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Generalized dual Fibonacci quaternions with dual coefficient

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Abstract: In this paper, we defined the generalized dual Fibonacci quaternions with dual coefficient. Also, we investigated the relations between the generalized dual Fibonacci quaternions with dual coefficient. Furthermore, we gave the Binet's formulas and Cassini identities for these quaternions.
Keywords: Fibonacci number, Generalized Fibonacci number, Fibonacci quaternion, Dual quaternion, Dual Fibonacci quaternion, Generalized dual Fibonacci quaternion.
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1 Introduction

The quaternions are a number system that extends the complex numbers. They were first described by the Irish mathematician William Rowan Hamilton in 1843. Hamilton [7] introduced the set of quaternions which can be represented as

$$H = \{ \, q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \;|\; q_0,\, q_1,\, q_2,\, q_3 \in \mathbb{R} \, \} \tag{1}$$

where

$$i^2 = j^2 = k^2 = -1\,, \;\; i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j\,.$$

Several authors worked on different quaternions and their generalizations (see [1, 3–6, 9, 10, 12, 13, 16]). In 1963, Horadam [10, 11] firstly introduced the n-th Fibonacci quaternion and generalized Fibonacci quaternions, which can be represented as

$$H_F = \{Q_n = F_n + i\,F_{n+1} + j\;F_{n+2} + k\,F_{n+3}\;\mid F_n,\, n\text{-th Fibonacci number,}\} \tag{2}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\;\; i\,j = -j\,i = k\,,\; j\,k = -k\,j = i\,,\, k\,i = -i\,k = j$$

and $n \geq 1$.

In 1969, Iyer [12, 13] derived many relations for the Fibonacci quaternions. Also, in 1973, Swamy [16] considered generalized Fibonacci quaternions as a new quaternion as follows:

$$P_n = H_n + i\,H_{n+1} + j\;H_{n+2} + k\,H_{n+3} \tag{3}$$

where

$$\begin{array}{l} H_n = H_{n-1} + \;H_{n-2}, \\ H_1 = p, H_2 = p+q, \\ \text{ or} \\ H_n = (p-q)F_n + qF_{n+1},\;\; n \geq 1. \end{array}.$$

Here, H_n is the n-th generalized Fibonacci number that defined in [10] (see [16] for generalized Fibonacci quaternions).

Clifford [3] published his work on dual numbers in 1873. The dual numbers extend to the real numbers has the form

$$d = a + \varepsilon\, a^*$$

where ε is the dual unit and ε^2, $\varepsilon \neq 0$. In 2009, Ata and Yaylı [2] defined dual quaternions with dual numbers ($a + \varepsilon\, b\,,\; a, b \in \mathbb{R}\,,\; \varepsilon^2 = 0\,,\; \varepsilon \neq 0$) coefficient as follows:

$$H(\mathbb{D}) = \{Q = A + Bi\; + Cj\; + Dk\; \mid\, A,\, B,\, C,\, D \in \mathbb{D}\,,\, i^2 = j^2 = k^2 = -1 = i\,j\,k\} \tag{4}$$

In 2014, Nurkan and Güven [15] defined dual Fibonacci quaternions as follows:

$$H(\mathbb{D}) = \{\tilde{Q}_n = \tilde{F}_n + i\tilde{F}_{n+1} + j\tilde{F}_{n+2} + k\tilde{F}_{n+3} \mid \tilde{F}_n = F_n + \epsilon F_{n+1},\, \epsilon^2 = 0, \epsilon \neq 0\}, \tag{5}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\;\; i\,j = -j\,i = k\,,\; j\,k = -k\,j = i\,,\, k\,i = -i\,k = j$$

$n \geq 1$ and $\tilde{Q}_n = Q_n + \varepsilon\, Q_{n+1}$. Essentially, these quaternions in equations (4) and (5) must be called dual coefficient quaternion and dual coefficient Fibonacci quaternions, respectively. Majernik [14] defined dual quaternions as follows:

$$H_{\mathbb{D}} = \left\{ \begin{array}{c} Q = a + b\,i\; + c\,j\; + d\,k\; \mid\; a\,,\, b\,,\, c\,,\, d \in \mathbb{R},\; i^2 = j^2 = k^2 = i\,j\,k = 0, \\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0 \end{array} \right\}. \tag{6}$$

For more details on dual quaternions, see [4].

It is clear that $H(\mathbb{D})$ and $H_{\mathbb{D}}$ are different sets.

In 2016, Yüce and Torunbalcı Aydın [17] defined dual Fibonacci quaternions as follows:

$$H_{\mathbb{D}} = \{\, Q_n = F_n + i\,F_{n+1} + j\;F_{n+2} + k\,F_{n+3} \;|\, F_n,\ n\text{-th Fibonacci number}\}, \tag{7}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\;\; i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

The Lucas sequence (L_n) and D_n^L which is the n-th term of the dual Lucas quaternion sequence (D_n^L) are defined by the following recurrence relations:

$$\begin{cases} L_{n+2} = L_{n+1} + L_n,\ \forall n \geq 0 \\ L_0 = 2,\ L_1 = 1 \end{cases} \tag{8}$$

and

$$D_n^L = L_n + i\,L_n + j\,L_{n+2} + k\,L_{n+3}, \tag{9}$$
$$i^2 = j^2 = k^2 = i\,j\,k = 0.$$

In 2016, Yüce and Torunbalcı Aydın [18] defined the generalized dual Fibonacci quaternions by using generalized Fibonacci numbers as follows

$$Q_{\mathbb{D}} = \{\mathbb{D}_{\mathbf{n}} = H_n + i\,H_{n+1} + j\;H_{n+2} + k\,H_{n+3} \;|\, H_n,\ n\text{-th Generalized Fibonacci number}\}, \tag{10}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\;\; i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

where H_n is the n-th generalized Fibonacci number that defined in [10]

$$\begin{cases} H_0 = & q,\ H_1 = p,\ H_2 = p+q,\ \ p,q \in \mathbb{Z} \\ H_n = & H_{n-1} +\ H_{n-2},\ \ n \geq 2 \\ \quad \text{or} & \\ H_n = & (p-q)F_n + q\,F_{n+1}. \end{cases} \tag{11}$$

Also, in 2016, Yüce and Torunbalcı Aydın [19] defined the generalized dual Fibonacci sequences as follows:

$$\begin{cases} \mathbb{D}_0 = & q + \varepsilon q,\ \mathbb{D}_1 = p + \varepsilon(p+q),\ \mathbb{D}_2 = (p+q) + \varepsilon(2p+q),\ \ p,q \in \mathbb{Z} \\ \mathbb{D}_n = & \mathbb{D}_{n-1} +\ \mathbb{D}_{n-2},\ \ n \geq 2 \\ \quad \text{or} & \\ \mathbb{D}_n = & (p-q+\varepsilon q)F_n +\ (q+\varepsilon p)\,F_{n+1}. \end{cases} \tag{12}$$

In this paper, we will define the generalized dual Fibonacci quaternions with dual coefficient as follows:

$$\widetilde{Q_{\mathbb{D}}} = \{\widetilde{\mathbb{D}_n} = (H_n + \varepsilon\,H_{n+1}) + i\,(H_{n+1} + \varepsilon\,H_{n+2}) + j\,(H_{n+2} + \varepsilon\,H_{n+2}) + k\,(H_{n+3} + \varepsilon\,H_{n+3})\ | \\ H_n,\ n\text{-th Generalized Fibonacci number}\} \tag{13}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0$$

and $n \geq 1$.

Also, we will give Binet's Formula and Cassini identities for the generalized dual Fibonacci quaternions with dual coefficient.

