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Some combinatorial identities via Stirling transform

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Abstract: The aim of this paper is to present some results on the use of the generalized Stirling transform. First, we establish a generalization of a recent Guo–Qi's identity for Bell numbers. Finally, a new explicit formula for Euler numbers are given.
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1 Introduction

Following the usual notations (see [3]), the falling factorial $x^{\underline{n}}$ $(x \in \mathbb{C})$ is defined by $x^{\underline{0}} = 1$, $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ for $n > 0$ and, the rising factorial denoted by $x^{\overline{n}}$, is defined by $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ with $x^{\overline{0}} = 1$. The (signed) Stirling numbers of the first kind $s(n,k)$ are the coefficients in the expansion

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)\, x^k.$$

The Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$, are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}.$$

The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count the number of ways to partition a set of n elements into exactly k nonempty subsets. The number of all partitions is the Bell number B_n, thus

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

The polynomials

$$B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$$

are called Bell polynomials or exponential polynomials. The exponential generating functions are respectively

$$\sum_{n \geq k} s(n,k) \frac{z^n}{n!} = \frac{1}{k!} \left(\ln(1+z) \right)^k,$$

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{z^n}{n!} = \frac{1}{k!} \left(e^z - 1 \right)^k$$

and

$$\sum_{n \geq 0} B_n(x) \frac{z^n}{n!} = \exp\left(x \left(e^z - 1 \right) \right).$$

Recently, the third author [6] developed a methodology for computing the Stirling transform and the inverse Stirling transform. More precisely, given a sequence $a_m := a_{0,m}$ $(m \geq 0)$, we construct an infinite matrix $\mathcal{S} := (a_{n,m})$ as follows:

1. The first row $a_{0,m}$ of the matrix is the initial sequence; the first column $b_n := a_{n,0}$ $(n \geq 0)$ is called the final sequence, and each entry $a_{n,m}$ is given recursively by

$$a_{n+1,m} = a_{n,m+1} + m a_{n,m}. \tag{1}$$

2. Conversely, if we start with the final sequence, the matrix $\mathcal{S}$ can be recovered by the recursive relations

$$a_{n,m+1} = a_{n+1,m} - m a_{n,m}. \tag{2}$$

Theorem 1. [6] *For $n, m \geq 0$, we have*

$$a_{n,m} = \sum_{k=0}^{m} s(m,k)\, b_{n+k} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_m a_{m+k}. \tag{3}$$

Recall that the r-Stirling numbers of the second kind (see [1] for more details) $\left\{ {n \atop k} \right\}_r$ count the number of partitions of a set of n objects into exactly k nonempty, disjoint subsets, such that the first r elements are in distinct subsets. The exponential generating function is given by

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{z^n}{n!} = \frac{1}{k!} e^{rz} \left(e^z - 1 \right)^k.$$

Theorem 2. [6] *Suppose that the initial sequence $a_{0,m+r}$ has the following exponential generating function $A_r(z) = \sum_{k\geq 0} a_{0,k+r}\frac{z^k}{k!}$. Then the sequence $\{a_{n,r}\}_n$ of the r-th columns of the matrix $\mathcal{S}$ has an exponential generating function $\mathcal{B}_r(z) = \sum_{n\geq 0} a_{n,r}\frac{z^n}{n!}$, given by*

$$B_r(z) = e^{rz}A_r(e^z - 1). \tag{4}$$

Theorem 3. [6] *Suppose that the final sequence $a_{n+r,0}$ has the following exponential generating function $\mathcal{B}_r(z) = \sum_{k\geq 0} a_{k+r,0}\frac{z^k}{k!}$. Then the sequence $\{a_{r,m}\}_m$ of the r-th rows of the matrix $\mathcal{S}$ has an exponential generating function $\mathcal{A}_r(z) = \sum_{m\geq 0} a_{r,m}\frac{z^m}{m!}$, given by*

$$\mathcal{A}_r(z) = \mathcal{B}_r(\ln(1+z)). \tag{5}$$

2 On Guo–Qi's identity for Bell numbers

The (unsigned) Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ are the coefficients expressing rising factorials in terms of falling factorials

$$x^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor x^{\underline{k}} \text{ and } x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \left\lfloor {n \atop k} \right\rfloor x^{\overline{k}}$$

or

$$\left\lfloor {n \atop k} \right\rfloor = \sum_{j=k}^{n} (-1)^{n-j} s(n,j) \left\{ {j \atop k} \right\}.$$

The (unsigned) Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ count the number of partitions of a set of n elements into exactly k ordered lists

