Some combinatorial identities via Stirling transform

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Abstract: The aim of this paper is to present some results on the use of the generalized Stirling transform. First, we establish a generalization of a recent Guo–Qi’s identity for Bell numbers. Finally, a new explicit formula for Euler numbers are given.

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1 Introduction

Following the usual notations (see [3]), the falling factorial $x^n \ (x \in \mathbb{C})$ is defined by $x^0 = 1$, $x^n = x(x-1)\cdots(x-n+1)$ for $n > 0$ and, the rising factorial denoted by $x^n$, is defined by $x^n = x(x+1)\cdots(x+n-1)$ with $x^0 = 1$. The (signed) Stirling numbers of the first kind $s(n,k)$ are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} s(n,k) x^k.$$
The Stirling numbers of the second kind, denoted by \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \), are the coefficients in the expansion
\[
x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k.
\]

The Stirling numbers of the second kind \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) count the number of ways to partition a set of \( n \) elements into exactly \( k \) nonempty subsets. The number of all partitions is the Bell number \( B_n \), thus
\[
B_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}.
\]

The polynomials
\[
B_n (x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k
\]
are called Bell polynomials or exponential polynomials. The exponential generating functions are respectively
\[
\sum_{n\geq k} s(n, k) \frac{z^n}{n!} = \frac{1}{k!} (\ln (1 + z))^k,
\]
\[
\sum_{n\geq k} \left\{ \begin{array}{c} n \\ k \end{array} \right\} \frac{z^n}{n!} = \frac{1}{k!} (e^z - 1)^k
\]
and
\[
\sum_{n\geq 0} B_n (x) \frac{z^n}{n!} = \exp (x (e^z - 1)).
\]

Recently, the third author [6] developed a methodology for computing the Stirling transform and the inverse Stirling transform. More precisely, given a sequence \( a_m := a_{0,m} (m \geq 0) \), we construct an infinite matrix \( S := (a_{n,m}) \) as follows:

1. The first row \( a_{0,m} \) of the matrix is the initial sequence; the first column \( b_n := a_{n,0} (n \geq 0) \) is called the final sequence, and each entry \( a_{n,m} \) is given recursively by
\[
a_{n+1,m} = a_{n,m+1} + ma_{n,m}.
\]
(1)

2. Conversely, if we start with the final sequence, the matrix \( S \) can be recovered by the recursive relations
\[
a_{n,m+1} = a_{n+1,m} - ma_{n,m}.
\]

Theorem 1. [6] For \( n, m \geq 0 \), we have
\[
a_{n,m} = \sum_{k=0}^{m} s(m, k) b_{n+k} = \sum_{k=0}^{n} \left\{ \begin{array}{c} n + m \\ k + m \end{array} \right\}_m a_{m+k}.
\]

(3)

Recall that the \( r \)-Stirling numbers of the second kind (see [1] for more details) \( \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r \) count the number of partitions of a set of \( n \) objects into exactly \( k \) nonempty, disjoint subsets, such that the first \( r \) elements are in distinct subsets. The exponential generating function is given by
\[
\sum_{n\geq k} \left\{ \begin{array}{c} n + r \\ k + r \end{array} \right\}_r \frac{z^n}{n!} = \frac{1}{k!} e^{rz} (e^z - 1)^k.
\]
Theorem 2. Suppose that the initial sequence \( a_{0,m+r} \) has the following exponential generating function \( A_r (z) = \sum_{k \geq 0} a_{0,k+r} \frac{z^k}{k!} \). Then the sequence \( \{a_{r,n}\}_n \) of the \( r \)-th columns of the matrix \( S \) has an exponential generating function \( B_r (z) = \sum_{n \geq 0} a_{n,r,m} \frac{z^n}{n!} \), given by
\[
B_r (z) = e^{rz} A_r (e^z - 1).
\] (4)

Theorem 3. Suppose that the final sequence \( a_{n+r,0} \) has the following exponential generating function \( B_r (z) = \sum_{k \geq 0} a_{r,n} \frac{z^k}{k!} \). Then the sequence \( \{a_{r,m}\}_m \) of the \( r \)-th rows of the matrix \( S \) has an exponential generating function \( A_r (z) = \sum_{m \geq 0} a_{r,m} \frac{z^m}{m!} \), given by
\[
A_r (z) = B_r (\ln(1 + z)).
\] (5)

2 On Guo–Qi’s identity for Bell numbers

The (unsigned) Lah numbers \( \left[\begin{array}{c}n \\ k\end{array}\right] \) are the coefficients expressing rising factorials in terms of falling factorials
\[
x^\pi = \sum_{k=0}^{n} \left[\begin{array}{c}n \\ k\end{array}\right] x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} (-1)^{n-k} \left[\begin{array}{c}n \\ k\end{array}\right] x^k
\]
or
\[
\left[\begin{array}{c}n \\ k\end{array}\right] = \sum_{j=k}^{n} \left(\begin{array}{c}n \\ j\end{array}\right) s(n, j) \left\{\begin{array}{c}j \\ k\end{array}\right\}.
\]