2 Generalized dual Fibonacci quaternions with dual coefficient

The generalized dual Fibonacci sequence $\mathbb{D}_n$ is defined as [19]

$$\begin{array}{ll} \mathbb{D}_n = & \{H_n + \varepsilon\, H_{n+1}\,,\ \varepsilon^2 = 0,\ \varepsilon \neq 0\} \\ \quad \text{or} & \\ \mathbb{D}_n = & (p - q + \varepsilon\, q) F_n + (q + \varepsilon\, p)\, F_{n+1}. \end{array} \tag{14}$$

where the elements of the generalized dual Fibonacci sequence are

$$(\mathbb{D}_\mathrm{n}):\ \ p + \varepsilon\,(p+q)\,,\ (p+q) + \varepsilon\,(2p+q),\ \ldots,\ (p - q + \varepsilon\, q)\, F_n + (q + \varepsilon\, p)\, F_{n+1}, \ldots \tag{15}$$

The generalized dual Fibonacci quaternion $Q_{\mathbb{D}}$ is defined as [18]

$$\begin{array}{l} Q_{\mathbb{D}} = \{\, \mathbb{D}_n = H_n + i\, H_{n+1} + j\, H_{n+2} + k\, H_{n+3}\ \mid \\ \qquad H_n,\ n\text{-th Generalized Fibonacci number}\} \end{array} \tag{16}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

We can define the generalized dual Fibonacci quaternions with dual coefficient by using generalized dual Fibonacci numbers as follows

$$\begin{array}{l} \widetilde{\mathbb{D}_n} = \{(H_n + \varepsilon\, H_{n+1}) + i\,(H_{n+1} + \varepsilon\, H_{n+2}) + j\,(H_{n+2} + \varepsilon\, H_{n+3}) + k\,(H_{n+3} + \varepsilon\, H_{n+4})\ \mid \\ \qquad H_n,\ n\text{-th Generalized Fibonacci number}\} \end{array} \tag{17}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0$$

and $\varepsilon^2 = 0,\ \varepsilon \neq 0$.

Let $\widetilde{\mathbb{D}_n}^1$ and $\widetilde{\mathbb{D}_n}^2$ be the n-th terms of the generalized dual Fibonacci quaternion with coefficient sequence $(\widetilde{\mathbb{D}_n}^1)$ and $(\widetilde{\mathbb{D}_n}^2)$ such that

$$\widetilde{\mathbb{D}_n}^1 = (H_n + \varepsilon\, H_{n+1}) + i\,(H_{n+1} + \varepsilon\, H_{n+2}) + j\,(H_{n+2} + \varepsilon\, H_{n+3}) + k\,(H_{n+3} + \varepsilon\, H_{n+4}) \tag{18}$$

and

$$\widetilde{\mathbb{D}_n}^2 = (K_n + \varepsilon\, K_{n+1}) + i\,(K_{n+1} + \varepsilon\, K_{n+2}) + j\,(K_{n+2} + \varepsilon\, K_{n+3}) + k\,(K_{n+3} + \varepsilon\, K_{n+4}) \tag{19}$$

Then, the addition and subtraction of the generalized dual Fibonacci quaternions with coefficient are defined by

$$
\begin{aligned}
\widetilde{\mathbb{D}_n}^1 \pm \widetilde{\mathbb{D}_n}^2 = & (H_n + \varepsilon\, H_{n+1}) + i\,(H_{n+1} + \varepsilon\, H_{n+2}) + j\,(H_{n+2} + \varepsilon\, H_{n+3}) + k\,(H_{n+3} + \varepsilon\, H_{n+4}) \\
& \pm (K_n + \varepsilon\, K_{n+1}) + i\,(K_{n+1} + \varepsilon\, K_{n+2}) + j\,(K_{n+2} + \varepsilon\, K_{n+3}) + k\,(K_{n+3} + \varepsilon\, K_{n+4}) \\
= & [(H_n \pm K_n) + \varepsilon(H_{n+1} \pm K_{n+1})] + i\,[(H_{n+1} \pm K_{n+1}) + \varepsilon(H_{n+2} \pm K_{n+2})] \\
& + j\,[(H_{n+2} \pm K_{n+2}) + \varepsilon(H_{n+3} \pm K_{n+3})] + k\,[(H_{n+3} \pm K_{n+3}) + \varepsilon(H_{n+4} \pm K_{n+4})]\,.
\end{aligned} \tag{20}
$$

The multiplication of the generalized dual Fibonacci quaternions with coefficient is defined by

$$
\begin{aligned}
\widetilde{\mathbb{D}_n}^1 \widetilde{\mathbb{D}_n}^2 = & \; [(H_n + \varepsilon\, H_{n+1}) + i\,(H_{n+1} + \varepsilon\, H_{n+2}) + j\,(H_{n+2} + \varepsilon\, H_{n+2}) + k\,(H_{n+3} + \varepsilon\, H_{n+3})] \\
& \; [(K_n + \varepsilon\, K_{n+1}) + i\,(K_{n+1} + \varepsilon\, K_{n+2}) + j\,(K_{n+2} + \varepsilon\, K_{n+2}) + k\,(K_{n+3} + \varepsilon\, K_{n+3})] \\
= & \; (H_n\, K_n) + \; i\,(H_n K_{n+1} + H_{n+1} K_n) + j\,(H_n K_{n+2} + H_{n+2} K_n) \\
& + k\,(H_n K_{n+3} + H_{n+3} K_n) \\
& + \varepsilon\{H_n K_{n+1} + H_{n+1} K_n + i\,(H_n K_{n+2} + H_{n+1} K_{n+1} + H_{n+1} K_{n+1} + H_{n+2} K_n) \\
& \quad + j\,(H_n K_{n+3} + H_{n+2} K_{n+1} + H_{n+1} K_{n+2} + H_{n+3} K_n) \\
& \quad + k\,(H_n K_{n+4} + H_{n+3} K_{n+1} + H_{n+1} K_{n+3} + H_{n+4} K_n)\} \\
= & \; [\, H_n K_n + \varepsilon(H_n K_{n+1} + H_{n+1} K_{n+1})\,] \\
& + \{\, [\, H_n\,(i\, K_{n+1} + j\, K_{n+2} + k\, K_{n+3})\,] \\
& \quad + \varepsilon[H_n\,(i\, K_{n+2} + j\, K_{n+3} + k\, K_{n+4}) + H_{n+1}\,(i\, K_{n+1} + j\, K_{n+2} + k\, K_{n+3})]\} \\
& + \{\, [\, K_n\,(i\, H_{n+1} + j\, H_{n+2} + k\, H_{n+3})\,] \\
& \quad + \varepsilon[K_{n+1}\,(i\, H_{n+1} + j\, H_{n+2} + k\, H_{n+3}) + K_n\,(i\, H_{n+2} + j\, H_{n+3} + k\, H_{n+4})]\}
\end{aligned} \tag{21}
$$

The scalar and the vector part of $\widetilde{\mathbb{D}_n}$ which is the n-th term of the generalized dual Fibonacci quaternion with $(\widetilde{\mathbb{D}_n})$ are denoted by

$$
S_{\widetilde{\mathbb{D}}_n} = H_n + \varepsilon\, H_{n+1} \;\text{ and }\; V_{\widetilde{\mathbb{D}}_n} = i\,(H_{n+1} + \varepsilon\, H_{n+2}) + j\,(H_{n+2} + \varepsilon\, H_{n+3}) + k\,(H_{n+3} + \varepsilon\, H_{n+4}). \tag{22}
$$

Thus, the generalized dual Fibonacci quaternion $\widetilde{\mathbb{D}_n}^1$ is given by $\widetilde{\mathbb{D}_n}^1 = S_{\widetilde{\mathbb{D}}_n^1} + V_{\widetilde{\mathbb{D}}_n^1}$. Then, relation (21) is defined by

$$
\widetilde{\mathbb{D}_n}^1\, \widetilde{\mathbb{D}_n}^2 = S_{\widetilde{\mathbb{D}}_n^1}\, S_{\widetilde{\mathbb{D}}_n^2} + S_{\widetilde{\mathbb{D}}_n^1} V_{\widetilde{\mathbb{D}}_n^2} + S_{\widetilde{\mathbb{D}}_n^2}\, V_{\widetilde{\mathbb{D}}_n^1}. \tag{23}
$$

The conjugate of generalized dual Fibonacci quaternion $\widetilde{\mathbb{D}_n}$ is denoted by $\overline{\widetilde{\mathbb{D}_n}}$ and it is

$$
\overline{\widetilde{\mathbb{D}_n}} = (H_n + \varepsilon H_{n+1}) - i\,(H_{n+1} + \varepsilon H_{n+2}) - j\,(H_{n+2} + \varepsilon H_{n+3}) - k\,(H_{n+3} + \varepsilon H_{n+4}) \tag{24}
$$

The norm of $\widetilde{\mathbb{D}_n}$ is defined as

$$
\|\widetilde{\mathbb{D}_n}\|^2 = \widetilde{\mathbb{D}_n}\; \overline{\widetilde{\mathbb{D}_n}} = (H_n)^2 + 2\varepsilon H_n H_{n+1}. \tag{25}
$$

Then, we give the following theorem using statements (14), (16) and the generalized Fibonacci number in [10] as follows

$$H_nH_m + H_{n+1}H_{m+1} = \begin{array}{l} p^2F_{n+m+1} + 2p\,q\,F_{n+m} + q^2F_{n+m-1} \\ = (2p-q)H_{n+m+1} - eF_{n+m+1} \end{array} . \tag{26}$$

where $e = p^2 - p\,q - q^2$.