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{n-1}{k-1} \text{ for } n \geq k \geq 1.$$

Let $L_{n,k}$ denote the (signed) Lah numbers [2]

$$L_{n,k} := (-1)^n \left\lfloor {n \atop k} \right\rfloor.$$

The exponential generating functions is

$$\sum_{n\geq k} \left\lfloor {n \atop k} \right\rfloor \frac{z^n}{n!} = \frac{1}{k!}\left(\frac{z}{1-z}\right)^k.$$

Setting the final sequence $a_{n,0} = (-1)^n B_n(x)$ in (2), we get the following matrix

$$S = \begin{pmatrix} 1 & -x & x^2+2x & -x^3-6x^2-6x & \cdots \\ -x & x^2+x & -x^3-4x^2-2x & x^4+9x^3+18x^2+6x & \cdots \\ x^2+x & -x^3-3x^2-x & x^4+7x^3+10x^2+2x & \vdots & \\ -x^3-3x^2-x & x^4+6x^3+7x^2+x & \vdots & & \\ x^4+6x^3+7x^2+x & \vdots & & & \\ \vdots & & & & \end{pmatrix}$$

Since

$$\begin{aligned}\mathcal{B}_0(z) &= \sum_{k\geq 0} B_k(x)\frac{(-z)^k}{k!}\\ &= \exp\left(x\left(e^{-z}-1\right)\right).\end{aligned}$$

It follows from (5) that the initial sequence has an exponential generating function, given by

$$\begin{aligned}\mathcal{A}_0(z) &= \mathcal{B}_0(\ln(1+z))\\ &= \exp\left(x\left(\frac{1}{1+z}-1\right)\right) \qquad (6)\\ &= \sum_{n\geq 0} L_n(x)\frac{z^n}{n!},\end{aligned}$$

where $L_n(x)$ denotes the (signed) Lah polynomials defined by

$$L_n(x) := \sum_{k=0}^{n} L_{n,k}\, x^k.$$

Now, from (6) we have

$$\begin{aligned}\sum_{n\geq 0} L_n(x)\frac{z^n}{n!} &= \exp\left(x\left(\frac{1}{1+z}-1\right)\right)\\ &= \frac{1}{e^x}+\frac{1}{e^x}\sum_{k\geq 1}\left(\frac{1}{1+z}\right)^k\frac{x^k}{k!}\\ &= \frac{1}{e^x}+\frac{1}{e^x}\left(\sum_{k\geq 1}\left(\sum_{n\geq 0}(-1)^n\frac{(k-1+n)!}{n!\,(k-1)!}z^n\right)\frac{x^k}{k!}\right)\\ &= \frac{1}{e^x}+\sum_{n\geq 0}\left(\frac{1}{e^x}\sum_{k\geq 0}(-1)^n\frac{(k+n)!}{k!\,(k+1)!}x^{k+1}\right)\frac{z^n}{n!}\end{aligned}$$

Equating the coefficients of $\dfrac{z^n}{n!}$, we get the Dobiński's formula for the n-th ($n \geq 1$) signed Lah polynomials

$$L_n(x) = \frac{(-1)^n}{e^x}\sum_{k\geq 0}\frac{(k+n)!}{k!\,(k+1)!}x^{k+1},\ \ n\geq 1.$$

A variation of Dobiński's formula in terms of Kummer confluent hypergeometric functions is given by

$$\begin{aligned}L_n(x) &= \frac{(-1)^n x}{e^x}\sum_{k\geq 0}\frac{(n+1)^{\overline{k}}}{2^{\overline{k}}}\frac{x^k}{k!}\\ &= \frac{(-1)^n n! x}{e^x}\,{}_1F_1\left(\begin{matrix} n+1\\ 2\end{matrix}\ ;x\right),\end{aligned}$$

where ${}_1F_1\left(\begin{matrix} a\\ b\end{matrix}\ ;z\right)$ denotes the Kummer confluent hypergeometric functions, defined by $\sum_{n\geq 0}\frac{a^{\overline{n}}}{b^{\overline{n}}}\frac{z^n}{n!}$.

Theorem 4. *For $n, m \geq 0$, we have*

$$\sum_{k=0}^{m} s(m,k)(-1)^{n+k} B_{n+k}(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m L_{m+k}(x). \tag{7}$$

If we set $m = 0$ in (7), we have

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^n L_k(x) \qquad (8) \\ &= \frac{x}{e^x} \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} n!\, {}_1F_1 \left(\begin{matrix} n+1 \\ 2 \end{matrix} ; x \right). \end{aligned}$$

Notice that the Guo–Qi's identity for Bell numbers [4,5] is obtained by setting $x = 1$ in (8).

