d\mu(y)\right)\\
&=\sum_{m=0}^{\infty}T_m(n,w)\frac{t^m}{m!}.
\end{aligned}\tag{16}$$

For $w_1, w_2 \in \mathbb{N}$, we set

$$I^{(m)}(w_1 w_2) = \frac{\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_1(\sum_m \mathbf{x} + w_2 x)}{2}} d\mathbf{x} \int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_2(\sum_m \mathbf{x} + w_1 x)}{2}} d\mathbf{x}}{\int_{\mathbb{Z}_p} (1-4t)^{\frac{w_1 w_2}{2} x} dx}, \tag{17}$$

where $\int_{\mathbb{Z}_p^m} f(x_1, x_2, \cdots, x_m) d\mathbf{x} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f(x_1, x_2, \cdots, x_m) dx_1 dx_2 \cdots dx_m$ and $\sum_m \mathbf{x} = x_1 + x_2 + \cdots + x_m$.

Note that $I^{(m)}(w_1, w_2)$ is symmetric in w_1 and w_2.

From (17), we have

$$\begin{aligned} I^{(m)}(w_1, w_2) &= \left(\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_1(\sum_m \mathbf{x})}{2}} d\mathbf{x} \right) (1-4t)^{\frac{w_1 w_2 x}{2}} \\ &\times \left(\frac{\int_{\mathbb{Z}_p} (1-4t)^{\frac{w_2 x_m}{2}} dx_m}{\int_{\mathbb{Z}_p} (1-4t)^{\frac{w_1 w_2}{2} x} dx} \right) \\ &\times \left(\int_{\mathbb{Z}_p^{m-1}} (1-4t)^{\frac{w_2(\sum_{m-1} \mathbf{x})}{2}} dx_1 dx_2 \cdots dx_{m-1} \right) (1-4t)^{\frac{w_1 w_2 y}{2}}. \end{aligned}$$

It is not difficult to show that

$$\begin{aligned} &\frac{2}{w \log(1-4t)} \left(\int_{\mathbb{Z}_p} (1-4t)^{\frac{w(x+n)}{2}} dx - \int_{\mathbb{Z}_p} (1-4t)^{\frac{wx)}{2}} dx \right) \\ &= \frac{n \int_{\mathbb{Z}_p} (1-4t)^{\frac{wx}{2}} dx}{\int_{\mathbb{Z}_p} (1-4t)^{\frac{nwx}{2}} dx} \\ &= \sum_{k=0}^{\infty} T_k(w) \frac{t^k}{k!}, \end{aligned}$$

and

$$\begin{aligned} &(1-4t)^{\frac{w_1 w_2}{2} x} \int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_1(\sum_m \mathbf{x})}{2}} d\mathbf{x} \\ &= \left(\frac{\frac{w_1}{2} \log(1-4t)}{(1-4t)^{\frac{w_1}{2}} - 1} \right)^m (1-4t)^{\frac{w_1 w_2}{2} x} \\ &= \sum_{n=0}^{\infty} d_{n,w_1}^{(m)}(w_2 x) t^n. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n,w_1}^{(m)}(w_2 x) t^n &= \left(\frac{\frac{w}{2} \log(1-4t)}{(1-4t)^{\frac{wx}{2}} - 1} \right)^m (1-4t)^{\frac{wx}{2}} \\ &= \left(\sum_{l=0}^{\infty} d_{l,w}^{(m)} t^l \right) \left(\sum_{k=0}^{\infty} \binom{\frac{wx}{2}}{k} (-4)^k t^k \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{\frac{wx}{2}}{k} (-1)^k 2^{2k} d_{n-k,w}^{(m)} \right) t^n. \end{aligned}$$

Hence

$$d_{n,w}^{(m)}(x)=\sum_{k=0}^{n}\binom{\frac{wx}{2}}{k}(-1)^k 2^{2k} d_{n-k,w}^{(m)}.$$