Theorem 1. Let H_n and $\widetilde{\mathbb{D}_n}$ be the n-th terms of generalized Fibonacci sequence (H_n) and the generalized dual Fibonacci quaternion sequence $(\widetilde{\mathbb{D}_n})$, respectively. In this case, for $n \geq 1$ we can give the following relations:

$$\widetilde{\mathbb{D}_n} + \widetilde{\mathbb{D}_{n+1}} = \widetilde{\mathbb{D}_{n+2}} \tag{27}$$

$$(\widetilde{\mathbb{D}_n})^2 = 2\,(H_n + \varepsilon H_{n+1})\,\widetilde{\mathbb{D}_n} - (H_n + \varepsilon H_{n+1})^2 \tag{28}$$

$$\widetilde{\mathbb{D}_n} - i\,\widetilde{\mathbb{D}_{n+1}} - j\,\widetilde{\mathbb{D}_{n+2}} - k\,\widetilde{\mathbb{D}_{n+3}} = H_n + \varepsilon H_{n+1} \tag{29}$$

$$\begin{aligned} \widetilde{\mathbb{D}_n}\,\widetilde{\mathbb{D}_m} + \widetilde{\mathbb{D}_{n+1}}\,\widetilde{\mathbb{D}_{m+1}} =& (2p-q)[(2\,\mathbb{D}_{n+m+1} - H_{n+m+1}) + 2\varepsilon(2\mathbb{D}_{n+m+2} - H_{n+m+2})] \\ & - e[(2\,Q_{n+m+1} - F_{n+m+1}) + 2\varepsilon(2Q_{n+m+2} - F_{n+m+2})]. \end{aligned} \tag{30}$$

where Q_{n+m+1} is the dual Fibonacci quaternion [17].

Proof. By

$$\widetilde{\mathbb{D}_n} = (H_n + \varepsilon\,H_{n+1}) + i\,(H_{n+1} + \varepsilon\,H_{n+2}) + j\,(H_{n+2} + \varepsilon\,H_{n+3}) + k\,(H_{n+3} + \varepsilon\,H_{n+4}) \tag{31}$$

and

$$\widetilde{\mathbb{D}_{n+1}} = (H_{n+1} + \varepsilon\,H_{n+2}) + i\,(H_{n+2} + \varepsilon\,H_{n+3}) + j\,(H_{n+3} + \varepsilon\,H_{n+4}) + k\,(H_{n+4} + \varepsilon\,H_{n+5}) \tag{32}$$

we see that,

$$\begin{aligned} & \widetilde{\mathbb{D}_n} + \widetilde{\mathbb{D}_{n+1}} = \\ =& (H_n + \varepsilon\,H_{n+1}) + i\,(H_{n+1} + \varepsilon\,H_{n+2}) + j\,(H_{n+2} + \varepsilon\,H_{n+3}) + k\,(H_{n+3} + \varepsilon\,H_{n+4}) \\ & + (H_{n+1} + \varepsilon\,H_{n+2}) + i\,(H_{n+2} + \varepsilon\,H_{n+3}) + j\,(H_{n+3} + \varepsilon\,H_{n+4}) + k\,(H_{n+4} + \varepsilon\,H_{n+5}) \\ =& (H_n + H_{n+1}) + \varepsilon\,(H_{n+1} + H_{n+2}) + i\,[H_{n+1} + H_{n+2} + \varepsilon\,(H_{n+2} + H_{n+3})] \\ & + j\,[H_{n+2} + H_{n+3} + \varepsilon\,(H_{n+3} + H_{n+4})] + k\,[H_{n+3} + H_{n+4} + \varepsilon\,(H_{n+4} + H_{n+5})] \\ =& (H_{n+2} + \varepsilon H_{n+3}) + i\,(H_{n+3} + \varepsilon H_{n+4}) + j\,(H_{n+4} + \varepsilon H_{n+5}) + k\,(H_{n+5} + \varepsilon H_{n+6}) \\ =& \widetilde{\mathbb{D}_{n+2}} \end{aligned}$$

So (27) holds. We observe

$$\begin{aligned} (\widetilde{\mathbb{D}_n})^2 =& \ (H_n + \varepsilon H_{n+1})^2 \\ & +2\,(H_n + \varepsilon H_{n+1})[i\,(H_{n+1} + \varepsilon H_{n+2}) + j\,(H_{n+2} + \varepsilon H_{n+3}) + k\,(H_{n+3} + \varepsilon H_{n+4})] \\ =& \ 2\,(H_n + \varepsilon H_{n+1})[(H_n + \varepsilon H_{n+1}) + \ i(H_{n+1} + \varepsilon H_{n+2}) \\ & + j(H_{n+2} + \varepsilon H_{n+3}) + \ k(H_{n+3} + \varepsilon H_{n+4}) - (H_n + \varepsilon H_{n+1})^2] \\ =& \ 2\,(H_n + \varepsilon H_{n+1})\,\widetilde{\mathbb{D}_n} - (H_n + \varepsilon H_{n+1})^2. \end{aligned}$$

So (28) holds. By using conditions and the equation (16), we see that

$$\begin{aligned}\widetilde{\mathbb{D}_n} - i\,\widetilde{\mathbb{D}_{n+1}} - j\,\widetilde{\mathbb{D}_{n+2}} - k\,\widetilde{\mathbb{D}_{n+3}} = &\ (H_n + \varepsilon\, H_{n+1}) + i\,[(H_{n+1} + \varepsilon\, H_{n+2}) - (H_{n+1} + \varepsilon\, H_{n+2})] \\ &+ j\,[(H_{n+2} + \varepsilon\, H_{n+3}) - (H_{n+2} + \varepsilon\, H_{n+3})] \\ &+ k\,[(H_{n+3} + \varepsilon\, H_{n+4}) - (H_{n+3} + \varepsilon\, H_{n+4})] \\ = &\ H_n + \varepsilon\, H_{n+1}.\end{aligned}$$

So (29) holds. By (21) and (26), we see that

$$\begin{aligned}\widetilde{\mathbb{D}_n}\,\widetilde{\mathbb{D}_m} = &\ [H_n\, H_m + \varepsilon(H_n H_{m+1} + H_{n+1} H_m)] \\ &+ i\,[(H_n\, H_{m+1} + H_{n+1}\, H_m) + \varepsilon(H_n H_{m+2} + H_{n+1} H_{m+1} + H_{n+1} H_{m+1} + H_{n+2} H_m)] \\ &+ j\,[(H_n\, H_{m+2} + H_{n+2}\, H_m) + \varepsilon(H_n H_{m+3} + H_{n+2} H_{m+1} + H_{n+1} H_{m+2} + H_{n+3} H_m)] \\ &+ k\,[(H_n\, H_{m+3} + H_{n+3}\, H_m) + \varepsilon(H_n H_{m+4} + H_{n+3} H_{m+1} + H_{n+1} H_{m+3} + H_{n+4} H_m)].\end{aligned} \tag{33}$$

and

$$\begin{aligned}\widetilde{\mathbb{D}_{n+1}}\,\widetilde{\mathbb{D}_{m+1}} = &\ [H_{n+1}\, H_{m+1} + \varepsilon(H_{n+1} H_{m+2} + H_{n+2} H_{m+1})] \\ &+ i\,[(H_{n+1}\, H_{m+2} + H_{n+2}\, H_{m+1}) \\ &+ \varepsilon(H_{n+1} H_{m+3} + H_{n+2} H_{m+2} + H_{n+2} H_{m+2} + H_{n+3} H_{m+1})] \\ &+ j\,[(H_{n+1}\, H_{m+3} + H_{n+3}\, H_{m+1}) \\ &+ \varepsilon(H_{n+1} H_{m+4} + H_{n+3} H_{m+2} + H_{n+2} H_{m+3} + H_{n+4} H_{m+1})] \\ &+ k\,[(H_{n+1}\, H_{m+4} + H_{n+4}\, H_{m+1}) \\ &+ \varepsilon(H_{n+1} H_{m+5} + H_{n+4} H_{m+2} + H_{n+2} H_{m+4} + H_{n+5} H_{m+1})].\end{aligned} \tag{34}$$