When $n = 0$ in (7) we get the inverse Stirling transform for Bell polynomials

$$L_m(x) = \sum_{k=0}^{m} s(m,k)(-1)^k B_k(x).$$

Now, setting $n = 1$ in (7), we get

$$\begin{aligned} \sum_{k=0}^{m} s(m,k)(-1)^{k+1} B_{k+1}(x) &= mL_m(x) + L_{m+1}(x) \\ &= \sum_{k=0}^{m} (-1)^{m+1} k! \binom{m}{k}^2 x^{m-k+1}. \end{aligned}$$

3 An explicit formula for Euler numbers

The Euler numbers E_n can be defined by the exponential generating function

$$\frac{1}{\cosh z} = \sum_{n \geq 0} E_n \frac{z^n}{n!}.$$

It is well-known that E_n are a sequence of integers with $E_{2n+1} = 0$ for $n \geq 0$. There are a number of explicit formulae for E_n, for example (see also [7])

$$E_{2n} = i \sum_{k=1}^{2n+1} \sum_{j=0}^{k} \frac{(-1)^j}{2^k i^k k} \binom{k}{j} (k-2j)^{2n+1},$$

where i denotes the imaginary unit with $i^2 = -1$. In this section, we propose a new explicit formula for the Euler numbers

$$E_n = -\sum_{k=0}^{n} \frac{k!}{2^k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \operatorname{Re}\left((i-1)^{k+1} \right), \tag{9}$$

where $\operatorname{Re}(z)$, denotes the real part of z.

Now, If we take the final sequence $a_{n,0} = E_n$, in (2), we get the following matrix

$$T = \begin{pmatrix} 1 & 0 & -1 & 3 & -6 & \cdots \\ 0 & -1 & 1 & 3 & -24 & \cdots \\ -1 & 0 & 5 & -15 & -6 & \cdots \\ 0 & 5 & -5 & -51 & 336 & \cdots \\ 5 & 0 & -61 & 183 & 714 & \cdots \\ 0 & -61 & 61 & 1263 & -7944 & \cdots \\ -61 & 0 & 1385 & -4155 & -35286 & \cdots \\ 0 & 1385 & -1385 & -47751 & 294816 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It follows from (5) that the initial sequence has an exponential generating function given by

$$\sum_{n\geq 0} R_n \frac{z^n}{n!} = \frac{1}{\cosh(\ln(1+z))} \tag{10}$$

$$= \frac{2(1+z)}{2+2z+z^2}. \tag{11}$$

Since $\dfrac{2(1+z)}{2+2z+z^2} = \dfrac{1}{z+1-i} + \dfrac{1}{z+1+i}$, we have

$$\sum_{n\geq 0} R_n \frac{z^n}{n!} = -\sum_{n\geq 0} \frac{1}{2^{n+1}} \left((-1+i)^{n+1} + (-1-i)^{n+1}\right) z^n$$

$$= -\sum_{n\geq 0} \frac{1}{2^n} \operatorname{Re}\left((i-1)^{n+1}\right) z^n.$$

Equating the coefficient of z^n, we get

$$R_n = -\frac{n!}{2^n} \operatorname{Re}\left((i-1)^{n+1}\right)$$

and by the binomial formula we have

$$R_n = \frac{n!}{2^n} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^{n-k} \binom{n+1}{2k},$$

where $\lfloor x \rfloor$ denotes the integral part of x, that is, the greatest integer not exceeding x.

For $n, m \geq 0$, we have

$$\sum_{k=0}^{m} s(m,k) E_{n+k} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m R_{m+k}. \tag{12}$$

Now, setting $m = 0$ in (12), we get (9). Putting $n = 0$ in (12), we have the following recursive formula Euler numbers involving the Stirling numbers of the first kind

$$\sum_{k=0}^{m} s(2m,2k) E_{2k} = \frac{(2m)!}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k}$$

$$= \begin{cases} (-1)^{\lfloor \frac{m}{2} \rfloor} \dfrac{(2m)!}{2^m}, & m \text{ even}, \\ (-1)^{\lfloor \frac{m-1}{2} \rfloor + 1} \dfrac{(2m)!}{2^m}, & m \text{ odd}. \end{cases}$$

Thus, for example, when $m = 0, 1, 2, 3$, we obtain

$$\begin{aligned} E_0 &= 1, \\ E_2 &= -1, \\ 11E_2 + E_4 &= -6, \\ 274E_2 + 85E_4 + E_6 &= 90. \end{aligned}$$

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