From (17), we have

$$\begin{aligned}
I^{(m)}(w_1,w_2) &= \left(\sum_{l=0}^{\infty} d_{l,w_1}^{(m)}(w_2x)t^l\right)\left(\sum_{k=0}^{\infty} T_k(w_1;w_2)\frac{t^k}{k!}\right)\left(\sum_{i=0}^{\infty} d_{l,w_1}^{(m-1)}(w_1y)t^i\right)\frac{1}{w_1}\\
&= \frac{1}{w_1}\left(\sum_{l=0}^{\infty} d_{l,w_1}^{(m)}(w_2x)t^l\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{T_k(w_1;w_2)}{k!}d_{j-k,w_2}^{(m-1)}(w,y)\right)t^i\right)\\
&= \sum_{l=0}^{\infty}\left(\frac{1}{w_1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{T_k(w_1;w_2)}{k!}d_{j-k,w_2}^{(m-1)}(w,y)d_{n-j,w_1}^{(m)}(w_2x)\right)t^n
\end{aligned}$$

On the other hand, by the symmetric property of $I^{(m)}(w_1,w_2)$.

$$\begin{aligned}
I^{(m)}(w_2,w_1) &= \left(\int_{\mathbb{Z}_p^m}(1-4t)^{\frac{w_2(\sum_m \mathbf{x})}{2}}d\mathbf{x}\right)(1-4t)^{\frac{w_1w_2x}{2}}\\
&\times\left(\frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{w_1x_m}{2}}dx_m}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{w_1w_2}{2}x}dx}\right)\\
&\times\left(\int_{\mathbb{Z}_p^{m-1}}(1-4t)^{\frac{w_1(\sum_{m-1}\mathbf{x})}{2}}dx_1dx_2\cdots dx_{m-1}\right)(1-4t)^{\frac{w_1w_2y}{2}}\\
&= \left(\sum_{l=0}^{\infty} d_{l,w_2}^{(m)}(w_1x)t^l\right)\left(\frac{1}{w_2}\sum_{k=0}^{\infty}T_k(w_2;w_1)\frac{t^k}{k!}\right)\\
&\times\left(\sum_{i=0}^{\infty} d_{i,w_2}^{(m-1)}(w_2y)t^i\right)\\
&= \frac{1}{w_2}\left(\sum_{l=0}^{\infty} d_{l,w_2}^{(m)}(w_1x)t^l\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{T_k(w_2;w_1)}{k!}d_{j-k,w_2}^{(m-1)}(w_2y)\right)t^j\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{w_2}\sum_{j=0}^{n}\sum_{k=0}^{j}\frac{T_k(w_2;w_1)}{k!}d_{n-j,w_2}^{(m)}(w_1x)d_{j-k,w_2}^{(m-1)}(w_2y)\right)t^n.
\end{aligned}$$

Therefore, by the symmetric property of $I^{(m)}(w_1,w_2)$ in w_1 and w_2, we obtain the following theorem.

Theorem 5. *For $m,w_1,w_2\in\mathbb{N}, n\geq 0$, we have*

$$\begin{aligned}
&\frac{1}{w_1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{T_k(w_1;w_2)}{k!}d_{j-k,w_2}^{(m-1)}(w,y)d_{n-j,w_1}^{(m)}(w_2x)\\
&\quad=\frac{1}{w_2}\sum_{j=0}^{n}\sum_{k=0}^{j}\frac{T_k(w_2;w_1)}{k!}d_{n-j,w_2}^{(m)}(w_1x)d_{j-k,w_2}^{(m-1)}(w_2y).
\end{aligned}$$