So (30) holds. Finally, adding equations (33) and (34) side by side, we obtain

$$\begin{aligned}\widetilde{\mathbb{D}_n}\,\widetilde{\mathbb{D}_m} + \widetilde{\mathbb{D}_{n+1}}\,\widetilde{\mathbb{D}_{m+1}} = &\ (2p-q)[(2\,\mathbb{D}_{n+m+1} - H_{n+m+1}) + 2\,\varepsilon\,(2\,\mathbb{D}_{n+m+2} - H_{n+m+2})] \\ &- e[(2\,Q_{n+m+1} - F_{n+m+1}) + 2\,\varepsilon\,(2\,Q_{n+m+2} - F_{n+m+2})]\end{aligned}$$

where Q_{n+m+1} is the dual Fibonacci quaternion [17]. □

Theorem 2. Let $\widetilde{\mathbb{D}_n}$ and $\widetilde{\mathbb{D}}_n^L$ be the n-th terms of the generalized dual Fibonacci quaternion sequence $(\widetilde{\mathbb{D}_n})$ and the dual Lucas quaternion sequence (D_n^L), respectively. The following relations are satisfied

$$\widetilde{\mathbb{D}_{n-1}} + \widetilde{\mathbb{D}_{n+1}} = p\,(D_n^L + \varepsilon D_{n+1}^L) + q\,(D_{n-1}^L + \varepsilon D_n^L) = \widetilde{\mathbb{D}_{n+2}} - \widetilde{\mathbb{D}_{n-2}}. \tag{35}$$

Proof. From equations (31), (32) and identities

$$H_n = (p-q)F_n + q\,F_{n+1}$$

and

$$H_{n+1} + H_{n-1} = p\,L_n + q\,L_{n-1}$$

between the generalized Fibonacci number and the Lucas number, we see that

$$\begin{aligned}&\widetilde{\mathbb{D}_{n-1}} + \widetilde{\mathbb{D}_{n+1}} \\ &\quad = [(H_{n-1} + H_{n+1}) + \varepsilon\,(H_n + H_{n+2})] + i\,[(H_n + H_{n+2}) + \varepsilon\,(H_{n+1} + H_{n+3})]\end{aligned}$$

$$
\begin{aligned}
&+ j\,[(H_{n+1}+H_{n+3})+\varepsilon\,(H_{n+2}+H_{n+4})]+k\,[(H_{n+2}+H_{n+4})+\varepsilon\,(H_{n+3}+H_{n+5})]\\
=&(p\,L_n+q\,L_{n-1})+i\,(p\,L_{n+1}+q\,L_n)+j\,(p\,L_{n+2}+q\,L_{n+1})+k\,(p\,L_{n+3}+q\,L_{n+2})\\
&+\varepsilon[(p\,L_{n+1}+q\,L_n)+i\,(p\,L_{n+2}+q\,L_{n+1})+j\,(p\,L_{n+3}+q\,L_{n+2})\\
&+k\,(p\,L_{n+4}+q\,L_{n+3})]\\
=&p\,\{(L_n+i\,L_{n+1}+j\,L_{n+2}+k\,L_{n+3})+\varepsilon(L_{n+1}+i\,L_{n+2}+j\,L_{n+3}+k\,L_{n+4})\}\\
&+q\,\{(L_{n-1}+i\,L_n+j\,L_{n+1}+k\,L_{n+2})+\varepsilon(L_n+i\,L_{n+1}+j\,L_{n+2}+k\,L_{n+3})\}\\
=&p\,(D^L_n+\varepsilon D^L_{n+1})+q\,(D^L_{n-1}+\varepsilon D^L_n)
\end{aligned}
$$

and

$$
\begin{aligned}
&\widetilde{\mathbb{D}_{n+2}}-\widetilde{\mathbb{D}_{n-2}}\\
=&[(H_{n+2}-H_{n-2})+\varepsilon\,(H_{n+3}-H_{n-1})]+i\,[(H_{n+3}-H_{n-1})+\varepsilon\,(H_{n+4}-H_n)]\\
&+j\,[(H_{n+4}-H_n)+\varepsilon\,(H_{n+5}-H_{n+1})]+k\,[(H_{n+5}-H_{n+1})+\varepsilon\,(H_{n+6}-H_{n+2})]\\
=&(p\,L_n+q\,L_{n-1})+i\,(p\,L_{n+1}+q\,L_n)+j\,(p\,L_{n+2}+q\,L_{n+1})+k\,(p\,L_{n+3}+q\,L_{n+2})\\
&+\varepsilon[(p\,L_{n+1}+q\,L_n)+i\,(p\,L_{n+2}+q\,L_{n+1})+j\,(p\,L_{n+3}+q\,L_{n+2})\\
&+k\,(p\,L_{n+4}+q\,L_{n+3})]\\
=&p\{(L_n+i\,L_{n+1}+j\,L_{n+2}+k\,L_{n+3})+\varepsilon\,(L_{n+1}+i\,L_{n+2}+j\,L_{n+3}+k\,L_{n+4})\\
&+q\{(L_{n-1}+i\,L_n+j\,L_{n+1}+k\,L_{n+2})+\varepsilon\,(L_n+i\,L_{n+1}+j\,L_{n+2}+k\,L_{n+3})\}\\
=&p\,(D^L_n+\varepsilon(D^L_{n+1})+q\,(D^L_{n-1}+\varepsilon D^L_n)\\
=&\widetilde{\mathbb{D}_{n-1}}+\widetilde{\mathbb{D}_{n+1}}. \qquad\square
\end{aligned}
$$

Theorem 3. Let $\widetilde{\mathbb{D}_n}$ be the n-th term of the generalized dual Fibonacci quaternion sequence $(\widetilde{\mathbb{D}_n})$. Then, we can give the following relations between these quaternions:

$$\widetilde{\mathbb{D}_n}+\overline{\widetilde{\mathbb{D}_n}}=2\,(H_n+\varepsilon H_{n+1}) \tag{36}$$

$$\widetilde{\mathbb{D}_n}\overline{\widetilde{\mathbb{D}_n}}+\widetilde{\mathbb{D}_{n-1}}\overline{\widetilde{\mathbb{D}_{n-1}}}=[(2p-q)H_{2n-1}-e\,F_{2n-1}]+2\varepsilon[(2p-q)H_{2n}-e\,F_{2n}] \tag{37}$$

$$\widetilde{\mathbb{D}_n}\overline{\widetilde{\mathbb{D}_n}}+\widetilde{\mathbb{D}_{n+1}}\overline{\widetilde{\mathbb{D}_{n+1}}}=[(2p-q)H_{2n+1}-e\,F_{2n+1}]+2\varepsilon[(2p-q)H_{2n+2}-e\,F_{2n+2}] \tag{38}$$

$$\widetilde{\mathbb{D}_{n+1}}\overline{\widetilde{\mathbb{D}_{n+1}}}+\widetilde{\mathbb{D}_{n-1}}\overline{\widetilde{\mathbb{D}_{n-1}}}=[(2p-q)H_{2n}-e\,F_{2n}]+2\varepsilon[(2p-q)H_{2n+1}-e\,F_{2n+1}] \tag{39}$$

$$
\begin{aligned}
(\widetilde{\mathbb{D}_n})^2+(\widetilde{\mathbb{D}_{n-1}})^2=\;&(2p-q)[(2\mathbb{D}_{2n-1}-H_{2n-1})+2\varepsilon(2\mathbb{D}_{2n}-H_{2n})]\\
&-e[(2\,Q_{2n-1}-F_{2n-1})+2\varepsilon(2\,Q_{2n}-F_{2n})]
\end{aligned} \tag{40}
$$

where Q_{2n-1} is the dual Fibonacci quaternion [17].