The (unsigned) Lah numbers \( \left[\begin{array}{c}n \\ k\end{array}\right] \) count the number of partitions of a set of \( n \) elements into exactly \( k \) ordered lists
\[
\left[\begin{array}{c}n \\ k\end{array}\right] = \frac{n!}{k! (n-k)!} \quad \text{for} \quad n \geq k \geq 1.
\]

Let \( L_{n,k} \) denote the (signed) Lah numbers [2]
\[
L_{n,k} := (-1)^n \left[\begin{array}{c}n \\ k\end{array}\right].
\]

The exponential generating functions is
\[
\sum_{n\geq k} \left[\begin{array}{c}n \\ k\end{array}\right] \frac{z^n}{n!} = \frac{1}{k!} \left( \frac{z}{1-z} \right)^k.
\]

Setting the final sequence \( a_{n,0} = (-1)^n B_n (x) \) in (2), we get the following matrix
\[
S = \begin{pmatrix}
1 & -x & x^2 + 2x & -x^3 - 6x^2 - 6x & \cdots \\
-x & x^2 + x & -x^3 - 4x^2 - 2x & x^4 + 9x^3 + 18x^2 + 6x & \cdots \\
x^2 + x & -x^3 - 3x^2 - x & x^4 + 7x^3 + 10x^2 + 2x & \cdots \\
-x^3 - 3x^2 - x & x^4 + 6x^3 + 7x^2 + x & \cdots \\
x^4 + 6x^3 + 7x^2 + x & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

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Since

\[ B_0(z) = \sum_{k \geq 0} B_k(x) \frac{(-z)^k}{k!} \]

\[ = \exp \left( x (e^{-z} - 1) \right). \]

It follows from (5) that the initial sequence has an exponential generating function, given by

\[ A_0(z) = B_0(\ln(1 + z)) \]

\[ = \exp \left( x \left( \frac{1}{1 + z} - 1 \right) \right) \]

\[ = \sum_{n \geq 0} L_n(x) \frac{z^n}{n!}, \]

where \( L_n(x) \) denotes the (signed) Lah polynomials defined by

\[ L_n(x) := \sum_{k=0}^{n} L_{n,k} x^k. \]

Now, from (6) we have

\[ \sum_{n \geq 0} L_n(x) \frac{z^n}{n!} = \exp \left( x \left( \frac{1}{1 + z} - 1 \right) \right) \]

\[ = \frac{1}{e^x} + \frac{1}{e^x} \sum_{k \geq 1} \left( \frac{1}{1 + z} \right)^k \frac{x^k}{k!} \]

\[ = \frac{1}{e^x} + \frac{1}{e^x} \left( \sum_{k \geq 1} \left( \sum_{n \geq 0} (-1)^n \frac{(k - 1 + n)!}{n! (k - 1)!} \frac{z^n}{n!} \right) \frac{x^k}{k!} \right) \]

\[ = \frac{1}{e^x} + \sum_{n \geq 0} \left( \frac{1}{e^x} \sum_{k \geq 0} (-1)^n \frac{(k + n)!}{k! (k + 1)!} \frac{z^n}{n!} \right) \frac{x^k}{k!} \]

Equating the coefficients of \( \frac{z^n}{n!} \), we get the Dobiński’s formula for the \( n \)-th \( (n \geq 1) \) signed Lah polynomials

\[ L_n(x) = \frac{(-1)^n}{e^x} \sum_{k \geq 0} \frac{(k + n)!}{k! (k + 1)!} x^{k+1}, \quad n \geq 1. \]

A variation of Dobiński’s formula in terms of Kummer confluent hypergeometric functions is given by

\[ L_n(x) = \frac{(-1)^n x}{e^x} \sum_{k \geq 0} \frac{(n + 1) \xi}{2^k} \frac{x^k}{k!} \]

\[ = \frac{(-1)^n n! x}{e^x} 1_F \left( \frac{n + 1}{2}; x \right), \]

where \( 1_F \left( \frac{a}{b}; z \right) \) denotes the Kummer confluent hypergeometric functions, defined by \( \sum_{n \geq 0} \frac{a^n z^n}{n!} \).
Theorem 4. For \( n, m \geq 0 \), we have
\[
\sum_{k=0}^{m} s(m, k) (-1)^{n+k} B_{n+k}(x) = \sum_{k=0}^{n} \left\{ \frac{n+m}{k+m} \right\} m_{m+k}(x). \quad (7)
\]

If we set \( m = 0 \) in (7), we have
\[
B_n(x) = \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} (-1)^k L_k(x) = \frac{x}{e^x} \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} n! F_1\left( \frac{n+1}{2}; x \right).
\]

Notice that the Guo–Qi’s identity for Bell numbers [4, 5] is obtained by setting \( x = 1 \) in (8).