Now we observe that

$$
\begin{aligned}
I^{(m)}(w_1, w_2) &= \left(\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_1(\sum_m \mathbf{x})}{2}} d\mathbf{x}\right) (1-4t)^{\frac{w_1 w_2 x}{2}} \\
&\times \left(\frac{\int_{\mathbb{Z}_p} (1-4t)^{\frac{w_2 x_m}{2}} dx_m}{\int_{\mathbb{Z}_p} (1-4t)^{\frac{w_1 w_2}{2} x} dx}\right) \\
&\times \left(\int_{\mathbb{Z}_p^{m-1}} (1-4t)^{\frac{w_2(\sum_{m-1} \mathbf{x})}{2}} dx_1 dx_2 \cdots dx_{m-1}\right) (1-4t)^{\frac{w_1 w_2 y}{2}} \\
&= \frac{1}{w_1} \left(\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_1(\sum_m \mathbf{x})}{2}} d\mathbf{x}\right) (1-4t)^{\frac{w_1 w_2 x}{2}} \left(\sum_{i=0}^{w_1-1} (1-4t)^{\frac{w_2 i}{2}}\right) \\
&\times \left(\int_{\mathbb{Z}_p^{m-1}} (1-4t)^{\frac{w_2(\sum_{m-1} \mathbf{x})}{2}} dx_1 dx_2 \cdots dx_{m-1}\right) (1-4t)^{\frac{w_1 w_2 y}{2}} \\
&= \frac{1}{w_1} \sum_{i=0}^{w_1-1} \left(\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_1(\sum_m \mathbf{x} + \frac{w_2}{w_1} i + w_2 x)}{2}} d\mathbf{x}\right) \\
&\times \left(\int_{\mathbb{Z}_p^{m-1}} (1-4t)^{\frac{w_2(\sum_{m-1} \mathbf{x})}{2}} dx_1 dx_2 \cdots dx_{m-1}\right) \\
&= \frac{1}{w_1} \left(\sum_{i=0}^{w_1-1} \sum_{k=0}^{\infty} d_{k,w_1}^{(m)}(w_2 x + \frac{w_2}{w_1} i) t^k\right) \left(\sum_{l=0}^{\infty} d_{l,w_2}^{(m-1)}(w_1 y) t^l\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{w_1} \sum_{k=0}^{n} \sum_{i=0}^{w_1-1} d_{k,w_1}^{(m)}(w_2 x + \frac{w_2}{w_1} i) d_{n-k,w_2}^{(m-1)}(w_1 y)\right) t^n
\end{aligned}
$$

On the other hand, we set

$$
\begin{aligned}
I^{(m)}(w_2, w_1) &= \left(\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_2}{2}(\sum_m \mathbf{x})} d\mathbf{x}\right) (1-4t)^{\frac{w_1 w_2}{2} x} \left(\frac{\int_{\mathbb{Z}_p} (1-4t)^{\frac{w_1}{2} x_m} dx_m}{\int_{\mathbb{Z}_p} (1-4t)^{\frac{w_1 w_2}{2} x} dx}\right) \\
&\times \left(\int_{\mathbb{Z}_p^{m-1}} (1-4t)^{\frac{w_1}{2}(\sum_{m-1} \mathbf{x})} dx_1 dx_2 \cdots dx_{m-1}\right) (1-4t)^{\frac{w_1 w_2}{2} y} \\
&= \frac{1}{w_2} \left(\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_2}{2}(\sum_m \mathbf{x})} d\mathbf{x}\right) (1-4t)^{\frac{w_1 w_2}{2} x} \left(\sum_{i=0}^{w_2-1} (1-4t)^{\frac{w_1}{2} i}\right) \\
&\times \left(\int_{\mathbb{Z}_p^{m-1}} (1-4t)^{\frac{w_1}{2}(\sum_{m-1} \mathbf{x} + w_2 y)} dx_1 dx_2 \cdots dx_{m-1}\right) \\
&= \frac{1}{w_2} \sum_{i=0}^{w_2-1} \left(\int_{\mathbb{Z}_p^m} (1-4t)^{\frac{w_2}{2}(\sum_m \mathbf{x} + \frac{w_1}{w_2} i + w_1 x)} d\mathbf{x}\right) \\
&\times \left(\int_{\mathbb{Z}_p^{m-1}} (1-4t)^{\frac{w_1}{2}(\sum_{m-1} \mathbf{x} + w_2 y)} dx_1 dx_2 \cdots dx_{m-1}\right)
\end{aligned}
$$

$$= \frac{1}{w_2}\left(\sum_{i=0}^{w_2-1}\sum_{k=0}^{\infty} d_{k,w_2}^{(m)}(w_1 x + \frac{w_1}{w_2} i) t^k\right)\left(\sum_{l=0}^{\infty} d_{l,w_1}^{(m-1)}(w_2 y) t^l\right)$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{w_2}\sum_{k=0}^{n}\sum_{i=0}^{w_2-1} d_{k,w_2}^{(m)}(w_1 x + \frac{w_1}{w_2} i) d_{n-k,w_1}^{(m-1)}(w_2 y)\right) t^n.$$

Therefore, by the symmetric property of $I^{(m)}(w_1, w_2)$ in w_1 and w_2, we obtain the following theorem.