Proof. By (24), we get

$$
\begin{aligned}
\widetilde{\mathbb{D}_n}+\overline{\widetilde{\mathbb{D}_n}}=\;&[(H_n+\varepsilon H_{n+1})+i\,(H_{n+1}+\varepsilon H_{n+2})+j\,(H_{n+2}+\varepsilon H_{n+3})+k\,(H_{n+3}+\varepsilon H_{n+4})]\\
&+[(H_n+\varepsilon H_{n+1})-i\,(H_{n+1}+\varepsilon H_{n+2})-j\,(H_{n+2}+\varepsilon H_{n+3})-k\,(H_{n+3}+\varepsilon H_{n+4})]\\
=\;&2\,(H_n+\varepsilon H_{n+1})\,.
\end{aligned}
$$

Then (36) holds. By (24) and (25), we get

$$\begin{aligned}
\widetilde{\mathbb{D}}_n\overline{\widetilde{\mathbb{D}}_n}+\widetilde{\mathbb{D}_{n-1}}\overline{\widetilde{\mathbb{D}_{n-1}}} &= (H_n+\varepsilon H_{n+1})^2+(H_{n-1}+\varepsilon H_n)^2\\
&= (H_n^2+H_{n-1}^2)+2\varepsilon(H_n\,H_{n+1}+H_{n-1}\,H_n)\\
&= [(2p-q)H_{2n-1}-eF_{2n-1}]+2\varepsilon[(2p-q)H_{2n}-eF_{2n}]
\end{aligned}$$

Then (37) holds. By (24) and (25), we get

$$\begin{aligned}
\widetilde{\mathbb{D}}_n\overline{\widetilde{\mathbb{D}}_n}+\widetilde{\mathbb{D}_{n+1}}\overline{\widetilde{\mathbb{D}_{n+1}}} &= (H_n+\varepsilon H_{n+1})^2+(H_{n+1}+\varepsilon H_{n+2})^2\\
&= (H_n^2+H_{n+1}^2)+2\varepsilon(H_n\,H_{n+1}+H_{n+1}\,H_{n+2})\\
&= [(2p-q)H_{2n+1}-eF_{2n+1}]+2\varepsilon[(2p-q)H_{2n+2}-eF_{2n+2}]
\end{aligned}$$

Then (38) holds. By (24) and (25), we get

$$\begin{aligned}
&\widetilde{\mathbb{D}_{n+1}}\overline{\widetilde{\mathbb{D}_{n+1}}}-\widetilde{\mathbb{D}_{n-1}}\overline{\widetilde{\mathbb{D}}_{n-1}}\\
&\qquad=(H_{n+1}+\varepsilon H_{n+2})^2-(H_{n-1}+\varepsilon H_n)^2\\
&\qquad=(H_{n+1}^2-H_{n-1}^2)+2\varepsilon(H_{n+1}\,H_{n+2}-H_{n-1}\,H_n)\\
&\qquad=[(2p-q)H_{2n}-eF_{2n}]+2\varepsilon[(2p-q)H_{2n+1}-eF_{2n+1}]
\end{aligned}$$

Then (39) holds. By (25), we get

$$\begin{aligned}
(\widetilde{\mathbb{D}}_n)^2+(\widetilde{\mathbb{D}_{n-1}})^2 &= [(H_n^2+H_{n-1}^2)+2\varepsilon(H_nH_{n+1}+H_{n-1}H_n)]\\
&\quad+2i[(H_nH_{n+1}+H_{n-1}H_n)+2\varepsilon(H_nH_{n+2}+H_{n-1}H_{n+1}+H_{n+1}^2+H_n^2)]\\
&\quad+2j[(H_nH_{n+2}+H_{n-1}H_{n+1})\\
&\quad+2\varepsilon(H_nH_{n+3}+H_{n-1}H_{n+2}+H_{n+1}H_{n+2}+H_nH_{n+1})]\\
&\quad+2k[(H_nH_{n+3}+H_{n-1}H_{n+2})\\
&\quad+2\varepsilon(H_nH_{n+4}+H_{n-1}H_{n+3}+H_{n+1}H_{n+3}+H_nH_{n+2})]\\
&= [(2p-q)H_{2n-1}-eF_{2n-1}]+2i[(2p-q)H_{2n}-eF_{2n}]\\
&\quad+2j[(2p-q)H_{2n+1}-eF_{2n+1}]\\
&\quad+2k[(2p-q)H_{2n+2}-eF_{2n+2}]\\
&\quad+2\varepsilon\{[(2p-q)H_{2n}-eF_{2n}]+2i[2(2p-q)H_{2n+1}-2eF_{2n+1}]\\
&\qquad+2j[2(2p-q)H_{2n+2}-2eF_{2n+2}]+2k[2(2p-q)H_{2n+3}-2eF_{2n+3}]\}\\
&= (2p-q)[(2\mathbb{D}_{2n-1}-H_{2n-1})+2\varepsilon(2\mathbb{D}_{2n}-H_{2n})]\\
&\quad-e[(2\,Q_{2n-1}-F_{2n-1})+2\varepsilon(2\,Q_{2n}-F_{2n})]
\end{aligned}$$

where Q_{2n-1} is the dual Fibonacci quaternion [17]. Then (40) holds. □

Theorem 4. Let $\widetilde{\mathbb{D}}_n$ be the n-th term of the generalized dual Fibonacci quaternion with dual coefficient sequence $(\widetilde{\mathbb{D}}_n)$. Then, we have the following identities

$$\sum_{s=1}^{n}\widetilde{\mathbb{D}}_s=\widetilde{\mathbb{D}_{n+2}}-\widetilde{\mathbb{D}}_2, \tag{41}$$

$$\sum_{s=0}^{p}\widetilde{\mathbb{D}_{n+s}}+\widetilde{\mathbb{D}_{n+1}}=\widetilde{\mathbb{D}_{n+p+2}}, \tag{42}$$

$$\sum_{s=1}^{n}\widetilde{\mathbb{D}_{2s-1}}=\widetilde{\mathbb{D}_{2n}}-\widetilde{\mathbb{D}_{0}}, \tag{43}$$

$$\sum_{s=1}^{n}\widetilde{\mathbb{D}_{2s}}=\widetilde{\mathbb{D}_{2n+1}}-\widetilde{\mathbb{D}_{1}}. \tag{44}$$

Proof. Since $\sum_{t=a}^{n}H_t=H_{n+2}-H_{a+1}$ [10], we get

$$\begin{aligned}\sum_{s=1}^{n}\widetilde{\mathbb{D}_{s}}=&\sum_{s=1}^{n}H_s+i\sum_{s=1}^{n}H_{s+1}+j\sum_{s=1}^{n}H_{s+2}+k\sum_{s=1}^{n}H_{s+3}\\&+\varepsilon(\sum_{s=1}^{n}H_{s+1}+i\sum_{s=1}^{n}H_{s+2}+j\sum_{s=1}^{n}H_{s+3}+k\sum_{s=1}^{n}H_{s+4})\\=&(H_{n+2}-H_2)+i\,(H_{n+3}-H_3)+j\,(H_{n+4}-H_4)+k\,(H_{n+5}-H_5)\\&+\varepsilon[(H_{n+3}-H_3)+i\,(H_{n+4}-H_4)+j\,(H_{n+5}-H_5)+k\,(H_{n+6}-H_6)]\\=&[(H_{n+2}+\varepsilon H_{n+3})-(H_2+\varepsilon H_3)]+i\,[(H_{n+3}+\varepsilon H_{n+4})-(H_3+\varepsilon H_4)]\\&+j\,[(H_{n+4}+\varepsilon H_{n+5})-(H_4+\varepsilon H_5)]+k\,[(H_{n+5})+\varepsilon H_{n+6})-(H_5+\varepsilon H_6)]\\=&\widetilde{\mathbb{D}_{n+2}}-\widetilde{\mathbb{D}_{2}}\,.\end{aligned}$$