When \( n = 0 \) in (7) we get the inverse Stirling transform for Bell polynomials
\[
L_m(x) = \sum_{k=0}^{m} s(m, k) (-1)^k B_k(x).
\]

Now, setting \( n = 1 \) in (7), we get
\[
\sum_{k=0}^{m} s(m, k) (-1)^{k+1} B_{k+1}(x) = mL_m(x) + L_{m+1}(x) = \sum_{k=0}^{m} (-1)^{m+1} k! \binom{m}{k} 2^{m-k+1} x^{m-k+1}.
\]

3 An explicit formula for Euler numbers

The Euler numbers \( E_n \) can be defined by the exponential generating function
\[
\frac{1}{\cosh z} = \sum_{n \geq 0} E_n \frac{z^n}{n!}.
\]

It is well-known that \( E_n \) are a sequence of integers with \( E_{2n+1} = 0 \) for \( n \geq 0 \). There are a number of explicit formulae for \( E_n \), for example (see also [7])
\[
E_{2n} = i \sum_{k=1}^{2n+1} \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} (k - 2j)^{2n+1},
\]

where \( i \) denotes the imaginary unit with \( i^2 = -1 \). In this section, we propose a new explicit formula for the Euler numbers
\[
E_n = -\sum_{k=0}^{n} \frac{k!}{2^k} \binom{n}{k} \text{Re} ((i-1)^{k+1}), \quad (9)
\]

where \( \text{Re}(z) \), denotes the real part of \( z \).
Now, if we take the final sequence $a_{n,0} = E_n$, in (2), we get the following matrix

$$T = \begin{pmatrix}
1 & 0 & -1 & 3 & -6 & \cdots \\
0 & -1 & 1 & 3 & -24 & \cdots \\
-1 & 0 & 5 & -15 & -6 & \cdots \\
0 & 5 & -5 & -51 & 336 & \cdots \\
5 & 0 & -61 & 183 & 714 & \cdots \\
0 & -61 & 61 & 1263 & -7944 & \cdots \\
-61 & 0 & 1385 & -4155 & -35286 & \cdots \\
0 & 1385 & -1385 & -47751 & 294816 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$  

It follows from (5) that the initial sequence has an exponential generating function given by

$$\sum_{n \geq 0} R_n \frac{z^n}{n!} = \frac{1}{\cosh (\ln(1 + z))} = \frac{2 (1 + z)}{2 + 2z + z^2}.$$  

Since \( \frac{2 (1 + z)}{2 + 2z + z^2} = \frac{1}{z + 1 - i} + \frac{1}{z + 1 + i} \), we have

$$\sum_{n \geq 0} R_n \frac{z^n}{n!} = -\sum_{n \geq 0} \frac{1}{2^{n+1}} ((-1 + i)^{n+1} + (-1 - i)^{n+1}) z^n$$

$$= -\sum_{n \geq 0} \frac{1}{2^n} \text{Re} \left((i - 1)^{n+1}\right) z^n.$$

Equating the coefficient of \( z^n \), we get

$$R_n = -\frac{n!}{2^n} \text{Re} \left((i - 1)^{n+1}\right)$$

and by the binomial formula we have

$$R_n = \frac{n!}{2^n} \sum_{k=0}^\lfloor (n+1)/2 \rfloor (-1)^{n-k} \binom{n+1}{2k}.$$  

where \( \lfloor x \rfloor \) denotes the integral part of \( x \), that is, the greatest integer not exceeding \( x \).

For \( n, m \geq 0 \), we have

$$\sum_{k=0}^m s(m,k) E_{n+k} = \sum_{k=0}^n \binom{n+m}{k+m} R_{m+k}.$$  

(12)

Now, setting \( m = 0 \) in (12), we get (9). Putting \( n = 0 \) in (12), we have the following recursive formula Euler numbers involving the Stirling numbers of the first kind

$$\sum_{k=0}^m s(2m, 2k) E_{2k} = \frac{(2m)!}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{2k}$$

$$= \begin{cases} 
-1 \left\lfloor \frac{m}{2} \right\rfloor \frac{(2m)!}{2^{2m}}, & m \text{ even}, \\
(-1)^{\left\lfloor \frac{m-1}{2} \right\rfloor + 1} \frac{(2m)!}{2^{2m}}, & m \text{ odd}.
\end{cases}$$
Thus, for example, when \( m = 0, 1, 2, 3 \), we obtain

\[
E_0 = 1, \\
E_2 = -1, \\
11E_2 + E_4 = -6, \\
274E_2 + 85E_4 + E_6 = 90.
\]

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**References**