Theorem 6. *For $w_1, w_2, m \in \mathbb{N}$ and $n \geq 0$, we have*

$$\frac{1}{w_1}\sum_{k=0}^{n}\sum_{i=0}^{w_1-1} d_{k,w_1}^{(m)}(w_2 x + \frac{w_2}{w_1} i) d_{n-k,w_2}^{(m-1)}(w_1 y)$$
$$= \frac{1}{w_2}\sum_{k=0}^{n}\sum_{i=0}^{w_2-1} d_{k,w_2}^{(m)}(w_1 x + \frac{w_1}{w_2} i) d_{n-k,w_1}^{(m-1)}(w_2 y).$$

Remark. Let $y = 0$ and $m = 1$, then we note that

$$\frac{1}{w_1}\sum_{i=0}^{w_1-1} d_{k,w_1}(w_2 x + \frac{w_2}{w_1} i)$$
$$= \frac{1}{w_2}\sum_{i=0}^{w_2-1} d_{k,w_2}(w_1 x + \frac{w_1}{w_2} i)$$

Taking $w_2 = 1$, we have

$$d_k(w_1 x) = \frac{1}{w_1}\sum_{i=0}^{w_1-1} d_{n,w-1}(x + \frac{1}{w_1} i).$$

4 Results and discussion

In this paper, we have defined the w-Catalan–Daehee polynomials and numbers,

$$\frac{\frac{1}{2}\log(1-4t)}{\sqrt{(1-4t)^w}-1}(1-4t)^{\frac{w}{2}x} = \sum_{n=0}^{\infty} d_{n,w}(x) t^n.$$

These are closely related Catalan, Daehee and Catalan–Daehee numbers. In Proposition 2, we gave an explicit formula for the w-Catalan–Daehee polynomials. In Propositions 1 and 3, we give relations between the w-Catalan–Daehee and Daehee numbers. Proposition 4 expresses a relation between the w-Catalan–Daehee and Catalan numbers.

In Section 3, we gave explicit expression for symmetric identities of the w-Catalan–Daehee polynomials, which are derived from p-adic invariant integral on $\mathbb{Z}_p$.

5 Conclusion

In this paper, we defined and investigated the symmetric property of the w-Catalan–Daehee polynomials. In addition, by using the p-adic integral on $\mathbb{Z}_p$, we explicitly showed that the w-Catalan–Daehee polynomials have symmetric identities from the p-adic invariant integral on $\mathbb{Z}_p$.

Acknowledgement

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018-01-019).

References

[1] Choi, S. (2018) Linear symmetry of the modified q-Euler polynomials. *Adv. Stud. Contemp. Math.* (Kyungshang), 28 (2), 201–206.

[2] Choi, S., Kim, T., Kwon, H.-I., & Kwon, J. (2018) Quadratic symmetry of modified q-Euler polynomials, *Adv. Difference Equ.*, Paper No. 38, 9 pages.

[3] Dolgy, D. V., Jang, G.-W., Kim, D. S., & Kim, T. (2017) Explicit Expressions for Catalan–Daehee numbers, *Proc. Jangjeon Math. Soc.*, 20(1), 1–9.

[4] Duran, U., Acikgoz, M., & Araci, S. (2015) Symmetric identities involving weighted q-Genocchi polynomials under S_4. *Proc. Jangjeon Math. Soc.*, 18 (4), 455–465.

[5] El-Desouky, B. S., & Mustafa, A. (2016) New results on higher-order Daehee and Bernoulli numbers and polynomials, *Adv. Difference Equ.*, 2016, Paper No. 32, 21 pages.