Then (41) holds. We can write

$$\begin{aligned}\sum_{s=0}^{p}\widetilde{\mathbb{D}_{n+s}}+\widetilde{\mathbb{D}_{n+1}}=\;&(H_{n+p+2}-H_{n+1}+H_{n+1})+i\,(H_{n+p+3}-H_{n+2}+H_{n+2})\\&+j\,(H_{n+p+4}-H_{n+3}+H_{n+3})+k\,(H_{n+p+5}-H_{n+4}+H_{n+4})\\&+\varepsilon[(H_{n+p+3}-H_{n+2}+H_{n+2})+i\,(H_{n+p+4}-H_{n+3}+H_{n+3})\\&\quad+j\,(H_{n+p+5}-H_{n+4}+H_{n+4})+k\,(H_{n+p+6}-H_{n+5}+H_{n+5})]\\=\;&(H_{n+p+2}+\varepsilon H_{n+p+3})+i\,(H_{n+p+3}+\varepsilon H_{n+p+4})\\&+j\,(H_{n+p+4}+\varepsilon H_{n+p+5})+k\,(H_{n+p+5}+\varepsilon H_{n+p+6})\\=\;&\widetilde{\mathbb{D}_{n+p+2}}\,.\end{aligned}$$

Then (42) holds. By

$$\sum_{i=1}^{n}H_{2i-1}=H_{2n}-q\;\text{ and }\;\sum_{i=1}^{n}H_{2i}=H_{2n+1}-p$$

(see [10]), we get

$$\begin{aligned}\sum_{s=1}^{n}\widetilde{\mathbb{D}_{2s-1}}=&(H_{2n}-q)+i\,(H_{2n+1}-p)+j\,(H_{2n+2}-q-p)+k\,(H_{2n+3}-2p-q)\\&+\varepsilon[(H_{2n+1}-p)+i\,(H_{2n+2}-q-p)+j\,(H_{2n+3}-2p-q)+k\,(H_{2n+4}-3p-2q)]\\=&[(H_{2n}+\varepsilon H_{2n+1})+i\,(H_{2n+1}+\varepsilon H_{2n+2})+j\,(H_{2n+2}+\varepsilon H_{2n+3})+k\,(H_{2n+3}+\varepsilon H_{2n+4})]\\&-[(q+\varepsilon p)+i\,(p+\varepsilon p+q)+j\,(p+q+\varepsilon 2p+q)+k\,(2p+q+\varepsilon 3p+2q)]\\=&\widetilde{\mathbb{D}_{2n}}-[(H_0+\varepsilon H_1)+i\,(H_1+\varepsilon H_2)+j\,(H_2+\varepsilon H_3)+k\,(H_3+\varepsilon H_4)]\\=&\widetilde{\mathbb{D}_{2n}}-\widetilde{\mathbb{D}_{0}}\,.\end{aligned}$$

Then (43) holds. By $\sum\limits_{i=1}^{n} H_{2i} = H_{2n+1} - p$ [10], we get

$$\begin{aligned}
&\sum_{s=1}^{n} \widetilde{\mathbb{D}_{2s}} = \\
=&(H_{2n+1} - p) + i\,(H_{2n+2} - p - q) + j\,(H_{2n+3} - 2p - q) + k\,(H_{2n+4} - 3p - 2q) \\
&+ \varepsilon[(H_{2n+2} - p - q) + i\,(H_{2n+3} - 2p - q) + j\,(H_{2n+4} - 3p - 2q) + k\,(H_{2n+5} - 5p - 3q)] \\
=&[(H_{2n+1} + \varepsilon H_{2n+2}) + i\,(H_{2n+2} + \varepsilon H_{2n+3}) + j\,(H_{2n+3} + \varepsilon H_{2n+4} + k\,(H_{2n+4} + \varepsilon H_{2n+5}] \\
&- \{[p + \varepsilon(p+q)] + i\,[(p+q) + \varepsilon(2p+q)] + j\,[(2p+q) + \varepsilon(3p+2q)] \\
&+ k\,[(3p+2q) + \varepsilon(5p+3q)]\} \\
=&\widetilde{\mathbb{D}_{2n+1}} - [(H_1 + \varepsilon H_2) + i\,(H_2 + \varepsilon H_3) + j\,(H_3 + \varepsilon H_4) + k\,(H_4 + \varepsilon H_5)] \\
=&\widetilde{\mathbb{D}_{2n+1}} - \widetilde{\mathbb{D}_1}\,.
\end{aligned}$$

Then (44) holds. □

Theorem 5. Let $\widetilde{\mathbb{D}_n}$ and Q_n be the n-th terms of the generalized dual Fibonacci quaternion sequence $(\widetilde{\mathbb{D}_n})$ and the dual Fibonacci quaternion sequence (Q_n), respectively. Then, we have

$$\widetilde{Q_n}\,\overline{\widetilde{\mathbb{D}_n}} - \overline{\widetilde{Q_n}}\,\widetilde{\mathbb{D}_n} = 2\,[H_nQ_n - F_n\mathbb{D}_n] + 2\varepsilon[H_nQ_{n+1} - F_n\mathbb{D}_{n+1} + H_{n+1}Q_n - F_{n+1}\mathbb{D}_n] \quad (45)$$

$$\widetilde{Q_n}\,\overline{\widetilde{\mathbb{D}_n}} + \overline{\widetilde{Q_n}}\,\widetilde{\mathbb{D}_n} = 2\,F_nH_n + 2\varepsilon(F_nH_{n+1} + F_{n+1}H_n) \quad (46)$$

$$\begin{aligned}
\widetilde{Q_n}\,\widetilde{\mathbb{D}_n} - \overline{\widetilde{Q_n}}\,\overline{\widetilde{\mathbb{D}_n}} =& 2\,[F_n\mathbb{D}_n + H_nQ_n - 2\,F_nH_n] \\
&+ 2\,\varepsilon\,\{[F_n\mathbb{D}_{n+1} + H_nQ_{n+1} - 2\,F_{n+1}H_n] \\
&+ [F_{n+1}\mathbb{D}_n + H_{n+1}Q_n - 2\,F_nH_{n+1}]\}
\end{aligned} \quad (47)$$

Proof. By (7) and (17), we get

$$\begin{aligned}
&\widetilde{Q_n}\,\overline{\widetilde{\mathbb{D}_n}} - \overline{\widetilde{Q_n}}\,\widetilde{\mathbb{D}_n} = \\
=&[(F_n + \varepsilon F_{n+1}) + i\,(F_{n+1} + \varepsilon F_{n+2}) + j\,(F_{n+2} + \varepsilon F_{n+3}) + k\,(F_{n+3} + \varepsilon F_{n+4})] \\
&[(H_n + \varepsilon H_{n+1}) - i\,(H_{n+1} + \varepsilon H_{n+2}) - j\,(H_{n+2} + \varepsilon H_{n+3}) - k\,(H_{n+3} + \varepsilon H_{n+4})] \\
=&(2F_nH_n - 2F_nH_n) + 2\,\varepsilon(F_nH_{n+1} + F_{n+1}H_n - F_nH_{n+1} - F_{n+1}H_n) \\
&+ 2i\,(-F_nH_{n+1} + F_{n+1}H_n) \\
&+ 2\,\varepsilon(-F_nH_{n+2} + F_{n+1}H_{n+1} - F_{n+1}H_{n+1} + F_{n+2}H_n) \\
&+ 2j\,(-F_nH_{n+2} + F_{n+2}H_n) \\
&+ 2\,\varepsilon(-F_nH_{n+3} + F_{n+2}H_{n+1} - F_{n+1}H_{n+2} + F_{n+3}H_n) \\
&+ 2k\,(-F_nH_{n+3} + F_{n+3}H_n) \\
&+ 2\,\varepsilon(-F_nH_{n+4} + F_{n+3}H_{n+1} - F_{n+1}H_{n+3} + F_{n+4}H_n) \\
=&-2\,F_n[H_n + i\,H_{n+1} + j\,H_{n+2} + k\,H_{n+3}] \\
&+ 2\,H_n[F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3}] \\
&+ \varepsilon\{[-2\,F_n[H_{n+1} + i\,H_{n+2} + j\,H_{n+3} + k\,H_{n+4}]
\end{aligned}$$

$$
\begin{aligned}
&+ 2\,H_n[F_{n+1} + i\,F_{n+2} + j\,F_{n+3} + k\,F_{n+4}]\\
&+ 2\,H_{n+1}[F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3}]\\
&- 2\,F_{n+1}[H_n + i\,H_{n+1} + j\,H_{n+2} + k\,H_{n+3}]\}\\
=& - 2\,[F_n\mathbb{D}_n + H_nQ_n] + 2\,\varepsilon\{[-F_n\mathbb{D}_{n+1} + H_nQ_{n+1} - F_{n+1}\mathbb{D}_n + H_{n+1}Q_n]\}.
\end{aligned}
$$

Then (45) holds. By (10) and (17), we get

$$
\begin{aligned}
\widetilde{Q_n}\,\overline{\widetilde{\mathbb{D}_n}} + \overline{\widetilde{Q_n}}\,\widetilde{\mathbb{D}_n} =\ & (F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3})\\
& (H_n - i\,H_{n+1} - j\,H_{n+2} - k\,H_{n+3})\\
& +(F_n - i\,F_{n+1} - j\,F_{n+2} - k\,F_{n+3})\\
& (H_n + i\,H_{n+1} + j\,H_{n+2} + k\,H_{n+3})\\
=\ & (F_nH_n + F_nH_n) + \varepsilon(F_nH_{n+1} + F_{n+1}H_n + F_nH_{n+1} + F_{n+1}H_n)\\
=\ & 2\,F_nH_n + 2\varepsilon(F_nH_{n+1} + F_{n+1}H_n).
\end{aligned}
$$

Then (46) holds. By (10) and (17), we get

$$
\begin{aligned}
&\widetilde{Q_n}\,\widetilde{\mathbb{D}_n} - \overline{\widetilde{Q_n}}\,\overline{\widetilde{\mathbb{D}_n}} =\\
=&[(F_n + \varepsilon F_{n+1}) + i\,(F_{n+1} + \varepsilon F_{n+2}) + j\,(F_{n+2} + \varepsilon F_{n+3}) + k\,(F_{n+3} + \varepsilon F_{n+4})]\\
&[(H_n + \varepsilon H_{n+1}) - i\,(H_{n+1} + \varepsilon H_{n+2}) - j\,(H_{n+2} + \varepsilon H_{n+3}) - k\,(H_{n+3} + \varepsilon H_{n+4})]\\
=&2\,(F_nH_n - F_nH_n) + 2\,\varepsilon(F_nH_{n+1} + F_{n+1}H_n - F_nH_{n+1} - F_{n+1}H_n)\\
&+ 2i\,(F_nH_{n+1} + F_{n+1}H_n) + 2\,\varepsilon(F_nH_{n+2} + F_{n+1}H_{n+1} + F_{n+1}H_{n+1} + F_{n+2}H_n)\\
&+ 2j\,(F_nH_{n+2} + F_{n+2}H_n) + 2\,\varepsilon(F_nH_{n+3} + F_{n+1}H_{n+2} + F_{n+2}H_{n+1} + F_{n+3}H_n)\\
&+ 2k\,(F_nH_{n+3} + F_{n+3}H_n) + 2\,\varepsilon(F_nH_{n+3} + F_{n+1}H_{n+3} + F_{n+3}H_{n+1} + F_{n+4}H_n)\\
=&2\,F_n[H_n + i\,H_{n+1} + j\,H_{n+2} + k\,H_{n+3}]\\
&+ 2\,H_n[F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3}] - 4\,F_nH_n\\
&+ \varepsilon[2\,F_n[H_{n+1} + i\,H_{n+2} + j\,H_{n+3} + k\,H_{n+4}]\\
&+ 2\,H_n[F_{n+1} + i\,F_{n+2} + j\,F_{n+3} + k\,F_{n+4}] - 4\,F_{n+1}H_n\\
&+ 2\,F_{n+1}[H_n + i\,H_{n+1} + j\,H_{n+2} + k\,H_{n+3}]\\
&+ 2\,H_{n+1}[F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3}] - 4\,F_nH_{n+1}]\\
=&2\,[F_n\mathbb{D}_n + H_nQ_n - 2\,F_nH_n]\\
&+ 2\,\varepsilon\{[F_n\mathbb{D}_{n+1} + H_nQ_{n+1} - 2\,F_{n+1}H_n] + [F_{n+1}\mathbb{D}_n + H_{n+1}Q_n - 2\,F_nH_{n+1}]\}.
\end{aligned}
$$

Then (47) holds. □

Theorem 6. (Binet's Formula). Let $\widetilde{\mathbb{D}_n}$ be the n-th terms of the generalized dual Fibonacci quaternion with dual coefficient sequence $(\widetilde{\mathbb{D}_n})$. For $n \geq 1$, the Binet's formulas for these quaternions are as follows:

$$
\widetilde{\mathbb{D}_n} = \frac{1}{\alpha - \beta}\left(\underline{\alpha}\ \alpha^n\,(1 + \varepsilon\alpha) - \underline{\beta}\ \beta^n\,(1 + \varepsilon\beta)\right) = \frac{1}{\alpha - \beta}\left(\hat{\alpha}\ \alpha^n - \hat{\beta}\ \beta^n\right) \tag{48}
$$

where $\hat{\alpha} = \underline{\alpha}\,(1 + \varepsilon\,\alpha)$ and $\hat{\beta} = \underline{\beta}\,(1 + \varepsilon\,\beta)$

$$\begin{aligned}\hat{\alpha} = \quad & [(p - q\beta)] + \varepsilon[p\,(1-\beta) + q] + i\,\{[p\,(1-\beta)+q] + \varepsilon[p\,(2-\beta)+q]\} \\ & +j\,\{[p\,(2-\beta)+q] + \varepsilon[p\,(3-2\beta)+q]\} \\ & +k\,\{[p\,(3-2\beta)+q] + \varepsilon[p\,(5-3\beta)+q]\}, \quad \alpha = \tfrac{1+\sqrt{5}}{2}\end{aligned}$$

and

$$\begin{aligned}\hat{\beta} = \quad & [(p - q\alpha)] + \varepsilon[p\,(1-\alpha) + q] + i\,\{[p\,(1-\alpha)+q] + \varepsilon[p\,(2-\alpha)+q]\} \\ & +j\,\{[p\,(2-\alpha)+q] + \varepsilon[p\,(3-2\alpha)+q]\} \\ & +k\,\{[p\,(3-2\alpha)+q] + \varepsilon[p\,(5-3\alpha)+q]\}, \quad \beta = \tfrac{1-\sqrt{5}}{2}\end{aligned}$$

Proof. The characteristic equation of recurrence relation $\widetilde{\mathbb{D}_{n+2}} = \widetilde{\mathbb{D}_{n+1}} + \widetilde{\mathbb{D}_n}$ is

$$t^2 - t - 1 = 0.$$

The roots of this equation are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ where $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$.

The Binet's formulas for Fibonacci sequence, generalized Fibonacci sequence and dual Fibonacci quaternion sequence, dual Fibonacci quaternion with dual coefficient sequence and generalized dual Fibonacci quaternion sequence, respectively, are as follows:

$$F_n = \frac{1}{\sqrt{5}}\,(\,\alpha^n - \,\beta^n), \quad H_n = \frac{1}{2\sqrt{5}}\,(\,l\,\alpha^n - \,m\,\beta^n), \quad Q_n = \frac{1}{\sqrt{5}}\left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right)$$

$$\widetilde{Q_n} = \frac{1}{\sqrt{5}}\left(\underline{\alpha}\alpha^n(1+\varepsilon\alpha) - \underline{\beta}\beta^n(1+\varepsilon\beta)\right), \quad \mathbb{D}_n = \frac{1}{\alpha-\beta}\left(\underline{\alpha}\;\alpha^n\; - \underline{\beta}\;\beta^n\,\right)$$

$$\underline{\alpha} = 1 + i\,\alpha^2 + j\,\alpha^3 + k\,\alpha^4, \quad \underline{\beta} = 1 + i\,\beta^2 + j\,\beta^3 + k\,\beta^4$$

(see [10, 11, 15, 17, 18]).