[6] He, Y. (2013) Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials. *Adv. Difference Equ.*, 2013:246, 10 pages.

[7] Jang, G.-W., Kwon, J., & Lee, J. G. (2017) Some identities of degenerate Daehee numbers arising from nonlinear differential equation. *Adv. Difference Equ.*, 2017, Paper No. 206, 10 pages.

[8] Jang, L.-C. (2011) A family of Barnes-type multiple twisted q -Euler numbers and polynomials related to Fermionic p-adic invariant integrals on Z_p, *J. Comput. Anal. Appl.*, 13 (2), 376–387.

[9] Khan, W. A., Nisar, K. S., Duran, U., Acikgoz, M., & Araci, S. (2018) Multifarious implicit summation formulae of Hermite-based poly-Daehee polynomials, *Appl. Math. Inf. Sci.*, 12 (2), 305–310.

[10] Kim, D. S., & Kim, T. (2017) Triple symmetric identities for w-Catalan polynomials, *J. Korean Math. Soc.*, 54 (4), 1243–1264.

[11] Kim, D. S., Lee, N., Na, J., & Park, K. H. (2012) Identities of symmetry for higher-order Euler polynomials in three variables (I). *Adv. Stud. Contemp. Math. (Kyungshang)*, 22 (1), 51–74.

[12] Kim, T. (2016) A note in Catalan numbers associated with p-adic integral in $\mathbb{Z}_p$, *Proc. Jangjeon Math. Soc.*, 19 (3), 493–501.

[13] Kim, T., & Kim, D. S. (2017) Differential equations associated with Catalan–Daehee numbers and their applications, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas* 111 (4), 1071–1081.

[14] Kim, T., Kim, D. S., & Seo, J.-J. (2016) Symmetric identities for an analogue of Catalan polynomials, *Proc. Jangjeon Math. Soc.*, 19 (3), 515–521.

[15] Koshy, T. (2009) *Catalan Numbers with Applications*. Oxford University Press, Oxford.

[16] Kwon, J., Sohn, G., & Park, J.-W. (2018) Symmetric identities for (h, q)-extensions of the generalized higher order modified q-Euler polynomials. *J. Comput. Anal. Appl.*, 24 (8), 1431–1438.

[17] Lim, D., Kwon, J. (2016) A note on poly-Daehee numbers and polynomials. *Proc. Jangjeon Math. Soc.*, 19 (2), 219—224.

[18] Liu, C., & Wuyungaowa, W. (2018) Application of probabilistic method on Daehee sequences, *Eur. J. Pure Appl. Math.*, 11 (1), 69–78.

[19] Moon, E.-J., Rim, S.-H., Jin, J.-H., & Lee, S.-J. (2010) On the symmetric properties of higher-order twisted q-Euler numbers and polynomials, *Adv. Difference Equ.*, 2010, Art. ID 765259, 8 pages.

[20] Park, J.-W. (2016) On the λ-Daehee polynomials with q-parameter. *J. Comput. Anal. Appl.*, 20 (1), 11–20.

[21] Pyo, S.-S., Kim, T., & Rim, S.-H. (2017) Identities of the degenerate Daehee numbers with the Bernoulli numbers of the second kind arising from nonlinear differential equation, *J. Nonlinear Sci. Appl.*, 10, 6219–6228.

[22] Pyo, S.-S., Kim, T., & Rim, S.-H. *Degenerate Daehee numbers of the third kind* (submitted).

[23] Simsek, Y. (2017) Identities on the Changhee numbers and Apostol-type Daehee polynomials, *Adv. Stud. Contemp. Math.* (Kyungshang), 27 (2), 199–212.

[24] Simsek, Y., & Yardimci, A. (2016) Applications on the Apostol–Daehee numbers and polynomials associated with special numbers, polynomials, and p-adic integrals. *Adv. Difference Equ.*, 2016, Paper No. 308, 14 pages.

[25] Stanley, R. P. (2015) *Catalan numbers*. Cambridge University Press, New York.