Using recurrence relation and initial values:

$$\widetilde{\mathbb{D}_0} = [q + \varepsilon p,\, p + \varepsilon(p+q),\, (p+q) + \varepsilon(2p+q), (2p+q) + \varepsilon(3p+2q)],$$

$$\widetilde{\mathbb{D}_1} = [p + \varepsilon(p+q),\, (p+q)\varepsilon(2p+q),\, (2p+q) + \varepsilon(3p+2q), (3p+2q) + \varepsilon(5p+3q)],$$

the Binet's formula for $\widetilde{\mathbb{D}_n}$ is

$$\widetilde{\mathbb{D}_n} = \frac{1}{\alpha - \beta}[\;\hat{\alpha}\;\alpha^n\; - \hat{\beta}\;\beta^n\;]$$

where $\hat{\alpha} = \underline{\alpha}(1 + \varepsilon\,\alpha)$, $\hat{\beta} = \underline{\beta}(1 + \varepsilon\,\beta)$. □

Theorem 7. (Cassini's Identity). Let $\widetilde{\mathbb{D}_n}$ be the n-th terms of the generalized dual Fibonacci quaternion sequence $(\widetilde{\mathbb{D}_n})$. For $n \geq 1$, the Cassini-like identity for $\widetilde{\mathbb{D}_n}$ is as follows:

$$\widetilde{\mathbb{D}_{n-1}}\widetilde{\mathbb{D}_{n+1}} - (\widetilde{\mathbb{D}_n})^2 = (-1)^n\,e\,(1 + i + 3j + 4k)\,(1+\varepsilon). \tag{49}$$

Proof. By (31) and (32) we get

$$\begin{aligned}
&\widetilde{\mathbb{D}_{n-1}}\,\widetilde{\mathbb{D}_{n+1}} - (\widetilde{\mathbb{D}_n})^2 = \\
=&(H_{n-1} + i\,H_n + j\,H_{n+1} + k\,H_{n+2})\,(H_{n+1} + i\,H_{n+2} + j\,H_{n+3} + k\,H_{n+4}) \\
&- [H_n + i\,H_{n+1} + j\,H_{n+2} + k\,H_{n+3}]^2 \\
=&[H_{n-1}H_{n+1} - H_n^2] + \varepsilon[H_{n-1}H_{n+2} + H_nH_{n+1} - 2\,H_nH_{n+1}] \\
&+ i\,\{[H_{n-1}H_{n+2} + H_nH_{n+1} - 2\,H_nH_{n+1}] + \varepsilon[H_{n-1}H_{n+3} + H_{n+1}H_{n+1} - 2\,H_{n+1}H_{n+1}]\} \\
&+ j\,\{[H_{n-1}H_{n+3} - 2\,H_nH_{n+2} + H_{n+1}^2] + \varepsilon[H_{n-1}H_{n+4} + H_nH_{n+3} - 2\,H_nH_{n+3}]\} \\
&+ k\,\{[H_{n-1}H_{n+4} + H_{n+1}H_{n+2} - 2\,H_nH_{n+3}] \\
&\quad + \varepsilon[H_{n-1}H_{n+5} + H_{n+2}H_{n+2} - H_nH_{n+4} - H_{n+1}H_{n+3}]\} \\
=&(-1)^n\,e\,(1 + i + 3j + 4k)\,(1 + \varepsilon),
\end{aligned}$$

where we use identity of the generalized Fibonacci number as follows:

$$H_{n+1}H_{n-1} - H_n^2 = (-1)^n\,e,$$

$$H_{n+2}H_{n-1} - H_nH_{n+1} = (-1)^n\,e,$$

$$H_{n+3}H_{n-1} - H_{n+1}H_{n+1} - 2\,H_nH_{n+2} = 3\,(-1)^n\,e,$$

$$H_{n+4}H_{n-1} - H_{n+2}H_{n+1} - 2\,H_nH_{n+3} = 4\,(-1)^n\,e,$$

$$H_{n-1}H_{n+5} + H_{n+2}H_{n+2} - H_nH_{n+4} - H_{n+3}H_{n+1} = 4\,(-1)^n\,e,$$

where $e = p^2 - p\,q - q^2$. So (49) holds. □

Example 1. Let $\widetilde{\mathbb{D}_1}$, $\widetilde{\mathbb{D}_2}$,$\widetilde{\mathbb{D}_3}$ and $\widetilde{\mathbb{D}_4}$ be the generalized dual Fibonacci quaternions with dual coefficient such that

$$\begin{aligned}
\widetilde{\mathbb{D}_1} =& [p + \varepsilon(p+q)] + i\,[(p+q) + \varepsilon(2p+q)] + j\,[(2p+q) + \varepsilon(3p+2q)] \\
&+k\,[(3p+2q) + \varepsilon(5p+3q)] \\
\widetilde{\mathbb{D}_2} =& [(p+q) + \varepsilon(2p+q)] + i\,[(2p+q) + \varepsilon(3p+2q)] + j\,[(3p+2q) + \varepsilon(5p+3q)] \\
&+k\,[(5p+3q) + \varepsilon(8p+5q)] \\
\widetilde{\mathbb{D}_3} =& [(2p+q) + \varepsilon(3p+2q)] + i\,[(3p+2q) + \varepsilon(5p+3q)] + j\,[(5p+3q) + \varepsilon(8p+5q)] \\
&+k\,(8p+5q) + \varepsilon(13p+8q)] \\
\widetilde{\mathbb{D}_4} =& [(3p+2q) + \varepsilon(5p+3q)] + i\,[(5p+3q) + \varepsilon(8p+5q)] + j\,(8p+5q) + \varepsilon(13p+8q)] \\
&+k\,(13p+8q) + \varepsilon(21p+13q)]\,.
\end{aligned}$$

In this case,

$$\begin{aligned}
&\widetilde{\mathbb{D}_1}\,\widetilde{\mathbb{D}_3} - (\widetilde{\mathbb{D}_2})^2 = \\
=&\{[p + \varepsilon(p+q)] + i\,[(p+q) + \varepsilon[2p+q)] + j\,[(2p+q) + \varepsilon(3p+2q)] \\
&+ k\,[(3p+2q) + \varepsilon(5p+3q)]\} \\
&\{[(2p+q) + \varepsilon(3p+2q)] + i\,[(3p+2q) + \varepsilon(5p+3q)] \\
&\quad + j\,[(5p+3q) + \varepsilon(8p+5q)] + k\,(8p+5q) + \varepsilon(13p+8q)]\}
\end{aligned}$$

$$
\begin{aligned}
&- \{[(p+q)+\varepsilon(2p+q)]+i\,[(2p+q)+\varepsilon(3p+2q)] \\
&\quad + j\,[(3p+2q)+\varepsilon(5p+3q)]+k\,[(5p+3q)+\varepsilon(8p+5q)]\}^2 \\
=&(p^2-p\,q-q^2)+i\,(p^2-p\,q-q^2)+j\,(3p^2-3\,p\,q-3\,q^2)+k\,(4p^2-4\,p\,q-4\,q^2) \\
=&\,(p^2-p\,q-q^2))(1+i+3j+4k) \\
=&(-1)^2\,e\,(1+i+3j+4k)\,(1+\varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\mathbb{D}_2}\,\widetilde{\mathbb{D}_4}-(\widetilde{\mathbb{D}_3})^2 =&\{[(p+q)+\varepsilon(2p+q)]+i\,[(2p+q)+\varepsilon(3p+2q)] \\
&\quad + j\,[(3p+2q)+\varepsilon(5p+3q)]+k\,[(5p+3q)+\varepsilon(8p+5q)]\} \\
&\{[(3p+2q)+\varepsilon(5p+3q)]+i\,[(5p+3q)+\varepsilon(8p+5q)] \\
&\quad + j\,(8p+5q)+\varepsilon(13p+8q)]+k\,[(13p+8q)+\varepsilon(21p+13q)]\} \\
&-\ \{[(2p+q)+\varepsilon(3p+2q)]+i\,[(3p+2q)+\varepsilon(5p+3q)] \\
&\quad + j\,[(5p+3q)+\varepsilon(8p+5q)]+k\,[(8p+5q)+\varepsilon(13p+8q)]\}^2 \\
&=(-p^2+p\,q+q^2)+i\,(-p^2+p\,q+q^2)+j\,(-3p^2+3\,p\,q+3\,q^2) \\
&+k\,(-4p^2+4\,p\,q+4\,q^2) \\
&=\ (-1)^3\,(p^2-p\,q-q^2))(1+i+3j+4k) \\
&=(-1)^3\,e\,(1+i+3j+4k)\,(1+\varepsilon).
\end{aligned}
$$

3 Conclusion

Dual numbers form two dimensional commutative, associative algebra over the real numbers. Also the algebra of dual numbers is a ring.

A quaternion with dual coefficient is an extension of dual numbers whereby the elements of that quaternion are dual numbers. The quaternions with dual coefficient are used as an appliance for expressing and analyzing the physical properties of rigid bodies. They are computationally efficient approach of representing rigid transforms like translation and rotation.

In this paper, we investigated the generalized dual Fibonacci quaternions with the dual coefficient.

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